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ПРАЦІ МІЖНАРОДНОГО ГЕОМЕТРИЧНОГО ЦЕНТРУ

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Hamiltonian operators and related differential-algebraic Balinsky-Novikov, Riemann and Leibniz type structures on nonassociative noncommutative algebras

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Abstract. We review main differential-algebraic structures lying in background of analytical constructing multi-component Hamiltonian operators as derivatives on suitably constructed loop Lie algebras, generated by nonassociative noncommutative algebras. The related Balinsky-Novikov and Leibniz type algebraic structures are derived, a new nonassociative "Riemann" algebra is constructed, deeply related with infinite multi-component Riemann type integrable hierarchies. An approach, based on the classical Lie-Poisson structure on coadjoint orbits, closely related with those, analyzed in the present work and allowing effectively enough construction of Hamiltonian operators, is also briefly revisited. As the compatible Hamiltonian operators are constructed by means of suitable central extentions of the adjacent weak Lie algebras, generated by the right Leibniz and Riemann type nonassociative and noncommutative algebras, the problem of their description requires a detailed investigation both of their structural properties and finite-dimensional representations of the right Leibniz algebras defined by the corresponding structural constraints. Subject to these important aspects we stop in the work mostly on the structural properties of the right Leibniz algebras, especially on their derivation algebras and their generalizations. We have also added a short Supplement within which we revisited the classical Poisson manifold approach, closely related to our construction of Hamiltonian operators, generated by nonassociative and noncommutative algebras. In particular, we presented its natural and simple generalization allowing effectively to describe a wide class of Lax type integrable nonlinear Kontsevich type Hamiltonian systems on associative noncommutative algebras.

Keywords: Hamiltonian operators, Lie-Poisson structure, differenetial algebras, integrability, derivatives, loop-algebra, cocycles, Balinsky-Novikov algebra, right Leibniz algebra, Riemann algebra, group agebras, π-metrized Lie alegbras, Kontsevich type systems

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Анотація. Дано огляд основних диференціально-алгебраїчних структур, що лежать в основі аналітичної побудови багато-компонентних операторів Гамільтона як диференціювань на відповідних алгебрах Лі петель, породжених неасоціативними некомутативними алгебрами. Введені алгебраїчні структури типу Балінського-Новікова і Ляйбница, побудована нова неасоціативна алгебра Рімана, глибоко пов'язана з нескінченними інтегровними багато-компонентними ієрархіями типу Рімана. Також коротко висвітлений підхід, що грунтується на класичних структурах Лі-Пуассона на ко-приєднаних орбітах, які дозволяють ефективно конструювати оператори Гамільтона. Враховуючи, що узгоджені оператори Гамільтона генеруються відповідними центральними розширеннями приєднаних слабо-визначених алгебр Лі, породжених неасоціативними некомутативними алгебрами, проблема їх опису вимагає детального дослідження як структурних властивостей свойств, так і скінченно-вимірних зображень правих алгебр Ляйбніца, визначених відповідними структурними обмеженнями. Стосовно цих важливих аспектів ми обмежились в работі в основному правими алгебрами Лі, зокрема на їхніх алгебрах диференціювань та їх узагальненнях. Ми також помістили Додаток, в которому ми коротко висвітлили класичний підхід, що грунтується на пуассонових многовидах і тісно звязаний з нашою побудовою операторів Гамільтона, породженних неасоціативними некомутативними алгебрами. Зокрема, ми представили його природнє та просте узагальнення, котре дозволяє ефективно будувати широкий клас нелінійних інтегровних за Лаксом гамільтонових систем типу Концевича на асоціативних некомутативних алгебрах.

1. Introduction

We present a short review of main differential-algebraic structures lying in background of analytical constructing multicomponent Hamiltonian operators as derivatives on suitably constructed loop Lie algebras, generated by nonassociative and noncommutative algebras. During the last decades there were discovered [38, 22, 20, 70] many integrable Hamiltonian systems, whose internal symmetyry structure was analytical nature was understood owing to the Lie-algebraic properties of their internal hidden symmetry structures. A first account of the Hamiltonian operators and related differential-algebraic structures, lying in the background of integrable systems, was given by I. Gelfand and I. Dorfman [43, 34] and later was extended by B. Dubrovin and S. Novikov [36, 37], and also by A. Balinsky and S. Novikov [11, 14, 12, 13]. Also some new special differential-algebraic techniques [80] were devised for studying the Lax integrability and the structure of related Hamiltonian operators for a wide class of the Riemann type hydrodynamic hierarchies. Just recently considerable work [8, 10, 9, 73] has been done devoted to the finite dimensional representations of the Balinsky-Novikov algebra. Their importance for constructing integrable multi-component nonlinear Camassa-Holm type dynamical systems on functional manifolds was demonstrated by I. Strachan and B. Szablikowski in [91], which in part suggested the Lie-algebraic imbedding of the Balinsky-Novikov algebra into the general Lie-Poisson orbits scheme of classifying Lax integrable Hamiltonian systems. It is also worth of mentioning the related work [46] by Holm and Ivanov in which integrable multicomponent nonlinear Camassa-Holm type dynamical systems on functional manifolds were constructed.

In our work here we describe a differential-algebraic reformulation of the classical Lie algebraic scheme and develop an effective approach to classification of the underlying algebraic structures of integrable multicomponent Hamiltonian systems. In particular, we have devised a simple algorithm allowing to construct new algebraic structures within which the corresponding Hamiltonian operators exist and generate integrable multicomponent dynamical systems. We show, as examples, that the well-known Balinsky-Novikov algebraic structure, obtained in [43, 11] as a condition for a matrix differential expression to be Hamiltonian and in [19, 27, 50, 61] as a flat torsion free left-invariant affine connection on affine manifolds, affine structures and convex homogeneous cones, appears in our approach as a derivation on the Lie-algebra naturally associated with a suitably constructed differential loop algebra. As a direct generalization of this example we obtain two new differentiations, whose underlying algebraic structures coincide, respectively, with the well-known [3, 40] right Leibniz algebra, introduced in [23, 24, 59], and with a new "*Riemann*" algebra, which naturally generate different Hamiltonian operators describing a wide class of multicomponent hierarchies [21, 80] of integrable Riemann hydrodynamic systems. As the compatible Hamiltonian operators, important for studying integrable multicomponent Hamiltonian systems, are constructed by means of suitable central extentions of the adjacent weak Lie algebras, determined by the right Leibniz and Riemann type nonassociative and noncommutative algebras, their description requires a detailed investigation both of the structural properties and finite-dimensional representations of the right Leibniz algebras defined by the corresponding structural constraints. Subject to these important aspects we stop in the work mostly on the structural properties of the right Leibniz algebras, especially on their derivation algebras and their generalizations.

In a supplement the classical Poisson manifold approach, closely related to our construction of Hamiltonian operators, generated by nonassociative and noncommutative algebras, is briefly revisited. In particular, its natural and simple generalization appeared to be useful [5, 6, 94, 96, 62, 66, 67] for describing a wide class of Lax type integrable nonlinear Hamiltonian systems on associative noncommutative algebras, initiated first in [25, 35, 76, 78] in case of the noncommutative operator algebras and continued later in [62, 51, 52, 53, 62, 66, 67, 71] in case of general associative noncommutative algebras.

2. The Hamiltonian operators and related algebraic structures via the differential-algebraic approach

Let $(\mathbb{A};\circ)$ be a finite-dimensional algebra of dimension $N=\dim\mathbb{A}\in\mathbb{Z}_+$ (in general noncommutive and nonassociative) over an algebraically closed field $\mathbb{K}$. Using the algebra $\mathbb{A}$ one can construct the related loop algebra $\widetilde{\mathbb{A}}$ of smooth mappings $u:\mathbb{S}^1\to\mathbb{A}$ and endow it with a suitably generalized natural convolution $\langle\cdot,\cdot\rangle$ on $\widetilde{\mathbb{A}}^*\times\widetilde{\mathbb{A}}\to\mathbb{K}$, where $\widetilde{\mathbb{A}}^*$ is the corresponding adjoint to $\widetilde{\mathbb{A}}$ space.

First, we shall consider a general scheme of constructing nontrivial ultra-local and local [38] Poisson structures on the adjoint space $\widetilde{\mathbb{A}}^*$, compatible with the internal multiplication in the loop algebra $\widetilde{\mathbb{A}}$. Consider a basis $\{e_s\in\mathbb{A}:s=\overline{1,N}\}$ of the algebra $\mathbb{A}$ and its dual $\{u^s\in\mathbb{A}^*:s=\overline{1,N}\}$ with respect to $\langle\cdot,\cdot\rangle$ on $\mathbb{A}^*\times\mathbb{A}$, that is $\langle u^j,e_i\rangle:=\delta_i^j$ for all $i,j=\overline{1,N}$, and such that for any $u(x)=\sum_{s=\overline{1,N}}u_s(x)u^s\in\widetilde{\mathbb{A}}^*$, $x\in\mathbb{S}^1$, the quantities $u_s(x):=\langle u(x),e_s\rangle\in\mathbb{K}$ for all $s=\overline{1,N}$, $x\in\mathbb{S}^1$. Put

$$\widetilde{\mathbb{A}}\wedge\widetilde{\mathbb{A}}:=\mathrm{Skew}(\widetilde{\mathbb{A}}\otimes\widetilde{\mathbb{A}})$$

and let $\vartheta^*:\widetilde{\mathbb{A}}\wedge\widetilde{\mathbb{A}}\to\widetilde{\mathbb{A}}$ be a skew-symmetric bilinear mapping. Then the expression

$$\{u_i(x),u_j(x)\}:=\langle u(x),\vartheta^*(e_i\wedge e_j)\rangle \tag{2.1}$$

defines for any $x,y\in\mathbb{S}^1$ and all $i,j=\overline{1,N}$ an ultra-local *linear* skew-symmetric pre-Poisson bracket on $\widetilde{\mathbb{A}}^*$. Since the algebra $\widetilde{\mathbb{A}}$ possesses its internal multiplicative structure "$\circ$", the important problem arises: *Under what conditions is the pre-Poisson bracket* (2.1) *Poisson and compatible with this internal structure on* $\widetilde{\mathbb{A}}$? To proceed with elucidating this question, we define a co-multiplication $\Delta:\widetilde{\mathbb{A}}^*\to\widetilde{\mathbb{A}}^*\otimes\widetilde{\mathbb{A}}^*$ on any element $u\in\widetilde{\mathbb{A}}^*$ by means of the relationship

$$\langle\Delta(u),(a\otimes b)\rangle:=\langle u,a\circ b\rangle \tag{2.2}$$

for arbitrary $a,b\in\widetilde{\mathbb{A}}$. Note that the co-multiplication $\Delta:\widetilde{\mathbb{A}}^*\to\widetilde{\mathbb{A}}^*\otimes\widetilde{\mathbb{A}}^*$, defined this way, is a *homomorphism* of the algebra $\widetilde{\mathbb{A}}^*$ with respect to the natural multiplication of functionals, and the linear pre-Poisson structure

$\{\cdot,\cdot\}$ (2.1) on $\widetilde{\mathbb{A}}^*$ is called *compatible* with the multiplication "$\circ$" on the algebra $\widetilde{\mathbb{A}}$, if the following symbolic invariance condition

$$\Delta\{u_i(x), u_j(x)\} = \{\Delta(u_i(x)), \Delta(u_j(x))\} \tag{2.3}$$

holds for any $x \in \mathbb{S}^1$ and all $i, j = \overline{1, N}$.

Taking into account that multiplication in the algebra $\mathbb{A}$ is given for any $i, j = \overline{1, N}$ by the condition

$$e_i \circ e_j := \sum_{s=\overline{1,N}} \sigma_{ij}^s e_s, \tag{2.4}$$

where the quantities $\sigma_{ij}^s \in \mathbb{K}$ for all i, j and $k = \overline{1, N}$ are constants, the related co-multiplication $\Delta : \widetilde{\mathbb{A}}^* \to \widetilde{\mathbb{A}}^* \otimes \widetilde{\mathbb{A}}^*$ acts on the basic functionals $u^s \in \widetilde{\mathbb{A}}^*$, $s = \overline{1, N}$, as

$$\Delta(u^s) = \sum_{i,j=\overline{1,N}} \sigma_{ij}^s u^i \otimes u^j. \tag{2.5}$$

Additionally, if the mapping $\vartheta^* : \widetilde{\mathbb{A}} \wedge \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ is given, for instance, in the simple linear form

$$\vartheta^* : (e_i \otimes e_j - e_j \otimes e_i) \to \sum_{s=\overline{1,N}} (c_{ij}^s - c_{ji}^s) e_s, \tag{2.6}$$

where quantities $c_{ij}^s \in \mathbb{K}$ are constant for all i, j and $s = \overline{1, N}$, then for the adjoint to (2.6) mapping $\vartheta : \mathrm{Symm}(\widetilde{\mathbb{A}}^*) \to \widetilde{\mathbb{A}}^* \wedge \widetilde{\mathbb{A}}^*$ one obtains the expression

$$\vartheta : u^s \to \sum_{i,j=\overline{1,N}} (c_{ij}^s - c_{ji}^s) u^i \otimes u^j. \tag{2.7}$$

For the pre-Poisson bracket (2.1) to be a Poisson bracket on $\widetilde{\mathbb{A}}^*$, it should must also satisfy the Jacobi identity. To find the corresponding additional constraints on the internal multiplication "$\circ$" on the algebra $\widetilde{\mathbb{A}}$, define for any $u(x) \in \widetilde{\mathbb{A}}^*$ the skew-symmetric linear mapping

$$\vartheta(u) : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}^*, \tag{2.8}$$

called [43] by the *Hamiltonian operator*, via the identity

$$\langle \vartheta(u) a, b \rangle := \langle \vartheta u(x), a \wedge b \rangle \tag{2.9}$$

for any $a, b \in \widetilde{\mathbb{A}}$, where the mapping $\vartheta : \widetilde{\mathbb{A}}^* \to \widetilde{\mathbb{A}}^* \wedge \widetilde{\mathbb{A}}^*$ is determined by the expression (2.7) to which it is adjoint. Then it is well known [43] that the pre-Poisson bracket (2.1) is Poisson iff the Hamiltonian operator (2.8) satisfies the well known [43] Schouten-σNijenhuis condition:

$$[[\vartheta(u), \vartheta(u)]] = 0 \tag{2.10}$$

for any $u(x) \in \tilde{\mathbb{A}}^*$. Since the mapping

$$\vartheta(u)e_i = \sum_{s,k=\overline{1,N}} (c_{ik}^s - c_{ki}^s)u_s(x)u^k \tag{2.11}$$

holds for any basis element $e_i \in \mathbb{A}, i = \overline{1,N}$, the resulting linear pre-Poisson bracket (2.1) becomes equal to

$$\begin{aligned}\{u_i(x)\ , u_j(x)\} = \langle \vartheta(u)e_i, e_j\rangle &= \sum_{s=\overline{1,N}} (c_{ij}^s\ - c_{ji}^s)u_s(x) \\ &= \langle u(x), \sum_{s=\overline{1,N}} (c_{ij}^s - c_{ji}^s)e_s\rangle\end{aligned} \tag{2.12}$$

for any $u(x) \in \tilde{\mathbb{A}}^*$. Now, defining on the algebra $\mathbb{A}$ the naturally adjacent to the algebra $\mathbb{A}$ Lie commutator structure

$$[e_i, e_j] = e_i \circ e_j - e_j \circ e_i := \sum_{s=\overline{1,N}} (\sigma_{ij}^s - \sigma_{ji}^s)e_s \tag{2.13}$$

for any basis elements $e_i, e_j \in \mathbb{A}, i, j = \overline{1,N}$, the expression (2.12) yields the well known [1, 4] classical Lie-Poisson bracket

$$\{u_i(x), u_j(x)\} = \langle u, [e_i, e_j]\rangle. \tag{2.14}$$

Concerning the adjacent Lie algebra structure condition (2.13), it can be easily rewritten as the set of relationships,

$$\sigma_{ij}^s - \sigma_{ji}^s = c_{ij}^s - c_{ji}^s \tag{2.15}$$

whose evident solution is

$$c_{ij}^s\ = \sigma_{ij}^s \tag{2.16}$$

for any $i, j, s = \overline{1,N}$. It is also clear that the compatibility condition (2.3) is completely unnecessary [14, 12] for the pre-Poisson bracket (2.1) to be a Poisson one. Moreover, as the bracket (2.14) is of the classical Lie-Poisson type, for the Hamiltonian operator (2.11) to satisfy the Schouten-Nijenhuis condition (2.10) is enough to check only the weak Jacobi identity for the weak Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, adjacent to to the algebra $\tilde{\mathbb{A}}$ via imposing the Lie structure (2.13), taking into account the relationships (2.16). Simple calculations for the special skew-symmetric case

$$e_i \circ e_j + e_j \circ e_i = 0 \tag{2.17}$$

for all $i, j = \overline{1,N}$ give rise to the constraints

$$e_i \circ e_j + e_j \circ e_i = 0, (e_i \circ e_j) \circ e_k + (e_j \circ e_k) \circ e_i + (e_k \circ e_i) \circ e_j = 0, \tag{2.18}$$

coinciding exactly with those in [43]. The corresponding Hamiltonian operator (2.8) then acts as

$$\vartheta(u)e_i = \sum_{s,k=\overline{1,N}} (\sigma_{ik}^s - \sigma_{ki}^s)u_s(x)u^k \tag{2.19}$$

on any basis element $e_i \in \mathbb{A}$. Since the bracket (2.14), owing to the constraints (2.17) and (2.18), satisfies the weak Jacobi identity implying that the mapping $\vartheta(u) : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}^*$ satisfies the Schouten-Nijenhuis condition (2.10), one has the following result.

Theorem 2.1. *The general linear pre-Poisson bracket* (2.1) *on* $\widetilde{\mathbb{A}}^*$ *under the constraints* (2.17) *and* (2.18) *on the algebra* $\mathbb{A}$*, which is of the Lie-Poisson type on the adjoint space* $\mathcal{L}^*_{\widetilde{\mathbb{A}}}$ *to the weak Lie algebra* $\mathcal{L}_{\widetilde{\mathbb{A}}}$ *adjacent to the loop algebra* $\widetilde{\mathbb{A}}$*, is a priori a Poisson and compatible with the internal algebraic structure of* $\mathbb{A}$*.*

Remark 2.2. Similarly, one can consider a simple ultra-local quadratic pre-Poisson bracket on $\widetilde{\mathbb{A}}^*$,

$$\{u_i(x), u_j(x)\} := \langle\, u(x) \otimes u(x),\ \vartheta^*(e_i \wedge e_j)\,\rangle, \tag{2.20}$$

where the skew-symmetric mapping $\vartheta^* : \widetilde{\mathbb{A}} \wedge \widetilde{\mathbb{A}} \to \mathrm{Symm}(\widetilde{\mathbb{A}} \otimes \widetilde{\mathbb{A}})$ is given for any $i, j = \overline{1,N}$ in the quadratic form

$$\vartheta^*(e_i \otimes e_j - e_j \otimes e_i) := \sum_{k,s=\overline{1,N}} (c_{ij}^{ks} - c_{ji}^{ks})(e_k \otimes e_s + e_s \otimes e_k). \tag{2.21}$$

In particular, if we assume that the coefficients $c_{ij}^{ks} = \sigma_{ij}^k \alpha^s$ for some constant numbers σ_{ij}^k and $\alpha^s \in \mathbb{K}$ for all i, j and $k, s = \overline{1,N}$, where

$$e_k \circ e_s := \sum_{k=\overline{1,N}} \sigma_{ij}^k e_k,$$

then the pre-Poissson bracket (2.20) yields a very compact form

$$\{u_i(x), u_j(x)\} := \langle u(x) \otimes u(x), \alpha \otimes [e_i, e_j]\ + [e_i, e_j] \otimes \alpha\rangle, \tag{2.22}$$

generalizing (2.14) and parametrically depending on the constant vector $\alpha := \sum_{s=\overline{1,N}} \alpha^s e_s \in \mathbb{A}$. For the pre-Poisson bracket (2.22) to be Poisson one can formulate suitable constraints on the algebraic structure of $\widetilde{\mathbb{A}}$, similar to those obtained in [14], which we shall not consider here.

Now, let $\widetilde{\mathbb{A}}(u) \subset \widetilde{\mathbb{A}}$ denote the polynomial differential ideal generated by an element $u \in \widetilde{\mathbb{A}}$ and its derivatives $D_x^n u \in \widetilde{\mathbb{A}}$, $n \in \mathbb{Z}_+$. The corresponding space of polynomial functions $\widetilde{\mathbb{A}}(u) \to \mathbb{K}$, constructed by means of some scalar form on $\widetilde{\mathbb{A}}(u)$, will be denoted by $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. Then the basic ultra-local and linear, with respect to an independent element $u(x) \in \widetilde{\mathbb{A}}, x \in \mathbb{S}^1$, pre-Poisson bracket (2.1) is easily generalized to a local pre-Poisson bracket for arbitrary functions $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$:

$$\{f, g\}(u) = \langle u(x), \vartheta^*(\nabla f(u(x)) \ \wedge \nabla g(u(x)\rangle, \tag{2.23}$$

in which the mapping

$$\vartheta^* : \widetilde{\mathbb{A}} \wedge \widetilde{\mathbb{A}} \to \mathrm{Symm}(\widetilde{\mathbb{A}} \otimes \widetilde{\mathbb{A}})$$

is invariantly reduced on the subspace $\widetilde{\mathbb{A}}(u) \wedge \widetilde{\mathbb{A}}(u)$ and depends nontrivially on the derivation $D_x : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$. In (2.23) we denoted the usual linear gradient mapping from $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ to the ideal $\widetilde{\mathbb{A}}(u) \subset \widetilde{\mathbb{A}}$ by "∇", that is for a given $h \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, $\nabla h(u(x)) \in \widetilde{\mathbb{A}}(u)$ and

$$\langle v(x), \nabla h(u(x)\rangle := dh(u + \varepsilon v)/d\varepsilon|_{\varepsilon=0}$$

for any $v(x) \in \widetilde{\mathbb{A}}^*$, $x \in \mathbb{S}^1$. Keeping in mind the problem of finding constraints on the multiplicative structure of the algebra $\widetilde{\mathbb{A}}$ under which the pre-Poisson bracket (2.23) is Poisson, it is very interesting to construct nontrivial examples of *linear* local pre-Poisson brackets on $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, compatible with the multiplication "$\circ$" on $\mathbb{A}$ and nontrivially depending on the usual differential operator $D_x : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ for $x \in \mathbb{S}^1$.

In particular, for arbitrary functions $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ one can consider the following nontrivial and simplest linear local pre-Poisson bracket

$$\{f, g\}(u) := \langle u(x), \ \vartheta^*(\nabla f(u(x)) \wedge \nabla g(u(x))\rangle, \tag{2.24}$$

where

$$\vartheta^* : (a(x) \wedge b(x)) \longrightarrow \\ \longrightarrow \sum_{j,k,s=\overline{1,N}} [c^s_{jk} D_x a^j(x) b^k(x) - c^s_{jk} D_x b^j(x) a^k(x)] e_s \tag{2.25}$$

for any

$$a(x) := \sum_{j=\overline{1,N}} a^j(x) e_j, b(x) := \sum_{j=\overline{1,N}} b^j(x) e_j \in \widetilde{\mathbb{A}},$$

$x \in \mathbb{S}^1$, and some arbitrarily chosen constant quantities $c^s_{jk} \in \mathbb{K}$ for all j, k and $s = \overline{1, N}$. If one also assumes that these constant quantities satisfy the condition (2.16), that is $c^s_{ij} = \sigma^s_{ij}$ for all i, j and $s = \overline{1, N}$, the mapping (2.25) can be recast as

$$\vartheta^* : (a(x) \wedge b(x)) \longrightarrow D_x a(x) \circ b(x) - D_x b(x) \circ a(x), \tag{2.26}$$

providing the pre-Poisson bracket (2.24) for arbitrary functions $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ with the canonical Lie-Poisson form

$$\begin{aligned} \{f,g\}(u) := \big\langle u(x),\ D_x\nabla f(u(x)) \circ \nabla g(u(x)) - \\ - D_x\nabla g(u(x)) \circ \nabla f(u(x)\big\rangle, \end{aligned} \tag{2.27}$$

which was recently presented in [91]. Thus, if the Lie structure

$$[a(x), b(x)]_D := D_x a(x) \circ b(x) - D_x b(x) \circ a(x) \tag{2.28}$$

for any $a(x), b(x) \in \widetilde{\mathbb{A}}$, $x \in \mathbb{S}^1$, generates the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, the pre-Poisson bracket (2.27) will automatically be Poisson on the space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. Moreover, the expression (2.27), rewritten in the tensor form

$$\begin{aligned} \{f,g\}(u) = \Big\langle \Delta u(x),\ D_x\nabla f(u(x)) \otimes \nabla g(u(x)) - \\ - D_x\nabla g(u(x)) \otimes \nabla f(u(x)\Big\rangle = \\ := (\vartheta(u)\nabla f(u(x)), \nabla g(u(x))), \end{aligned} \tag{2.29}$$

naturally defines a related bilinear form $(\cdot,\cdot)$ on the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, allowing to determine the corresponding Hamiltonian operator $\vartheta(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, whose matrix representation with respect to the basis $\{e_s \in \mathbb{A} \mid s = \overline{1,N}\}$ is

$$\vartheta(u) = \sigma(u) D_x + D_x \sigma(u)^{\intercal}, \tag{2.30}$$

where $\sigma(u) := \{\sum_{s=\overline{1,N}} \sigma_{ij}^s u_s \mid i,j = \overline{1,N}\}$. So, if the Hamiltonian operator (2.30) satisfies the Schouten-Nijenhuis condition (2.10), the pre-Poisson bracket (2.29) is Poisson. Yet, simultaneously, if the adjacent Lie algebra structure (2.28) satisfies the weak Jacobi condition

$$\begin{aligned} \big\langle u(x), [[a(x), b(x)]_D, c(x)]_D\big\rangle + \\ + \big\langle u(x), [[b(x), c(x)]_D, a(x)]_D\big\rangle + \\ + \big\langle u(x), [[c(x), a(x)]_D, b(x)]_D\big\rangle = 0 \end{aligned} \tag{2.31}$$

for any elements $a(x), b(x)$ and $c(x) \in \widetilde{\mathbb{A}}$, $x \in \mathbb{S}^1$, then the pre-Poisson bracket (2.27) equivalent to (2.29), being of the Lie-Poisson type, will be *a priori* Poisson. As the second criterion is easier to check, after some simple calculations one obtains the well-known [11, 43] Balinsky-Novikov algebra constraints

$$[R_a, R_b] = 0, [L_a, L_b] = L_{[a,b]} \tag{2.32}$$

on the multiplication structure of the algebra $\mathbb{A}$, where, by definition, for any $a, b \in \mathbb{A}$ the bracket $[a, b] := a \circ b - b \circ a$ and the mappings $L_a, R_a : \mathbb{A} \to \mathbb{A}$ are left and right multiplications, respectively: $L_a b := a \circ b = R_b a$,

Remark 2.3. As follows from Proposition 3.2, formulated below, if the algebra $\mathbb{A}$ is a Balinsky-Novikov algebra (2.32), then the constructed above Hamiltonian operator $\vartheta(u) : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ is a derivation of the weak adjacent Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ and vice versa: the operator $\vartheta(u) : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ is Hamiltonian if it is a derivation of the weak Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ adjacent to the Balinsky-Novikov algebra (2.32).

The next example of the bilinear, nonlocal (pseudodifferential) and weakly skew-symmetric mapping

$$\vartheta^* : (a(x) \wedge b(x)) \to D_x^{-1}a(x) \circ b(x) - D_x^{-1}b(x) \circ a(x), \tag{2.33}$$

where $D_x D_x^{-1} := 1 : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ is the identity mapping, generates the weak adjacent Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ structure

$$[a(x), b(x)]_D := D_x^{-1}a(x) \circ b(x) - D_x^{-1}b(x) \circ a(x) \tag{2.34}$$

for any $a(x), b(x) \in \tilde{\mathbb{A}}$. It is easy to check that the commutator structure (2.34) satisfies the weak Jacobi identity (2.31) iff the multiplicative structure of the algebra $\mathbb{A}$ satisfies the so called [59] *right Leibniz algebra* constraints:

$$R_{a\circ b} = [R_a, R_b], R_{a\circ b} + R_{b\circ a} = 0 \tag{2.35}$$

for arbitrary elements $a, b \in \mathbb{A}$. The corresponding matrix integro-differential Hamiltonian operator on the space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$ with respect to the basis

$$\{e_s \in \mathbb{A} \mid s = \overline{1, N}\}$$

for this case equals

$$\vartheta(u) = \sigma(u)D_x^{-1} + D_x^{-1}\sigma(u)^\intercal \tag{2.36}$$

for any $u(x) \in \tilde{\mathbb{A}}^*$, $x \in \mathbb{S}^1$.

Consider now the bilinear, nonlocal and weakly skew-symmetric mapping

$$\vartheta^* : (a(x) \wedge b(x)) \to D_x^{-1}b(x) \circ D_x a(x)) - D_x^{-1}a(x) \circ D_x b(x)), \tag{2.37}$$

which naturally generates the adjacent Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ structure

$$[a(x), b(x)]_D := D_x^{-1}b(x) \circ D_x a(x) - D_x^{-1}a(x) \circ D_x b(x). \tag{2.38}$$

Then it is easy to check that the commutator structure (2.38) satisfies the weak Jacobi identity (2.31), iff the following so called *Riemann algebra* $\mathbb{A}$ multiplicative structure

$$[R_a, R_b] = 0, \qquad L_{a\circ b} = R_{a\circ b} = L_{b\circ a} \tag{2.39}$$

holds for arbitrary elements $a, b \in \mathbb{A}$. For the related Hamiltonian operator on the functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$ with respect to the basis $\{e_s \in \mathbb{A} \mid s = \overline{1, N}\}$ one easily obtains from (2.37) the integro-differential expression

$$\vartheta(u) = D_x\sigma(u)D_x^{-1} - D_x^{-1}\sigma(u)^\intercal D_x \tag{2.40}$$

for any $u(x) \in \widetilde{\mathbb{A}}^*, x \in \mathbb{S}^1$. The above results can be reformulated as the following theorem.

Theorem 2.4. *An arbitrary linear pre-Poisson bracket* (2.29) *on the functional space* $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$*, which is of the Lie-Poisson type on the adjoint space* $\mathcal{L}^*_{\widetilde{\mathbb{A}}}$ *to the weak Lie algebra* $\mathcal{L}_{\widetilde{\mathbb{A}}}$ *adjacent to the loop algebra* $\widetilde{\mathbb{A}}$*, is a priori Poisson and compatible with the internal structure of the algebra* $\mathbb{A}$ *iff the related Lie algebra structure* $[\cdot,\cdot]_D$ *satisfies the weak Jacobi condition.*

Thus, all the operators (2.30), (2.36) and (2.40) are Hamiltonian and *a priori* satisfy the Schouten-Nijenhuis condition (2.10), as easily follows from Theorem 2.4. It is also clear that in contrast to the simple Hamiltonian operator criterion formulated in this theorem, direct and very cumbersome checking of the Schouten-Nijenhuis condition as in [43] for the cases considered above, would be required for the multiplicative structures on the algebra $\mathbb{A}$ coinciding with (2.32), (2.35) and (2.39).

3. Hamiltonian operators and related algebraic structures via the Lie-algebraic approach

Assume now that the loop algebra $(\widetilde{\mathbb{A}};\circ)$ allows a weak adjacent Lie algebra extension $(\mathcal{L}_{\widetilde{\mathbb{A}}};[\cdot,\cdot]_D)$ by means of a commutator operation

$$[\cdot,\cdot]_D : \widetilde{\mathbb{A}}\times\widetilde{\mathbb{A}} \to\widetilde{\mathbb{A}},$$

which can be endowed [17, 18, 38, 89] with a nondegenerate ad-invariant

$$([a,b]_D,c) = (a,[b,c]_D), \tag{3.1}$$

and symmetric

$$(a,b) = (b,a) \tag{3.2}$$

bilinear form $(\cdot,\cdot) : \mathcal{L}_{\widetilde{\mathbb{A}}}\times\mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathbb{K}$ for any a, b and $c \in \mathcal{L}_{\mathbb{A}}$.

Remark 3.1. If the Lie-structure $[\cdot,\cdot]_D$ on $\widetilde{\mathbb{A}}$ coincides with the usual commutator structure $[\cdot,\cdot]$ on $\widetilde{\mathbb{A}}$, with respect to the multiplication "$\circ$" and the symmetric bilinear form (3.2) also satisfies the shifting property

$$(a\circ b,c) = (a,b\circ c) \tag{3.3}$$

for any a, b and $c \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, then the ad-invariance condition (3.1) automatically holds.

The form (3.2) makes it possible to construct the natural identification $\mathcal{L}^*_{\widetilde{\mathbb{A}}} \simeq \mathcal{L}_{\widetilde{\mathbb{A}}}$. One can consider the subspace of polynomial functions $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$

of the space $\mathcal{F}(\mathcal{L}^*_{\tilde{\mathbb{A}}})$ of smooth functions on $\mathcal{L}^*_{\tilde{\mathbb{A}}}$, generated by an element $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}}$, jointly with its related Lie-Poisson bracket:

$$\{f, g\}_0 := \big(u, [\nabla f(u), \nabla g(u)]_D\big) \tag{3.4}$$

for any $f, g \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$. Owing to the construction [1, 4, 20, 17, 38], the Lie-Poisson bracket (3.4) satisfies *a priori* the classical Jacobi identity, and it can serve as a very powerful tool for constructing the related Hamiltonian operators on the functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$. In particular, following [43, 68], a smooth mapping $\vartheta(u) : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$ for a chosen element $u \in \widetilde{\mathbb{A}}$ is a Hamiltonian operator if the related pre-Poisson bracket

$$\{f, g\} := (\vartheta(u)\nabla f(u), \nabla g(u)),$$

determined for any $f, g \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, satisfies the Jacobi identity.

Taking into account that the canonical Lie-Poisson bracket (3.4) depends essentially on the loop Lie algebra structure of $\mathcal{L}_{\tilde{\mathbb{A}}}$, we proceed further to extending the Lie algebra structure on $\mathcal{L}_{\tilde{\mathbb{A}}}$ by means of the standard [38] central extension technique. Namely, let $\hat{\mathcal{L}}_{\tilde{\mathbb{A}}} := \mathcal{L}_{\tilde{\mathbb{A}}} \oplus \mathbb{K}$ denote the centrally extended Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ endowed with the bracket

$$[(a; \alpha), (b; \beta)]_D := ([a, b]_D; \varpi_2(a, b))$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ and $\alpha, \beta \in \mathbb{K}$, where the 2-cocycle $\varpi_2 : \mathcal{L}_{\tilde{\mathbb{A}}} \times \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathbb{K}$ is a skew-symmetric bilinear form satisfying the Jacobi identity:

$$\varpi_2([a, b]_D, c) + \varpi_2([b, c]_D, a) + \varpi_2([c, a]_D, b) = 0 \tag{3.5}$$

for any a, b and $c \in \mathcal{L}_{\tilde{\mathbb{A}}}$. It is evident that the existence of nontrivial central extensions on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ strongly depends on the algebraic structure of the algebra $\mathbb{A}$ underlying the whole construction presented above. Yet there exist some general algebraic properties which allow to proceed further with success. For example, assume that a smooth mapping

$$\mathrm{D} : \mathcal{L}_{\tilde{\mathbb{A}}} \to End\mathcal{L}_{\tilde{\mathbb{A}}}$$

defines for any $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}} \simeq \mathcal{L}_{\tilde{\mathbb{A}}}$ a *weak* derivation of the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, that is

$$(c, \mathrm{D}_u[a, b]_D) = (c, [\mathrm{D}_u a, b]_D + [a, \mathrm{D}_u b]_D) \tag{3.6}$$

for any a, b and $c \in \mathcal{L}_{\tilde{\mathbb{A}}}$. Then the following important proposition [69, 89] holds.

Proposition 3.2. *Let a linear mapping* $\mathrm{D}_u : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ *be for a fixed* $u \in \mathcal{L}^*_{\tilde{\mathbb{A}}}$ *a weak skew-symmetric derivation of the Lie algebra* $\mathcal{L}_{\tilde{\mathbb{A}}}$. *Then the expression*

$$\varpi_2(a, b) := (a, \mathrm{D}_u b) \tag{3.7}$$

for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ *and* $u \in \mathcal{L}^*_{\widetilde{\mathbb{A}}} \simeq \mathcal{L}_{\widetilde{\mathbb{A}}}$ *defines a nontrivial 2-cocycle on the Lie algebra* $\mathcal{L}_{\widetilde{\mathbb{A}}}$.

A proof is given by means of direct checking the Jacobi identity (3.5) and is omitted.

Remark 3.3. The expression (3.7) when linear in the element $u \in \mathcal{L}^*_{\widetilde{\mathbb{A}}}$ can be, evidently, represented for any $a, b \in \widetilde{\mathbb{A}}$ in the following scalar form:

$$(a, \mathrm{D}_u b) = (u, \vartheta^*(a \wedge b)), \tag{3.8}$$

where $\vartheta^* : \widetilde{\mathbb{A}} \wedge \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$ is some bilinear skew-symmetric mapping.

As follows from the results of Section (2) the right-hand side of expression (3.8) allows the equivalent form

$$(u, \vartheta^*(a \wedge b)) = (u, [a, b]_{\bar{D}}), \tag{3.9}$$

where the bracket $[a, b]_{\bar{D}}$ defines for any $a, b \in \widetilde{\mathbb{A}}$ a new adjacent Lie algebra structure on the loop algebra $\widetilde{\mathbb{A}}$, *a priori* compatible with the basic Lie structure $[\cdot, \cdot]_D$. In the case when these Lie structures coincide, that is $[\cdot, \cdot]_D = [\cdot, \cdot]_{\bar{D}}$, the cocycle (3.7) naturally determines on the phase space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ the equivalent to (3.4) Poisson bracket

$$\{f, g\}(u) := \big(\nabla f(u(x)), \mathrm{D}_u \nabla g(u(x))\big) \tag{3.10}$$

for any $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. Moreover, as follows from (3.9) and the ad-invariance property (3.1), the mapping $\mathrm{D}_u(\cdot) = -[u, \cdot]_D$ for any $u \in \mathcal{L}^*_{\widetilde{\mathbb{A}}}$ is automatically a derivation of the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$.

Example 3.4. As a natural example of the derivation $\mathrm{D}_u : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ one can take the mapping

$$\mathrm{D}_u := \Delta_1 u(x) D_x + D_x \Delta_2 u(x), \tag{3.11}$$

where for $u \in \mathcal{L}^*_{\widetilde{\mathbb{A}}}$ the expressions $\Delta_j u(x) : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}^*$, $j = 1, 2$, denote the convolution operators of the co-multiplication $\Delta : \widetilde{\mathbb{A}}^* \to \widetilde{\mathbb{A}}^* \otimes \widetilde{\mathbb{A}}^*$ with respect to its first and second tensor components, respectively. In particular, we have

$$(\Delta u, a \otimes b) := (u, a \circ b) = \sum_{s,i,j=1}^{N} \sigma_{ij}^{s} \int_{\mathbb{S}^1} u_s(x) a^i(x) b^j(x) dx$$

for a fixed $u = \sum_{s=1}^{N} u_s(x) u^j \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$ and any

$$a = \sum_{i=1}^{N} a^i(x) e_i, \ b = \sum_{j=1}^{N} b^j(x) e_j \in \mathcal{L}_{\widetilde{\mathbb{A}}},$$

as we have assumed by definition, that $(u^i, e_j) := \delta^i_j, i,j = \overline{1,N}$. If now the weak Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ is generated by the commutator Lie structure (3.4), that is

$$[a,b]_D = D_x a \circ b - D_x b \circ a \tag{3.12}$$

for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, it easy to check that the mapping (3.11) is a skew-symmetric with respect to the bilinear form $(\cdot,\cdot)$ on $\widetilde{\mathbb{A}}$ weak derivation of the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$. Moreover, the related weak Lie algebraic structure $[\cdot,\cdot]_{D'}$ on $\widetilde{\mathbb{A}}$, satisfying the condition $(u, [a,b]_{D'}) = (a, \mathrm{D}_u b)$ for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, coincides exactly with that (3.12).

There are also other strictly algebraic tools for constructing Poisson brackets on the functional space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. For instance, as a simple consequence of Proposition 3.2 the following result [34, 69, 89] holds.

Proposition 3.5. *Suppose that a nondegenerate linear skew-symmetric $\mathcal{R}$-matrix endomorphism $\mathcal{R} : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ satisfies the well-known weak Yang-Baxter commutator condition:*

$$\big(c, [\mathcal{R}a, \mathcal{R}b]_D\big) = \big(c, \mathcal{R}\left([\mathcal{R}a, b]_D + [a, \mathcal{R}b]_D\right)\big) = 0 \tag{3.13}$$

for any a, b and $c \in \mathcal{L}_{\widetilde{\mathbb{A}}}$. Then the inverse mapping $\mathcal{R}^{-1} : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ is a weak skew-symmetric derivation of the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ and the expression

$$\varpi_2(a,b) = (a, \mathcal{R}^{-1} b) \tag{3.14}$$

defines for any $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ a 2-cocycle on $\mathcal{L}_{\widetilde{\mathbb{A}}}$.

For any function $f \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ consider its differential $\delta f \in \Lambda^1(\widetilde{\mathbb{A}}(u))$ and define for a chosen element $h \in \widetilde{\mathbb{A}}(u)^*$ the vector field $\xi_h \in \Gamma(\widetilde{\mathbb{A}}(u))$ via the equality

$$\delta f(\xi_h) = (\nabla f(u), h).$$

As symplectic forms on the phase space $\widetilde{\mathbb{A}}(u)$ are dual objects to the Poisson brackets on the functional space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, one easily obtains the following [69, 85, 89] proposition.

Proposition 3.6. *Assume that the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ allows a skew-symmetric nondegenerate $\mathcal{R}$-structure homomorphism $\mathcal{R} : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, satisfying the weak Yang-Baxter condition*

$$\Big(c, \mathcal{R}\big([\mathcal{R}a, b]_D + [a, \mathcal{R}b]_D\big) - [\mathcal{R}a, \mathcal{R}b]_D\Big) = 0 \tag{3.15}$$

for any a, b and $c \in \mathcal{L}_{\widetilde{\mathbb{A}}}$. Then differential 2-forms

$$\omega_j^{(2)} \in \Lambda^2(\widetilde{\mathbb{A}}(u)), \quad j = 1, 2,$$

defined on the ideal $\widetilde{\mathbb{A}}(u) \subset \widetilde{\mathbb{A}}$ *by*

$$\omega_1^{(2)}(\xi_f, \xi_g) = \big(\mathcal{R}\xi_f, \xi_g\big) := -\big(\vartheta_1 \nabla f(u), \nabla g(u)\big), \tag{3.16}$$

where

$$\xi_f := \vartheta_1 \nabla f(u),\ \xi_g := \vartheta_1 \nabla g(u)\ \in\ \Lambda^1(\widetilde{\mathbb{A}}(u))$$

with $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, *and*

$$\omega_2^{(2)}(\xi_f, \xi_g) = \big(u, [\mathcal{R}\xi_f, \mathcal{R}\xi_f]_D\big) := -\big(\vartheta_2 \nabla f(u), \nabla g(u)\big), \tag{3.17}$$

where

$$\xi_f := \vartheta_2 \nabla f(u),\ \xi_g := \vartheta_2 \nabla g(u)\ \in\ \Lambda^1(\widetilde{\mathbb{A}}(u))$$

with $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, *are closed. Moreover, the corresponding Hamiltonian operators*

$$\vartheta_1, \vartheta_2 : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}^*_{\widetilde{\mathbb{A}}}$$

are compatible, that is for arbitrary $\lambda, \mu \in \mathbb{K}$ *the sum* $\lambda\vartheta_1 + \mu\vartheta_2 : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ *is also a Hamiltonian operator.*

Proof. (*Sketch*). The 2-form (3.16) is closed, as the expression (3.16) determines a 2-cocycle on the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ owing to the fact that the inverse mapping $\mathcal{R}^{-1} : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ is a weak derivation on $\mathcal{L}_{\widetilde{\mathbb{A}}}$, that is

$$(c, \mathcal{R}^{-1}[a, b]_D) = \big(c,\ [\mathcal{R}^{-1}a, b]_D + [a, \mathcal{R}^{-1}b]_D\big)$$

for any a, b and $c \in \mathcal{L}_{\widetilde{\mathbb{A}}}$. A proof of the second part of the proposition consists in direct checking the closedness of the 2-forms $\omega_2^{(2)} \in \Lambda^2(\widetilde{\mathbb{A}}(u))$, which is equivalent to the Yang-Baxter condition (3.15).

Concerning their compatibility, we observe that the Hamiltonian operator $\vartheta_2 : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, corresponding to the expression (3.17), is representable as the composition $\vartheta_2 = \vartheta_1(\vartheta_0^{-1}\vartheta_1)$, where the Hamiltonian operator $\vartheta_0 : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}^*_{\widetilde{\mathbb{A}}}$ is naturally determined from the canonical Lie-Poisson bracket (3.4) as

$$\big(u, [\nabla f(u), \nabla g(u)]\big) := \big(\vartheta_0 \nabla f(u), \nabla g(u)\big) \tag{3.18}$$

for any $f, g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. From this representation one easily derives [22, 20, 69, 43, 89] the compatibility of the Hamiltonian operators ϑ_2 and ϑ_1 on $\widetilde{\mathbb{A}}(u)$, following from the evident compatibility of operators ϑ_0 and ϑ_1 on $\widetilde{\mathbb{A}}(u)$, owing to 2-cocycle property of the bilinear form (3.14). □

The $\mathcal{R}$-structures on the weak Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ can be effectively exploited for constructing additional Hamiltonian operators on $\mathcal{L}^*_{\widetilde{\mathbb{A}}}$ owing to the fact that the bracket

$$\big(c, [a, b]_{(\mathcal{R})}\big) := \big(c, [a, \mathcal{R}b]_D + [\mathcal{R}a, b]_D\big) \tag{3.19}$$

generates for any $a, b, c \in \mathcal{L}_{\tilde{\mathbb{A}}}$ a new weak Lie structure on the linear space $\mathcal{L}_{\tilde{\mathbb{A}}}$, thus producing a new weak Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}^{(\mathcal{R})} := (\widetilde{\mathbb{A}}, [\cdot,\cdot]_{(\mathcal{R})})$.

Example 3.7. Consider the Rota-Baxter [86, 87] endomorphism

$$\mathcal{R} : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}},$$

where for any $a \in \mathcal{L}_{\tilde{\mathbb{A}}}$ we put

$$\mathcal{R}(a) := \frac{1}{2}\Big[\int_0^x a(y)dy - \int_x^{2\pi} a(y)dy\Big]$$

for any $x \in \mathbb{S}^1$, which satisfies the weak Yang-Baxter commutator condition (3.13). Then it is easy to check that the inverse mapping $\mathcal{R}^{-1} = d/dx$, $x \in \mathbb{S}^1$, is the natural skew-symmetric derivation of the weak Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$, generating a Poisson structure compatible with that of (3.10).

In a manner similar to the above [69, 57, 85, 89] one verifies the existence of the following so called "quadratic" Poisson brackets. Namely, the next proposition holds.

Proposition 3.8. *Suppose that the weak Yang-Baxter condition* (3.15) *hold. Then the brackets*

$$\{f,g\}_1 := \big(u \circ \nabla f(u),\ \mathcal{R}(u \circ \nabla g(u))\big) - \big(\nabla f(u) \circ u, \mathcal{R}(\nabla g(u) \circ u)\big) \quad (3.20)$$

and

$$\{f,g\}_2 := \big(u,\, [\mathcal{R}\nabla f(u), \nabla g(u)]_D + [\nabla f(u), \mathcal{R}\nabla g(u)]_D\big), \quad (3.21)$$

defined for any $f, g \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, are Poisson and compatible on $\widetilde{\mathbb{A}}(u)$.

As an interesting and also useful consequence of the $\mathcal{R}$-matrix construction, one has the fact that the following subspaces

$$\mathcal{L}_{\tilde{\mathbb{A}}}^{\pm} := \big(\mathbb{I} \pm \mathcal{R}\big)/\,2 \in \mathcal{L}_{\tilde{\mathbb{A}}} \quad (3.22)$$

are Lie subalgebras of $\mathcal{L}_{\tilde{\mathbb{A}}}$, which is equivalent to the condition that the mappings

$$\big(\mathbb{I} \pm \mathcal{R}\big)/2 : \mathcal{L}_{\tilde{\mathbb{A}}}^{(\mathcal{R})} \to \mathcal{L}_{\tilde{\mathbb{A}}}^{\pm} \subset \mathcal{L}_{\tilde{\mathbb{A}}} \quad (3.23)$$

are homomorphisms [89] of the Lie algebras $\mathcal{L}_{\tilde{\mathbb{A}}}^{(\mathcal{R})} := (\widetilde{\mathbb{A}}, [\cdot,\cdot]_{(\mathcal{R})})$ and $\mathcal{L}_{\tilde{\mathbb{A}}}^{\pm}$. In the special case when $\mathcal{L}_{\tilde{\mathbb{A}}}^{+} \cap \mathcal{L}_{\tilde{\mathbb{A}}}^{-} = \{0\}$, the operator $\mathcal{R} = P_+ - P_-$ and the linear operators $P_\pm := (\mathbb{I} \pm \mathcal{R})/2 : \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathcal{L}_{\tilde{\mathbb{A}}}$ are projectors and the weak Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ allows the direct sum splitting $\mathcal{L}_{\tilde{\mathbb{A}}} = \mathcal{L}_{\tilde{\mathbb{A}}}^{+} \oplus \mathcal{L}_{\tilde{\mathbb{A}}}^{-}$.

4. Integrable Riemann hydrodynamical systems and related multicomponent Hamiltonian operators

4.1. **General setting.** In preceding sections we have shown that any skew-symmetric Lie structure $[\cdot,\cdot]_D$ on the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, satisfying the Jacobi identity and nontrivially depending on the derivation $D_x : \widetilde{\mathbb{A}} \to \widetilde{\mathbb{A}}$, determines on the phase space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ the local Poisson bracket

$$\{f,g\}(u) := \big(u,\, [\nabla f(u(x)), \nabla g(u(x))]_D\big)$$

for all $u \in \widetilde{\mathbb{A}}^*$ and arbitrary functions $f,g \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$. Moreover, from the analysis provided above we know that if the Hamiltonian operator $\vartheta(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}^*_{\widetilde{\mathbb{A}}} \simeq \mathcal{L}_{\widetilde{\mathbb{A}}}$ related (2.29) corresponds to some 2-cocycle on the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, then it will be *a priori* Hamiltonian. Moreover, owing to Proposition 3.2, if this 2-cocycle is generated by a derivation $\mathrm{D}_u : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, $u \in \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$, on the Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, one need only check the related weak Leibniz property (3.6) in the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$. In Section 2 we already showed that the skew-symmetric structure

$$[a,b]_D = D_x a \circ b - D_x b \circ a$$

for any $a,b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, imposed on the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, gives rise to the Hamiltonian operator (2.30) on $\widetilde{\mathbb{A}}(u)$

$$\vartheta(u) = \sigma(u) D_x + D_x \sigma(u)^{\intercal} \tag{4.1}$$

and to the related multiplicative Balinsky-Novikov algebra structure (2.32) on $\mathbb{A}$

$$[R_a, R_b] = 0, [L_a, L_b] = L_{[a,b]} \tag{4.2}$$

for any a, b and $c \in \mathbb{A}$. Similarly, the skew-symmetric structure (2.34)

$$[a,b]_D = D_x^{-1} a \circ b - D_x^{-1} b \circ a$$

for any $a,b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ on the weak adjacent Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$ gives rise to the Hamiltonian operator (2.30) on $\widetilde{\mathbb{A}}(u)$

$$\vartheta(u) = \sigma(u) D_x^{-1} + D_x^{-1} \sigma(u)^{\intercal}$$

and to the related multiplicative right Leibniz algebra structure (2.32) on $\mathbb{A}$

$$R_{a\circ b} = [R_a, R_b], \qquad\qquad R_{a\circ b} + R_{b\circ a} = 0. \tag{4.3}$$

Moreover, the skew-symmetric structure (2.34)

$$[a,b]_D = D_x^{-1} b \circ D_x a - D_x^{-1} a \circ D_x b$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ on the weak adjacent Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ gives rise to the Hamiltonian operator (2.38) on $\widetilde{\mathbb{A}}(u)$

$$\vartheta(u) = D_x \sigma(u) D_x^{-1} - D_x^{-1} \sigma(u)^{\intercal} D_x \tag{4.4}$$

and to the related multiplicative Riemann algebra structure (2.39) on $\mathbb{A}$

$$[R_a, R_b] = 0, \qquad L_{a \circ b} = R_{a \circ b} = L_{b \circ a} \tag{4.5}$$

for all $a, b, c \in \mathbb{A}$.

Remark 4.2. Just as in Section 2, one can construct for all $a, b, c \in \mathbb{A}$ dual Balinsky-Novikov

$$R_{[a,b]} = [R_b, R_a], \qquad [L_a, L_b] = 0,$$

Leibniz

$$L_{a \circ b} = [L_a, L_b], \qquad L_{a \circ b} + L_{b \circ a} = 0$$

and Riemann

$$[L_a, L_b] = 0, \qquad R_{a \circ b} = L_{a \circ b} = R_{b \circ a}$$

algebra constraints, respectively related to the adjacent Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$ structures

$$\begin{aligned} [a, b]_D &= D_x a \circ b - D_x b \circ a, \\ [a, b]_D &= a \circ D_x^{-1} b - b \circ D_x^{-1} a, \\ [a, b]_D &= -D_x a \circ D_x^{-1} b + D_x b \circ D_x^{-1} a \end{aligned}$$

for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$.

As mentioned above, simultaneously we have shown that the expressions (4.1), (4.3) and (4.4) are true Hamiltonian operators on the functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$ satisfying the Schouten-Nijenhuis condition (2.10). Using the algebraic scheme in [91] and the right Leibniz algebra (4.3) and the new Riemann algebra (4.4), one can describe a wide class of multicomponent completely integrable dynamical systems containing, as follows from the recent results in [80], infinite hierarchies of the multicomponent hydrodynamical Riemann type systems.

As the expressions (2.30), (2.36) and (2.40) are true Hamiltonian operators on the functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$ satisfying the Schouten-Nijenhuis condition (2.10), following the algebraic scheme of [91] mentioned above and using the results of [80] and the right Leibniz algebra (2.35) and the new Riemann algebra (2.39), one can describe a wide class of multicomponent completely integrable dynamical systems containing the infinite hierarchies of multicomponent Riemann hydrodynamical flows. For instance, consider the generalized completely integrable Riemann type dynamical system

$$D_t u_1 = u_2,\ D_t u_2 = u_3,\ \ldots,\ D_t u_N = 0 \tag{4.6}$$

on the functional space $\widetilde{\mathbb{A}}(u)$ for some nonassociative and noncommutative finite-dimensional algebra $\mathbb{A}$, where

$$D_t := \partial/\partial t + u_1 D_x, \qquad\qquad D_x := \partial/\partial x,$$

$x \in \mathbb{S}^1$, $N \in \mathbb{Z}_+$, which was recently studied in detail in [77, 80]. The relationships (2.39) allow to calculate the corresponding representations of the Riemann algebra $\mathbb{A}$ for cases $N = 2$ and $N = 3$, giving rise to the corresponding Hamiltonian operators $\vartheta(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$, coinciding with those constructed in [80] modulo the trivial constant 2-cocycles on the weak adjacent loop Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$. In fact, for the case $N = 2$ one easily obtains from (2.40) the skew-symmetric two-dimensional matrix derivation representation

$$\vartheta_2(u) := \begin{pmatrix} 0 & u_{1,x} D_x^{-1} \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} \end{pmatrix},$$

coinciding, modulo the trivial constant 2-cocycle $\varpi_2(a, b) := f_2(D_x^{-1} a, b)$, determined for a suitable symmetric bilinear form $f_2 : \mathcal{L}_{\widetilde{\mathbb{A}}} \times \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathbb{K}$ and all $a, b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$, with the Hamiltonian operator

$$\eta_2(u) = \begin{pmatrix} D_x^{-1} & u_{1,x} D_x^{-1} \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} \end{pmatrix},$$

on the space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ for the Riemann type dynamical system (4.6), whose Hamiltonian representation

$$\frac{d}{dt}(u_1, u_2)^{\intercal} = -\eta_2(u) \nabla H_2(u_1, u_2)$$

holds for the Hamiltonian function $H_2 \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, equal to

$$H_2 := \frac{1}{2} \int_0^{2\pi} (u_2 u_{1,x} - u_1 u_{2,x}) dx.$$

Proceeding similarly for the case $N = 3$, one easily obtains from (2.40) the skew-symmetric three-dimensional matrix Hamiltonian operator

$$\vartheta_3(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$$

representation

$$\vartheta_3(u) = \begin{pmatrix} 0 & u_{1,x} D_x^{-1} & 0 \\ D_x^{-1} u_{1,x} & u_{2,x} D_x^{-1} + D_x^{-1} u_{2,x} & D_x^{-1} u_{3,x} \\ 0 & u_{3,x} D_x^{-1} & 0 \end{pmatrix},$$

coinciding, modulo the trivial constant 2-cocycle

$$\varpi_2(a,b) := f_3(D_x^{-1}a,b),$$

determined for a suitable symmetric bilinear form $f_3 : \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \to \mathbb{K}$ and all $a,b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ with the Hamiltonian operator

$$\eta_3(u) = \begin{pmatrix} D_x^{-1} & u_{1,x}D_x^{-1} & 0 \\ D_x^{-1}u_{1,x} & u_{2,x}D_x^{-1} + D_x^{-1}u_{2,x} & D_x^{-1}u_{3,x} \\ 0 & u_{3,x}D_x^{-1} & 0 \end{pmatrix}$$

on the functional space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ for the Riemann dynamical system (4.6), whose Hamiltonian representation

$$\frac{d}{dt}(u_1,u_2,u_3)^{\intercal} = -\eta_3(u)\nabla H_3[u_1,u_2,u_3]$$

holds for a suitably constructed Hamiltonian function $H_3 \in \mathcal{F}_{\widetilde{\mathbb{A}}}(u)$.

There is also an interesting observation concerning an infinite hierarchy [80] of the generalized Riemann hydrodynamic systems

$$D_t u_1 = u_2, \quad D_t u_2 = u_3, \quad \ldots, \quad D_t u_{N-1} = (\bar{u}_{N,x})^s, \quad D_t \bar{u}_N = 0 \tag{4.7}$$

on the functional space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$, where $s, N \in \mathbb{Z}_+$, with the algebra $\mathbb{A}$ generated by the constraints (2.39). For the case $s = 2$ and $N = 3$ the above skew-symmetric three-dimensional matrix Hamiltonian operator $\bar{\vartheta}_{3|2}(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$ representation

$$\bar{\vartheta}_{3|2}(u) = \begin{pmatrix} 0 & u_{1,x}D_x^{-1} & 0 \\ D_x^{-1}u_{1,x} & u_{2,x}D_x^{-1} + D_x^{-1}u_{2,x} & D_x^{-1}\bar{u}_{3,x} \\ 0 & \bar{u}_{3,x}D_x^{-1} & 0 \end{pmatrix}$$

proves to coincide, modulo the trivial constant 2-cocycle

$$\bar{\omega}_{2,\bar{\eta}}(a,b) := f_{\bar{\eta}}(D_x^{-1}a,b),$$

determined for any suitable symmetric bilinear form $f_{\bar{\eta}} : \mathcal{L}_{\widetilde{\mathbb{A}}} \times \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathbb{K}$ and all $a,b \in \mathcal{L}_{\widetilde{\mathbb{A}}}$ exactly with the Hamiltonian operator

$$\bar{\eta}_{3|2}(u) = \begin{pmatrix} D_x^{-1} & u_{1,x}D_x^{-1} & 0 \\ D_x^{-1}u_{1,x} & u_{2,x}D_x^{-1} + D_x^{-1}u_{2,x} & D_x^{-1}\bar{u}_{3,x} \\ 0 & \bar{u}_{3,x}D_x^{-1} & 0 \end{pmatrix} \tag{4.8}$$

on the functional space $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ for the Riemann type dynamical system (4.7), whose Hamiltonian representation

$$\frac{d}{dt}(u_1,u_2,\bar{u}_3)^{\intercal} = -\bar{\eta}_{3|2}(u)\nabla H_{3|2}(u_1,u_2,\bar{u}_3)$$

holds for the Hamiltonian function $H_{3|2} \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, equal to

$$H_{3|2} := \frac{1}{2}\int_0^{2\pi} [2u_1(\bar{u}_{3,x})^2 - u_2^2 - u_1^2 u_{2,x}]dx.$$

Moreover, one can calculate such a constant 2-cocycle on the Lie algebra $\mathcal{L}_{\tilde{\mathbb{A}}}$

$$\varpi_{2,\bar{\vartheta}}(a,b) := f_{\bar{\vartheta}}^{(1)}(D_x^{-1}a,b) + f_{\bar{\vartheta}}^{(2)}(a,b),$$

determined for any $a, b \in \mathcal{L}_{\tilde{\mathbb{A}}}$ by means of two suitably symmetric

$$f_{\bar{\vartheta}}^{(1)} : \mathcal{L}_{\tilde{\mathbb{A}}} \times \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathbb{K}$$

and skew-symmetric

$$f_{\bar{\vartheta}}^{(2)} : \mathcal{L}_{\tilde{\mathbb{A}}} \times \mathcal{L}_{\tilde{\mathbb{A}}} \to \mathbb{K}$$

bilinear forms, which naturally generates the (4.8)-compatible Hamiltonian operator

$$\bar{\vartheta}_{3|2}^{(0)}(u) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1/2D_x^{-1} \end{pmatrix}$$

on the functional space $\mathcal{F}_{\tilde{\mathbb{A}}}(u)$ for (4.7), whose Hamiltonian representation

$$\frac{d}{dt}(u_1, u_2, \bar{u}_3)^{\intercal} = -\bar{\vartheta}_{3|2}^{(0)}(u)\,\nabla H_{3|2}^{(0)}(u_1, u_2, \bar{u}_3)$$

holds for the Hamiltonian function $H_{3|2}^{(0)} \in \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, equal to

$$H_{3|2}^{(0)} := \frac{1}{2}\int_0^{2\pi} [u_1 u_{2,x} - u_2 u_{1,x} - 2(\bar{u}_{3,x})^2]dx.$$

It is worth noting here, as was already remarked in [81], that the generalized Riemann hydrodynamic system (4.7) for $s = 3, N = 3$ reduces to the well-known integrable Degasperis-Processi dynamical system [32, 31] for the function $u := u_1$:

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0.$$

Also, for $s = 2$ and $N = 3$, the system (4.7) for the function $u := u_1$ reduces to the well-known [29] integrable Camassa-Holm dynamical system

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} + uu_{xxx} = 0,$$

whose multicomponent extensions were recently extensively studied in [39, 46, 30, 91].

Now, returning to the case $N = 2$ of the system (4.6), it reduces under the substitutions $u_1 := u$, $u_2 := D_x^{-1}u_x^2/2$ to the well-known [47, 44, 74, 79] Hunter-Saxton nonlinear dynamical system

$$du/dt = -uu_x + D_x^{-1}u_x^2/2 \tag{4.9}$$

on the functional manifold $\widetilde{\mathbb{A}}(u)$, $u \in \widetilde{\mathbb{A}}^*$, describing propagation of short-waves in a relaxing medium with spatial memory effects. As shown in [74, 79, 75], the dynamical system (4.9) is a completely integrable bi-Hamiltonian flow on the functional manifold $\widetilde{\mathbb{A}}(u)$, $u \in \widetilde{\mathbb{A}}^*$, with respect to the compatible pair of scalar Hamiltonian operators

$$\vartheta_1(u), \vartheta_2(u) : T^*(\widetilde{\mathbb{A}}(u)) \to T(\widetilde{\mathbb{A}}(u)),$$

$$\vartheta_1(u) = D_x^{-1}, \qquad \vartheta_2(u) = uD_x^{-1} + D_x^{-1}u. \tag{4.10}$$

As we are interested in the corresponding multicomponent generalization of the dynamical system (4.9), we need to consider the functional space $\widetilde{\mathbb{A}}(u)$, $u \in \widetilde{\mathbb{A}}^*$, generated by a finite-dimensional noncommutive and nonassociative algebra $\mathbb{A}$, and construct the Poisson operators on $\mathcal{F}_{\widetilde{\mathbb{A}}}(u)$ in the form (2.36), related to the right Leibniz algebra structure (2.35) and reducing at $N = 1$ to the scalar Hamiltonian operator $\vartheta_2(u) : T^*(\widetilde{\mathbb{A}}(u)) \to T(\widetilde{\mathbb{A}}(u))$ from the pair (4.10).

Moreover, as the compatible Hamiltonian operators are generated by means of suitable central extentions of the adjacent weak Lie algebra, the problem of description them, as was noted above, requires a detailed investigation of the structural properties and finite-dimensional representations of the right Leibniz algebras defined by the constraints (2.35). In what will follow, we stop mostly on the structural properties of the right Leibniz algebras, defined by the constraints (2.35), in particular, we characterize in detail the related derivation algebras.

5. Preliminary algebraic setting

An algebra $(L, +, \cdot)$ over a field $\mathbb{K}$ is called a (right) *Leibniz algebra* if it satisfies the identity

$$x(yz) = (xy)z - (xz)y$$

for any $x, y, z \in L$. Any Lie algebra is clearly a Leibniz algebra.

Leibniz algebras were introduced by A.M. Bloh [23, 24] and rediscovered by J.-L. Loday [59]. Recall that a subalgebra $H \subseteq L$ is said to be *an ideal* of a Leibniz algebra L if $H \cdot L,\ L \cdot H \subseteq H$. A linear mapping $\delta : L \to L$ is called a *derivation* of L if

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in L$. The set $\mathrm{Der}L$ of all derivations of a Leibniz algebra L is a Lie algebra due to operations of the addition "+" and the commutation "$[-,-]$" of linear operators in L. The operator

$$r_a : L \ni x \mapsto xa \in L$$

of a right multiplication is a derivation of L (so-called an *inner derivation* of L induced by $a \in L$). The set

$$\mathrm{Inn}L := \{r_a \mid a \in L\}$$

of all inner derivations of L is an ideal of the Lie ring Der$L$ (see e.g. [41]. A general theory for inner derivations in nonassociative algebras is given in [88]. If L is a Leibniz algebra, then

$$\mathrm{Leib}(L) := \mathrm{spa}\{x^2 \mid x \in L\}$$

is the smallest ideal of L such that the quotient algebra $L/\mathrm{Leib}(L)$ is a Lie algebra (see e.g. [15, 33]. The center

$$Z(L) := \{z \in L \mid zL = 0 = Lz\}$$

and the right annihilator

$$\mathrm{rann}L := \{t \in L \mid Lt = 0\}$$

of L are ideals in L such that $Z(L), \mathrm{Leib}(L) \subseteq \mathrm{rann}L$.

A linear mapping $F : L \to L$ is called a *generalized derivation* of a Leibniz algebra L associated with a derivation $\delta \in \mathrm{Der}L$ (in the sense of Brežar [26] if

$$F(xy) = F(x)y + x\delta(y)$$

for all $x, y \in L$. Denote by GDer$L$ the set of all generalized derivations of L. *We will write* $(F, \delta) \in \mathrm{GDer}L$ *if and only if* F *is a generalized derivation of* L *associated with* $\delta \in \mathrm{Der}L$. Since $(\delta, \delta) \in \mathrm{GDer}L$ for any $\delta \in \mathrm{Der}L$, one concludes that

$$\mathrm{Inn}L \subseteq \mathrm{Der}L \subseteq \mathrm{GDer}L.$$

A generalized derivation F of L that is associated with an inner derivation $r_a \in \mathrm{Inn}L$ is called a *generalized inner derivation* of L. By IGDer$L$ we denote the set of all generalized inner derivations of L. Another various generalizations of Lie (and Leibniz) algebra derivations was introduced in [15, 28, 42, 56, 63] and others.

In what follows, let $D = \mathrm{Der}L$, $G = \mathrm{GDer}L$, Δ be a nonempty subset of D (respectively G). If I is an ideal of L and $\delta(I) \subseteq I$ for all $\delta \in \Delta$, then I is called a Δ-*ideal* of L. Inasmuch $(x + d(x))(x + d(x)) \in \mathrm{Leib}(L)$ for any $x \in L$ and $d \in D$, we deduce that $d(x^2) \in \mathrm{Leib}(L)$ and so $\mathrm{Leib}(L)$ is a D-ideal of L. For a Leibniz algebra $(L, +, \cdot)$, define the derived sequence as follows:

$$L^1 = L, \qquad L^2 = LL, \qquad L^{(k+1)} = L^{(k)}L, \quad (k \geqslant 1).$$

A Leibniz algebra L is called *nilpotent* if there exists a positive integer s such that $L^{(s)} = 0$ (see e.g. [2, 33, 42, 16]). For a (Lie or Leibniz) algebra

L, $X \in \{0, D, G\}$ and

$$W = \begin{cases} 0 & \text{if } Y = \mathrm{Leib}(L) \text{ and } \overline{X} = X \in \{0, D\}, \\ Y & \text{if } Y = \mathrm{rann}L \text{ and } X = G, \end{cases}$$

the algebra A/W is called:

- $\overline{X}$-*semisimple* if, for any X-ideal T of A, the condition $T^2 \subseteq Y$ implies that $T \subseteq Y$,
- $\overline{X}$-*prime* if, for any X-ideals T, Q of A, the condition $TQ \subseteq Y$ implies that $T \subseteq Y$ or $Q \subseteq Y$,
- $\overline{X}$-*simple* if $[A, A] \neq Y$ and it only has the following X-ideals: 0, Y and A (here 0 and Y are not necessarily different),
- $\overline{X}$-*primary* if, for any X-ideals T, Q of A, the condition $TQ \subseteq Y$ implies that $T \subseteq Y$ or $Q^m \subseteq Y$ for some positive integer m.

In particular, a $\overline{X}$-semisimple (respectively $\overline{X}$-prime, $\overline{X}$-simple or $\overline{X}$-primary) Lie (or Leibniz) algebra L is called *semisimple* (respectively *prime*, *simple* or *primary*) if $\overline{X} = 0$ and a 0-ideal is an ideal of L. A Leibniz algebra L is semisimple (respectively prime, simple or primary) if and only if the Lie algebra $L/\mathrm{Leib}(L)$ is the ones. If L is a simple Leibniz algebra, then $L/\mathrm{Leib}(L)$ is a simple Lie algebra, but the opposite is not true. It is not difficult to check that if a Leibniz algebra L is prime (respectively semisimple or simple), then $\mathrm{rann}L = \mathrm{Leib}(L)$. Every semisimple (respectively prime, simple or primary) Leibniz algebra is D-semisimple (respectively D-prime, D-simple or D-primary).

The Leibniz algebras are very popular in physics. Many authors have investigated derivations of Leibniz algebras in the context of geometric study of algebras (see e.g. [54, 72, 83, 82, 84]) and representations of Leibniz algebras (see e.g. [42, 41, 60, 63]). For example, A. Fialowski, A. Kh. Khudoyberdiyev and B. A. Omirov [42] have proved that a Leibniz algebra is nilpotent if and only if it admits an invertible Leibniz-derivation, B. A. Omirov [72], I. S. Rakhimov and A.-H. Al-Nashi [84] have studied derivation algebras of filiform Leibniz algebras, M. Ladra, I. M. Rikhsiboev and R. M. Turdibaev [54] have proved that a finite-dimensional Leibniz algebra L with a nonsingular derivation is nilpotent, I. S. Rakhimov, K. K. Masutova and B. A. Omiro [83] have proved, in particular, that any derivation of a simple finite-dimensional Leibniz algebra over a field of zero characteristic can be represented as sum of three derivations of special form. In this paper we study connections between Leibniz algebras L, their derivation algebras $\mathrm{Der}L$ and generalized derivation algebras $\mathrm{GDer}L$.

Our first result subject to these topics is the following proposition which will be proved in Section 6:

Proposition 5.1. *Let L be a Leibniz algebra. Then the following hold:*

(1) *D is a simple Lie algebra if and only if L is a simple Leibniz algebra and $D = \mathrm{Inn}L = [D, D]$,*
(2) *if D is a prime (respectively semisimple or primary) Lie algebra, then L is a D-prime (respectively D-semisimple or D-primary) Leibniz algebra.*

We also prove an analogue of the result of S. Tôgô [93] that is a finite-dimensional Leibniz algebra L such that $L \neq L^2$ and $Z(L) \neq 0$ has an outer derivation (see Proposition 6.5 below).

Obviously, a finite-dimensional Leibniz algebra L is semisimple if its maximal solvable ideal is equal to $\mathrm{Leib}(L)$. Semisimple Leibniz algebras have studied in [2, 33, 45, 83] and others. In this way we prove in Section 8 the next result.

Theorem 5.2. *If L is a D-prime (respectively D-semisimple or D-simple) Leibniz algebra, then $D/\mathrm{ADer}L$ is prime (respectively semisimple or simple) Lie algebra, where $\mathrm{ADer}L := \{\delta \in \mathrm{Der}L \mid \delta(L) \subseteq \mathrm{rann}L\}$.*

A linear mapping $T : L \to L$ is called a *multiplier* of a Leibniz algebra L whenever

$$T(xy) = T(x)y$$

for all $x, y \in L$. The set of all multipliers of L will be denoted by $\mathrm{M}L$. Obviously that, for any $T \in \mathrm{M}L$, $(T, 0) \in \mathrm{IGDer}L$ and so

$$\mathrm{M}L \subseteq \mathrm{IGDer}L \subseteq \mathrm{GDer}L.$$

Moreover, $\mathrm{M}L$ is an ideal of the Lie ring $\mathrm{GDer}L$. In Section 8 we will prove the following

Theorem 5.3. *Let L be a Leibniz algebra. Then the following hold:*

(1) *if $\mathrm{Der}L/\mathrm{ADer}L$ is a semisimple (respectively prime, simple or primary) Lie algebra, then $L/\mathrm{rann}L$ is a $\overline{G}$-semiprime (respectively $\overline{G}$-prime, $\overline{G}$-simple or $\overline{G}$-primary) Lie algebra,*
(2) *if $\mathrm{Inn}L/\mathrm{AInn}L$, where $\mathrm{AInn}L = \mathrm{ADer}L \cap \mathrm{Inn}L$, is a semisimple (respectively prime, simple or primary) Lie algebra, then $L/\mathrm{rann}L$ is a semisimple (respectively prime, simple or primary) Lie algebra.*

For basic definitions and properties of Leibniz and Lie algebras we refer to [2, 7, 48, 49, 59].

6. Properties of derivation algebras

At first we will present some information about derivation algebras. If $A \subseteq L$, then

$$\mathrm{Inn}_A L := \{r_a \mid a \in A\}$$

and, in particular, $\mathrm{Inn}L = \mathrm{Inn}_L L$.

Lemma 6.1. *Let L be a Leibniz algebra, A its ideal. Then the following hold:*

(i) *if A is a Δ-ideal of L, then $\mathrm{Inn}_A L$ is an ideal of D,*
(ii) *$\mathrm{Inn}_A L$ is an ideal of $\mathrm{Inn}L$,*
(iii) *$\mathrm{Inn}_A L = 0$ if and only if $A \subseteq \mathrm{rann}L$,*
(iv) *$\mathrm{Inn}L = 0$ if and only if $L^2 = 0$,*
(v) *$\mathrm{Inn}_A L = \mathrm{Inn}L$ if and only if $L = A + \mathrm{rann}L$,*
(vi) *$\mathrm{rann}L$ is a D-ideal of L,*
(vii) *there is the Lie algebra isomorphism*

$$\mathrm{Inn}L \ni r_a \mapsto a + \mathrm{rann}L \in L/\mathrm{rann}L,$$

(viii) *if Φ is an ideal of $\mathrm{Inn}L$, then*

$$\Delta_\Phi = \{a \in L \mid r_a \in \Phi\}$$

is an ideal of L,
(ix) *if Φ is an ideal of D, then Δ_Φ is a D-ideal of L,*
(x) *if $B, C \subseteq L$, then $[\mathrm{Inn}_B L, \mathrm{Inn}_C L] = \mathrm{Inn}_{CB} L$.*

Proof. By routine calculations. □

Lemma 6.2. *Let A be a Leibniz algebra and Φ an ideal of D. Then we have:*

(i) *$\mathrm{ADer}L$ is an ideal of D,*
(ii) *$[\Phi, \mathrm{Inn}L] = 0$ if and only if $\Phi \subseteq \mathrm{ADer}L$,*
(iii) *if $\Phi \cap \mathrm{Inn}L = 0$, then $\Phi \subseteq \mathrm{ADer}L$.*

Proof. (i) Immediately.

(ii) Let $\delta \in \Phi$ and $a \in L$.

Sufficiency. Since

$$r_{\delta(a)} = [\delta, r_a] \tag{6.1}$$

and $[\delta, r_a] = 0$, we deduce that $\delta(a) \in \mathrm{rann}L$.

Necessity. Inasmuch $r_{\delta(a)} = 0$, we conclude that the assertion holds in view of Eq. (6.1)).

(iii) It follows from (ii). □

Corollary 6.3. *Let L be a Leibniz algebra. Then* Inn$L$ *is a simple (respectively prime, semisimple or primary) Lie algebra if and only if L is a simple (respectively prime, semisimple or primary) Leibniz algebra.*

Proof of Proposition 5.1. (1)⇒(2). Let D be a simple Lie algebra. If Inn$L = 0$, then $L^2 = 0$ and any endomorphism of the additive group L^+ is a derivation of L. If p is a prime, then

$$E_p = \{\sigma_m : L^+ \to L^+ \mid \sigma_m(a) = ma, \text{where } a \in L \text{ and } p \text{ is a divisor of an integer } m\},$$

is an ideal of D. Therefore $E_p = D$ if the characteristic charakt$\mathbb{K} = 0$ or $E_q = D = E_r$ if the characteristic charakt$\mathbb{K} = p$, where p, q, r are pair wise distinct primes. This leads to a contradiction. Hence

$$0 \neq \mathrm{Inn}L = D.$$

Then $L/\mathrm{rann}L$ is a simple Lie algebra by Corollary 6.3, $L = L^2 + \mathrm{rann}L$ and

$$D = [D, D] = [\mathrm{Inn}L, \mathrm{Inn}L] = \mathrm{Inn}L.$$

(2)⇒(1). By Corollary 6.3, Inn$L$ is a simple Lie algebra. Consider three cases.

(2a) Let D be a prime Lie algebra and A, B be D-ideals of L such that

$$AB \subseteq \mathrm{rann}L. \tag{6.2}$$

Then

$$[\mathrm{Inn}_B L, \mathrm{Inn}_A L] = 0 \tag{6.3}$$

by Lemma 6.1(x) and, as a consequence, $\mathrm{Inn}_B L = 0$ or $\mathrm{Inn}_A L = 0$. This means that $B \subseteq \mathrm{rann}L$ or $A \subseteq \mathrm{rann}L$. Thus L is a D-prime Leibniz algebra.

(2b) If D is a semisimple Lie algebra, then we can obtain the assertion by the same argument as in the part (2a).

(2c) Assume that D is a primary Lie algebra and A, B are D-ideals of L satisfying Eq. (6.2). Then, as in (2a), we have Eq. (6.3) and so $\mathrm{Inn}_B L = 0$ or $\mathrm{Inn}_A L$ is a nilpotent ideal of D. Consequently $B \subseteq \mathrm{rann}L$ or $A^n \subseteq \mathrm{rann}L$. Hence L is a D-primary Leibniz algebra. □

Lemma 6.4. *Let L be a Leibniz algebra and M be its ideal of codimension 1. If $0 \neq Z(L) \subseteq M$, then $Z(M)$ is an ideal of L and $Z(M) \neq L \cdot Z(M)$.*

Proof. It is easy to see that $L = M \oplus a\mathbb{K}$ is a direct sum of subspaces, where $a \in L$. Since

$$m(ux) = 0, \qquad m(xu) = 0, \qquad (ux)m = 0, \qquad (xu)m = 0$$

for all $x \in L$, $m \in M$ and $u \in Z(M)$, we deduce that $Z(M)$ is an ideal of L. Moreover,

$$l_a : Z(M) \ni u \mapsto au \in Z(M)$$

is an endomorphism of the additive group $Z(M)^+$.

Since $0 \neq Z(L) \subseteq Z(M)$, we see that the kernel $\mathrm{Ker} l_a \neq 0$ and so

$$\dim Z(M) > \dim(a \cdot Z(M)).$$

Hence $L \cdot Z(M) \neq Z(M)$. □

Proposition 6.5. *Let L be a finite-dimensional Leibniz algebra such that $Z(L) \neq 0$ and $L^2 \neq L$. Then L has an outer derivation.*

Proof. Since $L^2 \neq L$, we deduce that there exists a subspace M of codimension 1 of L such that $L^2 \subseteq M$ and $L = M \oplus a\mathbb{K}$ is a direct sum of subspaces for some $a \in L$. Obviously that M is an ideal of L. Suppose that $0 \neq z_0 \in L$ and there exists a linear map

$$\delta : L \ni m + \lambda a \mapsto \lambda z_0 \in L,$$

where $\lambda \in \mathbb{K}$, $\delta(a) = z_0$ and $\delta(m) = 0$ for any $m \in M$. If, moreover, $\delta = r_u$ is an inner derivation for some $u \in L$, then $0 = \delta(M) = Mu$ and

$$z_0 = \delta(a) = r_u(a) = au. \tag{6.4}$$

1) If $Z(L) \nsubseteq M$, $z_0 \in Z(L) \backslash M$ and $a \in Z(L)$, then $0 \neq \delta \in \mathrm{Der} L$ and $z_0 = au \in M$ by Eq. (6.4), a contradiction.

2) If $Z(L) \nsubseteq L^2$ and $z_0 \in Z(L) \backslash L^2$, then $0 \neq \delta \in \mathrm{Der} L$ and $z_0 = au \in L^2$, a contradiction.

3) Assume that $Z(L) \subseteq L^2$ and so $Z(L) \subseteq Z(M)$.

a) Suppose $Z(M) \neq Z(L)$ and $z_0 \in Z(M) \backslash Z(L)$. Then $0 \neq \delta \in \mathrm{Der} L$. If $u = m_0 + \lambda_0 a$ for some $m_0 \in M$ and $\lambda_0 \in \mathbb{K}$, then

$$z_0 u = \lambda_0 z_0 a, \qquad u z_0 = \lambda_0 a z_0$$

and

$$z_0 u + u z_0 = \lambda_0 \delta(a^2) = 0.$$

This yields that $uz_0 = -z_0 u = 0$ and consequently $z_0 \in Z(L)$, which gives a contradiction.

Now assume that $Z(L) = Z(M)$.

b) If $z_0 \in Z(M) \backslash L \cdot \mathrm{rann} M$, then $0 \neq \delta \in \mathrm{Der} L$. Indeed, $a^2 \in M$ and therefore $\delta(a^2) = 0$. Then

$$\begin{aligned}
\delta((m_1 + \lambda_1 a)(m_2 + \lambda_2 a)) = 0 &= \lambda_1 \lambda_2 \delta(a^2) \\
&= \lambda_1 \lambda_2 \delta(a) a + \lambda_1 \lambda_2 a \delta(a) = \\
&= \lambda_1 z_0 (m_2 + \lambda_2 a) + (m_1 + \lambda_1 a) \lambda_2 z_0 = \\
&= \delta(m_1 + \lambda_1 a) \cdot (m_2 + \lambda_2 a) + (m_1 + \lambda_1 a) \cdot \delta(m_2 + \lambda_2 a)
\end{aligned}$$

for any $m_1, m_2 \in M$ and $\lambda_1, \lambda_2 \in \mathbb{K}$. Moreover, $z_0 = au \in L \cdot \mathrm{rann} M$, which gives a contradiction.

c) Now assume that $Z(M) = L \cdot \mathrm{rann} M$. In view of Lemma 6.4,

$$Z(M) \neq L \cdot Z(M)$$

and so $Z(M) \neq \mathrm{rann} M$. If $z_0 \in Z(M) \backslash \mathrm{rann} M$, then $0 \neq \delta \in \mathrm{Der} L$ and $z_0 = au \in \mathrm{rann} M$, which gives a contradiction.

It now follows from 1)-3) that δ is an outer derivation of L. □

Corollary 6.6. *If L is a finite-dimensional nilpotent Leibniz algebra, then it admits an outer derivation.*

7. Generalized derivations

Lemma 7.1. *Let L be a Leibniz algebra. Then:*

(i) $\mathrm{GDer} L$ *is a Lie ring with respect to the point-wise addition "+" and the point-wise Lie multiplication "$[-,-]$" given by the rules*

$$(H + K)(x) = H(x) + K(x)$$

and

$$[H, K](x) = H(K(x)) - K(H(x))$$

for all $x \in L$ and $H, K \in \mathrm{GDer} L$;

(ii) *if A is a D-ideal of L, then*

$$\mathrm{I_A GDer} L = \{F \in \mathrm{GDer} L \mid F \textit{ is associated with some } r_a \in \mathrm{Inn}_A L\}$$

is an ideal of $\mathrm{GDer} L$*. In particular,*

$$\mathrm{IGDer} L = \mathrm{I_L GDer} L, \qquad \mathrm{M} L = \mathrm{I_O GDer} L.$$

(iii) $\mathrm{GDer} L = \mathrm{M} L + \mathrm{Der} L$*, where* $\mathrm{M} L$ *is an ideal of* $\mathrm{GDer} L$*, and*

$$\mathrm{M} L \bigcap \mathrm{Der} L \subseteq \mathrm{ADer} L;$$

(iv) $\mathrm{IGDer} L = \mathrm{M} L + \mathrm{Inn} L$*, where* $\mathrm{M} L$ *is an ideal of* $\mathrm{IGDer} L$*, and*

$$\mathrm{M} L \bigcap \mathrm{Inn} L \subseteq \mathrm{ADer} L;$$

(v) *if $(F, \delta), (F, d) \in \mathrm{GDer} L$, then $\delta + \mathrm{ADer} L = d + \mathrm{ADer} L$.*

Proof. Assume that $(F, \delta), (K, d) \in \mathrm{GDer} L$, $T \in \mathrm{M} L$ and $x, y \in L$.

(i) We see that $(F - K, \delta - d) \in \mathrm{GDer} L$,

$$[F, K](xy) = F(K(x)y + xd(y)) - K(F(x)y + x\delta(y)) =$$
$$= [F, K](x)y + x[\delta, d](y)$$

and so $([F, K], [\delta, d]) \in \mathrm{GDer} L$.

(ii) Evident.

(iii) The equality

$$[F,T](xy) = [F,T](x)y$$

implies that $[F,T] \in \mathrm{M}L$ and, as a consequence, $\mathrm{M}L$ is an ideal of $\mathrm{GDer}L$. From

$$(\delta - F)(xy) = \delta(x)y + x\delta(y) - F(x)y - x\delta(y) = (\delta - F)(x)y$$

it follows that $\delta - F \in \mathrm{M}L$. If $g \in \mathrm{Der}L \cap \mathrm{M}L$, then

$$g(x)y = g(xy) = g(x)y + xg(y).$$

Hence $xg(y) = 0$ whih implies that $g(L) \subseteq \mathrm{rann}L$.

(iv) By the same argument as in the part (iii).

(v) If $(F,\delta), (F,d) \in \mathrm{GDer}L$ for some $\delta, d \in \mathrm{Der}L$, then

$$x\delta(y) = xd(y)$$

and consequently $x(\delta - d)(y) = 0$. This means that $(\delta - d)(L) \subseteq \mathrm{rann}L$. □

Lie algebras L with abelian derivation algebras $\mathrm{Der}L$ was studied by S. Tôgô [92].

Lemma 7.2. *Let L be a Leibniz algebra and $(F,d) \in \mathrm{GDer}L$. Then we have:*

(i) *if $F = 0$, then $d \in \mathrm{ADer}L$,*
(ii) *if $d \in \mathrm{ADer}L$, then $F \in \mathrm{M}L$,*
(iii) *if $\mathrm{GDer}L$ is an abelian Lie algebra, then $\mathrm{Der}L$ is abelian,*
(iv) *if $L \neq 0$, then $\mathrm{IGDer}L \neq 0$.*

Proof. Assume that $x, y \in L$.

(i) In fact,

$$0 = F(xy) = F(x)y + xd(y) = xd(y)$$

and so $d(y) \in \mathrm{rann}L$.

(ii) Since $F(xy) = F(x)y$, we deduce that $F \in \mathrm{M}L$.

(iii) We have $\mathrm{Der}L \subseteq \mathrm{GDer}L$ and therefore the assertion holds.

(iv) Straightforward. □

Lemma 7.3. *Let L be a Leibniz algebra and $(H, r_a) \in \mathrm{IGDer}L$. Then the following hold:*

(i) *if $H = 0$, then $a \in \mathrm{rann}L$,*
(ii) *if $a \in \mathrm{rann}L$, then $H \in \mathrm{M}L$,*
(iii) *if $\mathrm{IGDer}L$ is an abelian Lie algebra, then $L^2 \subseteq \mathrm{rann}L$,*
(iv) *if L is abelian, then $\mathrm{IGDer}L = \mathrm{M}L$.*

Proof. It is easy to check by using the same argument as in the proof of Lemma 7.2. □

Let $\Phi \subseteq \mathrm{GDer}L$, $\Gamma \subseteq \mathrm{Der}L$,

$$T_\Phi = \{d \in \mathrm{Der}L \mid \text{ there is } H \in \Phi \text{ that is associated with } d \in \mathrm{Der}L\},$$
$$U_\Gamma = \{H \in \mathrm{GDer}L \mid H \text{ is associated with some } d \in \Gamma\},$$
$$\Sigma_\Phi = \{a \in L \mid \text{ there is } H \in \Phi \text{ that is associated with } r_a \in \mathrm{Inn}L\}.$$

Lemma 7.4. *Let L be a Leibniz algebra. If Φ is an ideal of $\mathrm{GDer}L$ such that $[\Phi, \mathrm{IGDer}L] = 0$, then $T_\Phi \subseteq \mathrm{ADer}L$ (and so $\Phi \subseteq \mathrm{M}L$).*

Proof. Indeed, if $(F, d) \in \Phi$, then $r_{d(a)} = [d, r_a] \in \mathrm{ADer}L$ for any $a \in L$ and so $d(a) \in \mathrm{rann}L$. □

Lemma 7.5. *Let L be a Leibniz algebra. Then the following hold:*

(1) *if Φ is an ideal of $\mathrm{IGDer}L$ (respectively $\mathrm{GDer}L$), then Σ_Φ is an ideal (respectively a D-ideal) of L,*
(2) *if Γ is an ideal of $\mathrm{Der}L$, then U_Γ is an ideal of $\mathrm{GDer}L$ (in particular, $U_0 = \mathrm{I_O GDer}L = \mathrm{M}L$),*
(3) *if Φ is an ideal of $\mathrm{GDer}L$, then T_Φ is an ideal of $\mathrm{Der}L$.*

Proof. (1) Let $a, b \in \Sigma_\Phi$, $t \in L$, $(H, r_a), (K, r_b) \in \Phi$, $(S, r_t) \in \mathrm{IGDer}L$ and $(M, \delta) \in \mathrm{GDer}L$. Since

$$(H - K, r_{a-b}),\ ([M, H], r_{\delta(a)}),\ ([H, S], r_{ta}),\ ([S, H], r_{at}) \in \Phi,$$

we conclude that $a - b, \delta(a), ta, at \in \Sigma_\Phi$ and therefore Σ_Φ is an ideal of L.
(2)-(3) By the same argument as in the part (1). □

Lemma 7.6. *Let L be a Leibniz algebra and A its ideal. Then the following conditions are equivalent:*

(1) $\mathrm{I_A GDer}L \subseteq \mathrm{M}L$,
(2) $A \subseteq \mathrm{rann}L$,
(3) $\mathrm{Inn}_A L = 0$.

Proof. For proof see Lemmas 6.1 and 7.1. □

Lemma 7.7. *Let L be a Leibniz algebra and $a \in L$. If L is D-semisimple, then $r_a \in \mathrm{ADer}L$ if and only if $a \in \mathrm{rann}L$.*

Proof. We have that

$$La = r_a(L) \subseteq \mathrm{rann}L.$$

Moreover, $L\delta^n(a) \subseteq \mathrm{rann}L$ for any $\delta \in D$ and a non-negative integer n. If $l, t \in L$, then

$$0 = t(l\delta^n(a)) = (tl)\delta^n(a) - (t\delta^n(a))l$$

and so $(t\delta^n(a))l \in L\delta^n(a)$. Hence

$$L_0 := La + \sum_{n=1}^{\infty} L\delta^n(a)$$

is a D-ideal of L and $L_0 \cdot L_0 = 0$. Consequently $La = 0$. □

Lemma 7.8. *Let L be a Leibniz algebra and A its ideal. Then there exist Lie algebra isomorphisms:*

(1) $\mathrm{Der}L/\mathrm{ADer}L \ni d + \mathrm{ADer}L \longmapsto F + \mathrm{M}L \in \mathrm{GDer}L/\mathrm{M}L$, *where* $(F, d) \in \mathrm{GDer}L$,
(2) $\mathrm{Inn}L/\mathrm{AInn}L \ni r_a + \mathrm{AInn}L \longmapsto H + \mathrm{M}L \in \mathrm{IGDer}L/\mathrm{M}L$, *where* $(H, r_a) \in \mathrm{IGDer}L$.

Proof. Straightforward. □

8. Proofs

Proof of Theorem 5.2. (a) Let L be a D-prime Leibniz algebra and Ψ, Ω be ideals of D such that $[\Psi, \Omega] = 0$. Then $\mathrm{rann}L = \mathrm{Leib}(L)$. If

$$\Phi := \Psi \bigcap \mathrm{Inn}L \quad \text{and} \quad \Lambda := \Omega \bigcap \mathrm{Inn}L,$$

then

$$\Phi = \mathrm{Inn}_{\Delta_\Phi} L, \qquad \Lambda = \mathrm{Inn}_{\Delta_\Lambda} L, \qquad [\Phi, \Lambda] = 0.$$

Lemma 6.1(x) and (iii) imply that

$$\Delta_\Lambda \Delta_\Phi \subseteq \mathrm{rann}L.$$

Since Δ_Λ and Δ_Φ are D-ideals by Lemma 6.1(ix), we deduce that

$$\Delta_\Lambda \subseteq \mathrm{rann}L \quad \text{or} \quad \Delta_\Phi \subseteq \mathrm{rann}L$$

by the D-primeness of L. This gives that $\Lambda = 0$ or $\Phi = 0$ by Lemma 6.1(iii). As a consequence of Lemma 6.2(iii), the quotient Lie algebra $D/\mathrm{ADer}L$ is prime.

(b) If L is a D-semisimple Leibniz algebra, then we can obtain that $D/\mathrm{ADer}L$ is semisimple analogously as in (a).

(c) Assume that L is a D-simple Leibniz algebra and Ψ is an ideal of D. Then $\mathrm{rann}L = \mathrm{Leib}(L)$. If $\Phi := \Psi \cap \mathrm{Inn}L$, then Δ_Φ is a D-ideal of L and so $\Delta_\Phi \subseteq \mathrm{rann}L$. By Lemma 6.1(iii), $\Phi = \mathrm{Inn}_{\Delta_\Phi} L = 0$ and $\Psi \subseteq \mathrm{ADer}L$ by Lemma 6.1(iii). Thus $D/\mathrm{ADer}L$ is a simple Lie algebra. □

Proposition 8.1. *Let L be a Leibniz algebra. Then the following hold:*

(1) *if* $\mathrm{GDer}L/\mathrm{M}L$ *is a semisimple (respectively prime, simple or primary) Lie algebra, then* $L/\mathrm{rann}L$ *is a* $\overline{G}$*-semiprime (respectively* $\overline{G}$*-prime,* $\overline{G}$*-simple or* $\overline{G}$*-primary) Lie algebra;*
(2) *if* $\mathrm{IGDer}L/\mathrm{M}L$ *is a semisimple (respectively prime, simple or primary) Lie algebra, then* $L/\mathrm{rann}L$ *is a semiprime (respectively prime, simple or primary) Lie algebra.*

Proof. (1a) Assume that $\mathrm{GDer}L/\mathrm{M}L$ is a prime Lie algebra and A, B are G-ideals of L such that $AB \subseteq \mathrm{rann}L$. Then

$$[\mathrm{I_B}\mathrm{GDer}L, \mathrm{I_A}\mathrm{GDer}L] \subseteq \mathrm{M}L \tag{8.1}$$

and so $\mathrm{I_B}\mathrm{GDer}L \subseteq \mathrm{M}L$ or $\mathrm{I_A}\mathrm{GDer}L \subseteq \mathrm{M}L$ what forces that $B \subseteq \mathrm{rann}L$ or $A \subseteq \mathrm{rann}L$ by Lemma 7.6. Hence $L/\mathrm{rann}L$ is a $\overline{G}$-prime Lie algebra.

(1b) If $\mathrm{GDer}L/\mathrm{M}L$ is a semisimple Lie algebra, then we can prove by the same argument as in the case (1a).

(1c) Assume that $\mathrm{GDer}L/\mathrm{M}L$ is a simple Lie algebra and A is a G-ideal of L. By Lemma 7.1(ii), $\mathrm{I_A}\mathrm{GDer}L$ is an ideal of $\mathrm{GDer}L$ and so

$$\mathrm{GDer}L = \mathrm{I_A}\mathrm{GDer}L$$

or

$$\mathrm{M}L = \mathrm{I_A}\mathrm{GDer}L$$

what implies that $L = A + \mathrm{rann}L$ or $A \subseteq \mathrm{rann}L$. Consequently $L/\mathrm{rann}L$ is a $\overline{G}$-simple Lie algebra.

(1d) Let $\mathrm{GDer}L/\mathrm{M}L$ be a primary Lie algebra and A, B be G-ideals of L satisfying the condition $AB \subseteq \mathrm{rann}L$.

Then (8.1) holds and so $\mathrm{I_A}\mathrm{GDer}L \subseteq \mathrm{M}L$ or

$$[\underbrace{\mathrm{I_B}\mathrm{GDer}L, \dots, \mathrm{I_B}\mathrm{GDer}L}_{m \text{ times}}] \subseteq \mathrm{M}L$$

for some positive integer m. Therefore $A \subseteq \mathrm{rann}L$ or $B^m \subseteq \mathrm{rann}L$. Hence $L/\mathrm{rann}L$ is a $\overline{G}$-primary Lie algebra.

(2) By the analogues argument as in the proof of the part (1). □

Proof of Theorem 5.3. This theorem is a consequence of Proposition 8.1 and Lemma 7.8. □

9. Supplement: The classical Poisson manifolds approach revisited

9.1. **Poisson structures on noncommutative functional manifolds.** It is interesting to look at the construction of the Hamiltonian operators presented above and revisit it from the standard point of view, considering them as those defined on the naturally associated [1, 4, 22, 20, 68, 70, 78] cotangent space $T^*(M)$ to some linear functional manifold $M \simeq \widetilde{\mathbb{A}}^* \simeq \widetilde{\mathbb{A}}$.

Then, a Hamiltonian operator on M is defined [1] as a smooth mapping $\vartheta : M \to Hom(T^*(M); T(M))$, such that for any fixed $u \in M$ the bracket

$$\{f, g\} := (\nabla f(u), \vartheta(u)\nabla g(u)), \tag{9.1}$$

where $f, g : M \to \mathbb{K}$ are arbitrary smooth mappings from the functional space $\mathcal{D}(M) \simeq \mathcal{F}_{\tilde{\mathbb{A}}}(u)$, satisfies the Jacobi identity. The bracket (9.1) is determined on M by means of the natural convolution $(\cdot, \cdot)$ on the product $T^*(M) \times T(M)$, and respectively, the gradient $\nabla f(u) \in T^*(M)$ of a function $f \in \mathcal{D}(M)$ is calculated as

$$(\nabla f(u), h) := df[u + \varepsilon h]/d\varepsilon|_{\varepsilon=0} \tag{9.2}$$

for any $h \in T(M)$. It is well known [43, 58] that a linear operator

$$\vartheta(u) : T^*(M) \to T(M),$$

determined at any point $u \in M$, is Hamiltonian iff the suitably defined [43] Schouten-Nijenhuis bracket

$$[[\vartheta(u), \vartheta(u)]] = 0 \tag{9.3}$$

identically on M. Namely, this condition (9.3) was used in the investigations [43, 90] to formulate criteria for the operator $\vartheta(u) : T^*(M) \to T(M)$ to be Hamiltonian on the functional manifold M. Yet these criteria appear to be very complicated and involve a large amount of cumbersome calculations even in the case of fairly simple differential expressions. So, we have reanalyzed this problem from a slightly different point of view. First, recall that the Jacobi identity for the bracket (9.1) is completely equivalent to the fact that the bracket operator defined as $D_f(g) := \{f, g\}$ for a fixed $f \in \mathcal{D}(M)$ and arbitrary $g \in \mathcal{D}(M)$ acts as a derivation on the space $(\mathcal{D}(M); \{\cdot, \cdot\})$:

$$D_f\{g, h\} = \{D_f(g), h\} + \{g, D_f(h)\}, \tag{9.4}$$

where $g, h \in \mathcal{D}(M)$ are taken arbitrary. This can be easily reformulated as follows: take any element $\varphi \in T^*(M)$, such that the Fréchet derivative $\varphi'(u) = \varphi'^{,*}(u)$ at any $u \in M$ with respect to the convolution $(\cdot, \cdot)$ on $T^*(M) \times T(M)$, and construct a vector field $K : M \to T(M)$ as

$$K(u) := \vartheta(u)\varphi(u).$$

Then the derivation condition (9.4) can be equivalently rewritten [1, 68, 20, 70, 78] as the strong Lie derivative

$$L_K\vartheta := \vartheta' \cdot K - \vartheta K'^{,*} - K'\vartheta = 0 \tag{9.5}$$

along the vector field $K(u) = \vartheta(u)\varphi(u) \in T(M)$ at any $u \in M$ for all "self-adjoint" elements $\varphi \in T^*(M)$. Equivalently, a given linear skew-symmetric operator $\vartheta(u) : T^*(M) \to T(M), u \in M$, is Hamiltonian iff the Lie derivative (9.5) vanishes for all "self-adjoint" elements $\varphi \in T^*(M)$. Moreover, as was observed in [64], it suffices to check the condition (9.5) only on the subspace of elements $\varphi \in T^*(M)$ satisfying the condition $\varphi'(u) = 0$ for any $u \in M$.

As an example, one can check that a skew-symmetric matrix-differential operator on M of the form

$$\vartheta(u) := \sigma(u)D_x + D_x\sigma^{\intercal}(u), \tag{9.6}$$

where, an n-dimensional square matrix

$$\sigma(u) := \Big(\sum_{s=1}^{n} u_s\sigma_{ij}^{s} \mid i,j = \overline{1,n}, n \in \mathbb{Z}_+, u \in M\Big),$$

satisfies the condition (9.5) iff the linearly independent elements from

$$\underset{\mathbb{K}}{span}\{e_j \in \mathbb{A} \mid j = \overline{1,n}\}$$

generate the finite dimensional nonassociative Balinsky-Novikov algebra (4.2) and satisfy the conditions $e_i \circ e_j = \sum_{s=1}^{n} \sigma_{ij}^{s} e_s$ for all $i,j = \overline{1,n}$. Similarly, one can verify that the skew-symmetric inverse-differential operator

$$\vartheta(u) := \sigma(u)D_x^{-1} + D_x^{-1}\sigma(u)^{\intercal}, \tag{9.7}$$

where, as above $\sigma(u) := \Big(\sum_{s=1}^{n} u_s\sigma_{ij}^{s} \mid i,j = \overline{1,n}, n \in \mathbb{Z}_+, u \in M\Big)$, the sign "$\intercal$" means the usual matrix transposition, is Hamiltonian iff the basic nonassociative algebra $\mathbb{A} := \underset{\mathbb{K}}{span}\{e_j : j = \overline{1,n}\}$ coincides with the right Leibniz algebra (4.3) and the condition $e_i \circ e_j = \sum_{s=1}^{n} \sigma_{ij}^{s} e_s$ holds for any $i,j = \overline{1,n}$. The skew-symmetric inverse-differential operator (9.7) can be naturally generalized to the expression

$$\vartheta(u) := D_x\sigma(u)D_x^{-1} - D_x^{-1}\sigma(u)^{\intercal}D_x,$$

which can be rewritten as

$$\vartheta(u) = \sigma(D_x u)D_x^{-1} + D_x^{-1}\sigma(D_x u)^{\intercal} + \sigma(u) - \sigma(u)^{\intercal}. \tag{9.8}$$

The condition (9.5) for the operator (9.8) to be Hamiltonian reduces to the constraints on the related nonassociative algebra

$$\mathbb{A} := \underset{\mathbb{K}}{span}\{e_j : j = \overline{1,n}\}$$

exactly coinciding with that of (4.4), and analyzed in some detail in Section 3.

As it was already mentioned, based on the matrix representations of the right Leibniz algebra (4.3) and the new nonassociative Riemann algebra (4.5), one can construct many nontrivial Hamiltonian operators

$$\vartheta(u) : \mathcal{L}_{\widetilde{\mathbb{A}}} \to \mathcal{L}_{\widetilde{\mathbb{A}}}$$

on the associated weak Lie algebra $\mathcal{L}_{\widetilde{\mathbb{A}}}$, related with diverse types of nonassociative algebras $\mathbb{A}$. These Hamiltonian operators prove to be very useful [21, 80, 81] for describing a wide class of multicomponent hierarchies of integrable Riemann type hydrodynamic systems and their various physically reasonable reductions.

9.2. **Poisson structures on manifolds generated by associative noncommutative algebras.** Proceed now to a slightly generalized construction of Hamiltonian operators on a phase space, generated by associative noncommutative algebra A-valued matrices, which was first studied in [25, 35, 76, 78] in case of the noncommutative operator algebras and continued later in [62, 51, 52, 53, 62, 66, 67, 71] in case of general associative noncommutative algebras. This natural and simple generalization appeared to be very useful [5, 6, 94, 96, 62, 66, 67] for describing a wide class of new Lax type integrable nonlinear Hamiltonian systems on associative noncommutative algebras, interesting for diverse applications in modern quantum physics.

We start here with a free associative noncommutative algebra

$$A = \mathbb{K}\langle u_1, u_2, \ldots, u_m \rangle,$$

generated by a finite set of elements $\{u_j \in A : j = \overline{1,m}\}$, and define its "abelianization" $A_\natural := A/[A, A]$ and the projection $\pi : A \to A_\natural$, where $[A, A] := \{uv - vu \in A : u, v \in A\}$. Consider now a naturally related with A n-dimensional matrix Lie algebra $\mathcal{G} := gl(n; A)$ over the field $\mathbb{K}$ with entries in A subject to the usual matrix commutator $[a \,|\, b] := ab - ba$ for all $a, b \in \mathcal{G}$. Being first interested in the Lie-algebraic studying [22, 20, 38, 85] of co-adjont orbits on the adjoint space $\mathcal{G}^*$, let us construct a bi-linear form $\langle \cdot \,|\, \cdot \rangle : \mathcal{G} \times \mathcal{G} \to A_\natural$ on the Lie algebra $\mathcal{G}$ by means of the trace-type expression

$$\langle a \,|\, b \rangle := \pi \mathrm{tr}(a^\intercal b) \tag{9.9}$$

for any $a, b \in \mathcal{G}$. The following important lemma holds.

Lemma 9.3. *The bilinear form* (9.9) *on* $\mathcal{G}$ *is symmetric, nondegenerate and ad-invariant.*

Proof. *Symmetricity.* We have:

$$\begin{aligned}\langle a \,|\, b\rangle &= \sum_{i,j=\overline{1,n}} \pi(a_{ij}b_{ij}) \\ &= \sum_{i,j=\overline{1,n}} \pi(a_{ij}b_{ij} - b_{ij}a_{ij}) + \sum_{i,j=\overline{1,n}} \pi(b_{ij}a_{ij}) \\ &= \sum_{i,j=\overline{1,n}} \pi(b_{ij}a_{ij}) = \langle b \,|\, a\rangle\end{aligned} \tag{9.10}$$

for any $a, b \in \mathcal{G}$.

Nondegeneracy. Assume that $\langle a \,|\, b\rangle = 0_\natural \in A_\natural$ for a fixed $a \in \mathcal{G}$ and all $b \in \mathcal{G}$. To state that a = 0, let us put then b = a and obtain

$$\langle a \,|\, a\rangle = \sum_{i,j=\overline{1,n}} \pi(a_{ij}a_{ij}) = 0_\natural.$$

Taking into account that the associative algebra is generated by the finite set of elements $\{u_j \in A \mid j = \overline{1,m}\}$, it is easy to deduce from n^2 expansions of elements

$$\begin{aligned} a_{ij} := c_{k(i.j)} = &\sum_{|s^{(1)}| \in \mathbb{Z}_+} \sum_{\sigma_1 \in \sigma_n} C^{\left(s_1^{(1)} s_2^{(1)} \ldots s_m^{(1)}\right)}_{(k;\sigma_1(1),\sigma_1(2),\ldots,\sigma_1(m))} \left(u^{s_1^{(1)}}_{\sigma_1(1)} u^{s_2^{(1)}}_{\sigma_1(2)} \ldots u^{s_m^{(1)}}_{\sigma_1(m)}\right) + \\ &+ \sum_{|s^{(1)}|,|s^{(2)}| \in \mathbb{Z}_+} \sum_{\sigma_1,\sigma_2 \in \sigma_n} C^{\left(s_1^{(1)} s_2^{(1)} \ldots s_m^{(1)}; s_1^{(2)} s_2^{(2)} \ldots s_m^{(2)}\right)}_{(k;\sigma_1(1),\sigma_1(2),\ldots,\sigma_1(m);\sigma_2(1),\sigma_2(2),\ldots,\sigma_2(m))} \times \\ &\times \left(u^{s_1^{(1)}}_{\sigma_1(1)} u^{s_2^{(1)}}_{\sigma_1(2)} \ldots u^{s_m^{(1)}}_{\sigma_2(m)}\right) \times \left(u^{s_1^{(2)}}_{\sigma_1(1)} u^{s_2^{(2)}}_{\sigma_2(2)} \ldots u^{s_m^{(2)}}_{\sigma_2(m)}\right) + \ldots \end{aligned}$$

from A that the sum

$$\sum_{k=\overline{1,n}} \pi(c_k c_k) = 0_\natural \tag{9.11}$$

iff $c_k = 0$ for all $k = \overline{1, n^2}$. Indeed, the sum of (9.11) under the π-mapping can be now rewritten, respectively, as

$$\begin{aligned} \sum_{k=\overline{1,n^2}} (c_k c_k) = &\sum_{|s^{(1)}| \in \mathbb{Z}_+} \sum_{\sigma_1 \in \sigma_n} D^{\left(s_1^{(1)} s_2^{(1)} \ldots s_m^{(1)}\right)}_{(\sigma_1(1),\sigma_1(2),\ldots,\sigma_1(m))} \left(u^{s_1^{(1)}}_{\sigma_1(1)} u^{s_2^{(1)}}_{\sigma_1(2)} \ldots u^{s_m^{(1)}}_{\sigma_1(m)}\right) + \\ &+ \sum_{|s^{(1)}|,|s^{(2)}| \in \mathbb{Z}_+} \sum_{\sigma_1,\sigma_2 \in \sigma_n} D^{\left(s_1^{(1)} s_2^{(1)} \ldots s_m^{(1)}; s_1^{(2)} s_2^{(2)} \ldots s_m^{(2)}\right)}_{(\sigma_1(1),\sigma_1(2),\ldots,\sigma_1(m);\sigma_2(1),\sigma_2(2),\ldots,\sigma_2(m))} \times \\ &\times \left(u^{s_1^{(1)}}_{\sigma_1(1)} u^{s_2^{(1)}}_{\sigma_1(2)} \ldots u^{s_m^{(1)}}_{\sigma_2(m)}\right) \times \left(u^{s_1^{(2)}}_{\sigma_1(1)} u^{s_2^{(2)}}_{\sigma_2(2)} \ldots u^{s_m^{(2)}}_{\sigma_2(m)}\right) + \ldots \end{aligned}$$

with some D-coefficients from $\mathbb{K}$ for all $\sigma_j \in S_n$, depending quadratically on coefficients of expansions, staying at uniform and symmetric basis elements of the algebra A. As the π-mapping sends all of them, by definition, to zero, the resulting system (9.11) reduces to the set of algebraic equations

$$D^{\left(s_1^{(1)}s_2^{(1)}...s_m^{(1)}\right)}_{(\sigma_1(1),\sigma_1(2),...,\sigma_1(m))} = 0,$$

$$D^{\left(s_1^{(1)}s_2^{(1)}...s_m^{(1)};s_1^{(2)}s_2^{(2)}...s_m^{(2)}\right)}_{(\sigma_1(1),\sigma_1(2),...,\sigma_1(m);\sigma_2(1),\sigma_2(2),...,\sigma_2(m))} = 0,$$

$$\ldots,$$

reducing successively for all $\sigma_j \in S_n$ to the conditions

$$C^{\left(s_1^{(1)}s_2^{(1)}...s_m^{(1)}\right)}_{(k;\sigma_1(1),\sigma_1(2),...,\sigma_1(m))} = 0,$$

$$C^{\left(s_1^{(1)}s_2^{(1)}...s_m^{(1)};s_1^{(2)}s_2^{(2)}...s_m^{(2)}\right)}_{(k;\sigma_1(1),\sigma_1(2),...,\sigma_1(m);\sigma_2(1),\sigma_2(2),...,\sigma_2(m))} = 0,$$

$$\ldots,$$

being equivalent to the equalities $c_k = 0$ for all $k = \overline{1, n^2}$. □

As a simple consequence from Lemma 9.3 one derives the following proposition.

Proposition 9.4. *The constructed Lie algebra $\mathcal{G}$ is ad-invariant and π-metrized.*

Proof. Really, from the symmetry property (9.10) one easily obtains that

$$\langle \mathrm{a} \,|\, [\mathrm{b}, \mathrm{c}] \rangle = \langle [\mathrm{a}, \mathrm{b}] \,|\, \mathrm{c} \rangle \tag{9.12}$$

modulo π-mapping for any elements a, b and $\mathrm{c} \in \mathcal{G}$. As the bilinear form (9.9) is non-degenerate, one has $\mathcal{G}^* \simeq \mathcal{G}$, that jointly with the ad-invariance property (9.12) means that the Lie algebra $\mathcal{G}$ is metrized. □

Being interested in constructing integrable noncommutative dynamical systems on the algebra A, we need to introduce into our analysis a "spectral" parameter $\lambda \in \mathbb{C}$, responsible for the existence of infinite hierarchies of the corresponding dynamical systems invariants, guaranteeing their integrability. This wil be done in next Section, devoted to the Lie-algebraic analysis on loop-Lie-algebras, related with the Lie algebra $\mathcal{G}$, introduced above.

Consider now the Lie algebra $\{\mathcal{G}, [\cdot,\cdot]\}$, constructed above, and the related loop Lie algebra

$$\left\{\tilde{\mathcal{G}} := \mathcal{G} \otimes \mathbb{C}\{\{\lambda, \lambda^{-1}\}\},\ [\cdot,\cdot]\right\}$$

of the corresponding $\mathcal{G}$-valued Laurent series with respect to the parameter $\lambda \in \mathbb{C}$,

$$\tilde{\mathcal{G}} := \underset{N\in\mathbb{Z}}{\cup}\Big\{\tilde{a} = \sum_{j\leqslant N} a_j\lambda^j \mid a_j \in \mathcal{G}, j = \overline{1,N}\Big\},$$

and define on it the corresponding to (9.9) modulo π-mapping bilinear form $(\cdot\,|\,\cdot) : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to A$:

$$(\tilde{a}\,|\,\tilde{b}) := res_\lambda\langle \tilde{ma}\,|\,\tilde{b}\rangle \tag{9.13}$$

for any elements $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$. It is easy to observe that the bilinear form (9.13) is also symmetric and non-degenerate. Thus, the following proposition holds.

Proposition 9.5. *The loop Lie algebra $\tilde{\mathcal{G}}$ is ad-invariant and π-metrized.*

As the loop Lie algebra $\tilde{\mathcal{G}}$ allows natural direct sum splitting $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_+ \oplus \tilde{\mathcal{G}}_-$ into two Lie subalgebras $\tilde{\mathcal{G}}_+$ and $\tilde{\mathcal{G}}_-$, where

$$\tilde{\mathcal{G}}_+ : \bigcup_{N\in\mathbb{Z}_+}\Big\{\tilde{a} = \sum_{j=\overline{0,N}} a_j\lambda^j \mid a_j \in \mathcal{G}, j = \overline{1,N}\Big\},$$

$$\tilde{\mathcal{G}}_- := \bigcup_{N\in\mathbb{Z}_+}\Big\{\tilde{a} = \sum_{j\in\mathbb{Z}_+} a_j\lambda^{-(j+1)} \mid a_j \in \mathcal{G}, j \in \mathbb{Z}_+\Big\},$$

their adjoint spaces with respect to the bilinear form (9.13)) split the adjoint loop space $\tilde{\mathcal{G}}^* = \tilde{\mathcal{G}}_+^* \oplus \tilde{\mathcal{G}}_-^*$ and satisfy the equivalences

$$\tilde{\mathcal{G}}_+^* \simeq \tilde{\mathcal{G}}_-, \qquad\qquad \tilde{\mathcal{G}}_-^* \simeq \tilde{\mathcal{G}}_+.$$

Let now a linear endomorphism $\mathcal{R} : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ equals $\mathcal{R} = (P_+ - P_-)/2$, where, by definitions, $P_\pm : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}_\pm \subset \tilde{\mathcal{G}}$ are the projections on the corresponding subspaces $\tilde{\mathcal{G}}_\pm \subset \tilde{\mathcal{G}}$. It is a well known property [22, 20, 38, 85] that for any $\tilde{a}, \tilde{b} \in \tilde{\mathcal{G}}$ the deformed Lie product

$$[\tilde{a}, \tilde{b}]_R := [\mathcal{R}\tilde{a}, \tilde{b}] + [\tilde{a}, \mathcal{R}\tilde{b}]$$

satisfies the Jacobi condition and generates on the loop Lie algebra $\tilde{\mathcal{G}}$ a new Lie algebra structure.

Within the classical Adler-Kostant-Symes Lie-algebraic approach, or its $\mathcal{R}$-matrix structure generalization [22, 20, 38, 85], the adjoint loop space $\tilde{\mathcal{G}}^*$ is then endowed with the modified Lie-Poisson structure

$$\{\tilde{l}(\tilde{a}), \tilde{l}(\tilde{b})\}_\natural := \big(\tilde{l}\,|\,[\tilde{a}, \tilde{b}]_\mathcal{R}\big), \tag{9.14}$$

for any basic functionals $\tilde{l}(\tilde{a}), \tilde{l}(\tilde{b}) \in D(\tilde{\mathcal{G}}^*)$ subject to which the whole set

$$I(\tilde{\mathcal{G}}^*) = \big\{\gamma \in D(\tilde{\mathcal{G}}^*) \mid (\tilde{l}\,|\,[\mathrm{grad}\gamma(\tilde{l}), \tilde{a}]) = 0_\natural, \tilde{a} \in \tilde{\mathcal{G}}^*\big\}$$

of smooth Casimir functionals on $\tilde{\mathcal{G}}^*$ is commutative with respect to the deformed Lie-Poisson structure (9.14) on $\tilde{\mathcal{G}}^*$, that is $\{\gamma,\mu\}_\natural = 0_\natural \in A_\natural$ for all $\gamma,\mu \in I(\tilde{\mathcal{G}}^*)$ and, by definition,

$$(\tilde{q}\,|\,\mathrm{grad}\gamma(\tilde{l})) := \frac{d}{d\varepsilon}\gamma(\tilde{l}+\varepsilon\tilde{q})\bigg|_{\varepsilon=0}.$$

The latter makes it possible to construct integrable Hamiltonian flows on the associative algebra A as Poissonian flows on the co-adjoint orbits on the adjoint space $\tilde{\mathcal{G}}^*$, generated by a suitable loop Lie algebra $\tilde{\mathcal{G}}$ of Casimir gradient elements. Namely, if an element $\tilde{l} \in \tilde{\mathcal{G}}^*$ is fixed, the corresponding Hamiltonian flow on $\tilde{\mathcal{G}}^*$ subject to the deformed Poisson bracket (9.14) and a Casimir functional $\gamma \in I(\tilde{\mathcal{G}}^*)$ possesses the well known Lax type [55, 65, 85] representation

$$d\tilde{l}/dt = [P_+\mathrm{grad}\gamma(\tilde{l}),\tilde{l}], \tag{9.15}$$

where $t \in \mathbb{K}$ is a related evolution parameter. The example of this construction and its Lie algebraic properties are discussed in the next Subsection.

9.6. **Kontsevich type integrable systems on unital finitely generated free associative noncommutative algebras.** Let a free unital finitely generated associative non-commutative algebra $A := \mathbb{K}\langle u^\pm, v^\pm\rangle$ be the corresponding group algebra of a group $G\{u,v\}$, generated by two elements $u,v \in G$. The algebra A is infinite dimensional with the countable basis

$$L_A\langle 1,\ u^jv^{s_1-j},\ v^ju^{s_1-j},\ u^jv^{s_2-j}u^{j-k}v^{k-q},\ v^ju^{s_2-j}v^{j-k}u^{k-q},\ldots \mid s_1,s_2,\ldots \in \mathbb{Z}\rangle,$$

the related two-dimensional matrix loop Lie algebra $\tilde{\mathcal{G}} = \mathcal{G}\otimes\mathbb{C}\{\{\lambda,\lambda^{-1}\}\}$, $\mathcal{G} := gl(2;A)$, is metrized subject to the bi-li near product (9.13) and generated by affine elements

$$a = \sum_{j=\overline{0,3}}\sigma_k\sum_{j\ll\infty}a_j^{(k)}\lambda^j$$

with four basis Pauli matrix elements $\sigma_k \in gl(2;\mathbb{K})$, $k=\overline{0,3}$, and algebra components $a_j^{(k)} \in A$, $j \ll \infty$, $k=\overline{0,3}$. The corresponding Casimir functionals $\gamma \in I(\tilde{\mathcal{G}}^*)$ generates a Hamiltonian flow on points $\tilde{l} \in \tilde{\mathcal{G}}^*$ with respect to the Poisson bracket (9.14) in the Lax type form (9.15). To analyze this flow in detail, let us put, by definition, that the seed orbit point $\tilde{l} \in \tilde{\mathcal{G}}^*$ is given by the following λ-squared expression

$$\tilde{l} = \sum_{j=\overline{0,3}}\sum_{k=\overline{0,2}}\sigma^j\lambda^{k-3}u_j^{(k)}, \tag{9.16}$$

where

$$\left\{\sigma^j \in gl^*(2;\mathbb{K}) \mid \operatorname{tr}(\sigma^j \sigma_k) = \delta^j_k, j,k=\overline{0,3}\right\}$$

is the dual basis of the matrix space $gl^*(2;\mathbb{K}) \simeq gl(2;\mathbb{K})$ and elements

$$\{u_j^{(k)} \in A \mid j=\overline{0,3}, k=\overline{0,2}\}$$

are coordinates of some A-algebra valued phase space $M_A^{(0\,|\,2)}$ in a general position. In particular, we will choose the following dual bases:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $gl(2;\mathbb{K})$ and

$$\sigma^0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

in $gl(2;\mathbb{K})^*$. Moreover, we also will assume that A-algebra valued coefficients of the phase space $M_A^{(0\,|\,2)}$ and (9.16) are representable subject to the basis of A as

$\sigma\backslash\lambda$	λ^{-3}	λ^{-2}	λ^{-1}
σ^0	$u_0^{(0)}=1$	$u_0^{(1)} = v+v^{-1}+u+u^{-1}+v^{-1}u^{-1}$	$u_0^{(2)}=0$
σ^1	$u_1^{(0)}=u$	$u_1^{(1)}=v^{-1}$	$u_1^{(2)}=0$
σ^2	$u_2^{(0)}=0$	$u_2^{(1)}=v^{-1}u^{-1}+u^{-1}+1$	$u_2^{(2)}=v$
σ^3	$u_3^{(0)}=-1$	$u_3^{(1)}=-v+v^{-1}+u-u^{-1}-v^{-1}u^{-1}$	$u_3^{(2)}=0$

, (9.17)

following the result obtained in [96].

As a first important task, we will calculate the corresponding Poisson structure on the related A-algebra valued phase space $M_A^{(0\,|\,2)}(\tilde{l})$, generated by coefficients, presented in the expression (9.17). To do this, we need to take into account that the phase space $M_A^{(0\,|\,2)}(\tilde{l})$, being endowed with the $\mathcal{R}$-modified Poisson structure (9.14), is strongly reduced via the Dirac scheme [38, 78] subject to the set

$$\Phi := \Big\{\varphi_1 = u_0^{(0)} - 1 = 0,\ \varphi_2 = u_0^{(2)} = 0,\ \varphi_3 = u_2^{(0)} = 0,$$
$$\varphi_4 = u_3^{(2)} = 0,\ \varphi_5 = u_3^{(0)} + 1 = 0\Big\}$$

of algebraic constraints, imposed on the phase space $M_A^{(0\,|\,2)}$. The latter means that the true Poisson structure on the reduced phase space $M_A^{(0\,|\,2)}(\tilde{l}) := M_A^{(0\,|\,2)}/\Phi$ coincides with the corresponding Dirac type reduction of the $\mathcal{R}$-modified Poisson structure, defined on the full phase space

$M_A^{(0\,|\,2)}$. As a result of simple enough yet cumbersome calculations we arrive at the following Poisson brackets

$$\{u,v\}_\natural = -uv, \qquad \{u,u\}_\natural = 0_\natural = \{v,v\}_\natural$$

on the reduced phase space $M_A^{(0\,|\,2)}(\tilde{l}) \simeq A := \mathbb{K}\langle u^{\pm}, v^{\pm}\rangle$.

Having taken as a Hamiltonian operator $h := res\lambda^2 \mathrm{tr}(\tilde{l}^2) \in I(\tilde{\mathcal{G}}^*)$, one easily obtains the following [51] nonlinear integrable Kontsevich dynamical system

$$\left.\begin{array}{l} du/dt := \{h,u\}_\natural = uv - uv^{-1} - v^{-1} \\ dv/dt := \{h,v\}_\natural = -vu + vu^{-1} + u^{-1} \end{array}\right\} := K(u,v) \tag{9.18}$$

on the reduced phase space $A = \mathbb{K}\langle u^{\pm}, v^{\pm}\rangle$. Moreover, owing to the Lax type representation (9.14), the Kontsevich dynamical system (9.18) proves to be equivalent to the following matrix commutator equation

$$d\tilde{l}/dt = [\tilde{l},\, p(\tilde{l})]$$

for any $\lambda \in \mathbb{K}$ in the Lie algebra $\tilde{\mathcal{G}}$, where the A-valued matrix

$$p(\tilde{l}) = P_+ \mathrm{grad} h(\tilde{l})/2 = \sigma_0(v^{-1} - v + u + 1)/2 + \\ + \sigma_1 \lambda v + \sigma_2 v^{-1} + \sigma_3(v^{-1} - v + u - 1)/2 \in \tilde{\mathcal{G}}.$$

Taking as Hamiltonian functions the algebraic expressions

$$h^{(m,n)} := res\lambda^m \mathrm{tr}(\tilde{l}^n) \in I(\tilde{\mathcal{G}}^*), \qquad m,n \in \mathbb{Z},$$

one can obtain a complete set of π-commuting to each other conservation laws of the Kontsevich dynamical system (9.18), thus proving its generalized integrability. Moreover, choosing both another group algebra and orbit elements $\tilde{l} \in \tilde{\mathcal{G}}^*$, one can construct the same way many other integrable Hamiltonian systems on the associative noncommutative phase space A, that is planned to be a topic of a next investigation.

10. Conclusion

In this work we succeeded in formal tensor and differential-algebraic reformulating the criteria [43, 90, 64] for a given differential expression to be Hamiltonian and developed an effective approach to classification of the algebraic Poisson structures lying in the background of the integrable multicomponent Hamiltonian systems. We have devised a simple algorithm allowing to construct new algebraic structures within which the corresponding Hamiltonian operators exist and generate integrable multicomponent dynamical systems. We also showed, as examples, that the well known Balinsky-Novikov algebraic structure, obtained before in [43, 11] as a condition for a matrix differential expression to be Hamiltonian, appears within the devised approach as a derivation on the adjacent Lie algebra, naturally associated with a suitably constructed differential loop algebra. By means of a direct generalization of this example it is obtained new Lie algebraic relationships, whose background algebraic structures coincide, respectively, with the right Leibniz algebra, introduced in [23, 24, 59] and with a new Riemann type nonassociative algebra. The constructed Hamiltonian operators describe a wide class of multi-component hierarchies [21, 80] of integrable multicomponent hydrodynamic Riemann type systems. Their reductions appeared to be closely related both to the integrable Camassa-Holm and with the Degasperis-Processi dynamical systems, and are of special interest from the equivalence transformation point of view, devised recently in [95].

Taking into account that the compatible Hamiltonian operators, important for studying integrable multicomponent Hamiltonian systems on functional manifolds, are constructed by means of suitable central extentions of the adjacent weak Lie algebras, determined by the right Leibniz and Riemann type nonassociative and noncommutative algebras, the problem of their description requires a detailed investigation both of their structural properties and finite-dimensional representations of the right Leibniz algebras defined by the corresponding structural constraints. Subject to these important aspects we stopped in the work mostly on the structural properties of the right Leibniz algebras, especially on their derivation algebras and their generalizations. We added also a Supplement in which we revisited the classical Poisson manifolds approach to Hamiltonian operators on functional noncommutative manifolds, as well as presented it simple and natural realization, generated by associative noncommutative group algebra. The latter appeared to be very useful for describing a wide class of new Lax type integrable nonlinear Hamiltonian systems on associative noncommutative algebras, interesting for diverse applications in modern quantum physics.

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Dynamics and exact solutions of the generalized Harry Dym equation

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Abstract. The Harry Dym equation is the third-order evolutionary partial differential equation. It describes a system in which dispersion and nonlinearity are coupled together. It is a completely integrable nonlinear evolution equation that may be solved by means of the inverse scattering transform. It has an infinite number of conservation laws and does not have the Painleve property. The Harry Dym equation has strong links to the Korteweg-de Vries equation and it also has many properties of soliton solutions. A connection was established between this equation and the hierarchies of the Kadomtsev-Petviashvili equation. The Harry Dym equation has applications in acoustics: with its help, finite-gap densities of the acoustic operator are constructed. The paper considers a generalization of the Harry Dym equation, for the study of which the methods of the theory of finite-dimensional dynamics are applied. The theory of finite-dimensional dynamics is a natural development of the theory of dynamical systems. Dynamics make it possible to find families that depends on a finite number of parameters among all solutions of evolutionary differential equations. In our case, this approach allows us to obtain some classes of exact solutions of the generalized equation, and also indicates a method for numerically constructing solutions.

Анотація. Рівняння Гаррі Діма є еволюційним рівняння в частинних похідних третього порядку і описує системи з нелінійною дисперсією. Це цілком інтегровне нелінійне рівняння, яке може бути розв'язане за допомогою зворотнього перетворення розсіювання. Воно має нескінченну кількість законів збереження і не володіє властивостю Пенлеве. Рівняння Гаррі Діма тісно пов'язане з рівнянням Кортевега-де Вріса та має багато властивостей рівнянь з солітонними рішеннями. Раніше було встановлено зв'язок між даним рівнянням та ієрархіями рівняння Кадомцева-Петвіашвілі. Воно також має застосування в акустиці: з його допомогою будуються скінченно-вимірні щільності акустичного оператора.

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В даній роботі розглядається узагальнення рівняння Гаррі Діма, для вивчення якого застосовуються методи теорії скінченновимірної динаміки еволюційних рівнянь в частинних похідних, що є природнім розвитком теорії динамічних систем. Динаміка дає змогу знайти сім'ї розв'язків рівнянь в частинних похідних, які залежать від скінченного числа параметрів. У нашому випадку такий підхід дозволяє отримати деякі класи точних розв'язків узагальненого рівняння Гаррі Діма, а також вказує на можливість побудови числових методів для побудови їх розв'язків.

1. Introduction

The Harry Dym equation has the following form:

$$u_t = u^3 u_{xxx}. \tag{1.1}$$

Apparently the first time this equation was given in paper [4] by M. Kruskal, which referred to an unpublished work by Harry Dym. This equation describes nonlinear dispersion processes and is closely related to the Korteweg-de Vries equation and the Kadomtsev-Petviashvili equation. The Harry Dym equation is used in acoustics to construct finite-gap densities of an acoustic operator [8]. It is a completely integrable nonlinear evolution equation, which can be solved using the inverse scattering problem.

This article is devoted to a generalized Harry Dym equation (GHD) of the form

$$u_t = f(u) u_{xxx} \tag{1.2}$$

for some function f. We suppose that the function f belongs to a class C^∞ in its domain of definition.

For such equations, first and second order dynamic will be constructed, which will then be used to construct their exact solutions.

The theory of finite-dimensional dynamics is a natural extension of the theory of dynamical systems to evolutionary partial differential equations. It describes a system in which dispersion and nonlinearity are coupled together.

A detailed description of this theory is presented in [3, 7]. Here we give only the necessary definitions, ideas and results.

A method for constructing attractors for second-order evolutionary differential equations was proposed in [1], on the basis of which an algorithm for the numerical solution of such equations was developed in [6].

2. Finite-dimensional dynamics

Consider an ordinary differential equation of $(k+1)$-th order

$$y^{(k+1)} = h\left(x, y, y', y'', \ldots, y^{(k)}\right). \tag{2.1}$$

This equation generates a one-dimensional distribution P on the jet space $J^k(\mathbb{R})$ such that its integral curves are prolongations of the solutions graphs into the space $J^k(\mathbb{R})$. The distribution P is generated by the vector field

$$\mathcal{D} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \cdots + y_k \frac{\partial}{\partial y_{k-1}} + h \frac{\partial}{\partial y_k},$$

where $x, y_0, y_1, \ldots, y_k$ are coordinates on $J^k(\mathbb{R})$.

A vector field X on $J^k(\mathbb{R})$ is called an *infinitesimal symmetry* of equation (2.1) if translations along X save P.

Infinitesimal symmetries form Lie algebra $\mathsf{Symm}\,\mathsf{P}$ with respect to the Lie bracket. An infinitesimal symmetry is called *characteristic* if translations along it save each integral curve of the distribution P. Characteristic symmetries form an ideal in $\mathsf{Symm}\,\mathsf{P}$ which we denote by $\mathsf{Char}\,\mathsf{P}$.

The quotient Lie algebra $\mathsf{Shuff}\,\mathsf{P} := \mathsf{Symm}\,\mathsf{P}/\mathsf{Char}\,\mathsf{P}$ is called the Lie algebra of *shuffling* symmetries.

Each shuffling symmetry can be identified with a vector field of the form

$$S_\phi = \phi \frac{\partial}{\partial y_0} + \mathcal{D}(\phi) \frac{\partial}{\partial y_1} + \mathcal{D}^2(\phi) \frac{\partial}{\partial y_2} + \cdots + \mathcal{D}^k(\phi) \frac{\partial}{\partial y_k},$$

where ϕ is a function on $J^k(\mathbb{R})$ that is called a *generating function* of the corresponding shuffling symmetry.

Let $y = y(x)$ be a solution of equation (2.1), ϕ a generating function of a shuffling symmetry, and Φ_t the translation along the vector field S_ϕ. Then the function $u(t,x) = \left(\Phi_t^{-1}\right)^* (y(x))$ is a solution of the evolutionary partial differential equation

$$\frac{\partial u}{\partial t} = \phi\left(x, u, u_1, u_2, \ldots, u_k\right) \tag{2.2}$$

with the initial data $u(0,x) = y(x)$, where $u_j = \dfrac{\partial^j u}{\partial x^j}$.

Equation (2.1) is called a *finite-dimensional dynamics* of equation (2.2). The number $k+1$ is called an *order* of the dynamics.

The following theorem (see [1]) provides a method for calculating finite-dimensional dynamics of evolutionary equations.

Theorem 2.1. *The ordinary differential equation*

$$F = y_{k+1} - h(x, y_0, y_1, \ldots, y_k) = 0$$

is a dynamics of evolutionary equation (2.2) *if and only if*

$$[\phi, F] = 0 \bmod \mathbf{DF}, \tag{2.3}$$

where $\mathbf{DF} = \langle F, D(F), D^2(F), \dots \rangle$ *is the generated by the function* F *differential ideal,*

$$D = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + y_2 \frac{\partial}{\partial y_1} + \cdots \tag{2.4}$$

is the operator of total derivative, and

$$[\phi, F] = \sum_{i \geqslant 0} \left(\frac{\partial \phi}{\partial y_i} D^i(F) - \frac{\partial F}{\partial y_i} D^i(\phi) \right)$$

is a prolongation of the classical Poisson-Lie bracket into the jet space (see, for example, [5]*).*

3. First order dynamics

Find first order dynamics of GHD equation (1.2) in the following form:

$$F = y_1 + A(y_0), \tag{3.1}$$

where A is a function. Then the Poisson-Lie bracket is

$$[\phi, F] = f(y_0)A'''(y_0)y_1^3 + 3f(y_0)A''(y_0)y_1y_2 + f'(y_0)y_3A(y_0).$$

Applying the operator of total differentiation by x to the equation $F = 0$, we obtain the following expressions of derivatives of the second and third order:

$$\begin{aligned} y_2 &= -\, A'(y_0)y_1, \\ y_3 &= -\, A''(y_0)y_1^2 - A'(y_0)y_2. \end{aligned}$$

Then equation (2.3) has the form

$$A^2 \left(f'AA'' + f'A'^2 + AfA''' + 3fA'A'' \right) = 0. \tag{3.2}$$

This equation has the trivial solution $A(y_0) = 0$ and nontrivial one

$$\begin{aligned} A(y_0) &= \pm\sqrt{-C_1 \left(\int \frac{y_0 dy_0}{f(y_0)} - y_0 \int \frac{dy_0}{f(y_0)} \right) C_2 y_0 + C_3} \\ &= \pm\sqrt{C_1 \int\int \frac{1}{f(y_0)} dy_0 dy_0 + C_2 y_0 + C_3}. \end{aligned}$$

If $A(y_0) \neq 0$ and $C_1 \neq 0$ for some constants a, b then equation (3.2) can be solved with respect to the function f:

$$f(y_0) = \frac{C}{(A^2(y_0))''},$$

where C is arbitrary constant.

The restriction of the function ϕ to equation $F = 0$ is

$$\overline{\phi} = -f(y_0)\left(A^2(y_0)A''(y_0) - A(y_0)\left(A'(y_0)\right)^2\right)$$

Example 3.1 (Classical Harry Dym Equation)**.** The function $f(u) = u^3$ corresponds to the classical Harry Dym Equation. In this case

$$A(y_0) = \pm\sqrt{\frac{C_1 + 2C_2y_0^2 + 2C_3y_0}{2y_0}}.$$

To simplify the calculations, we put $C_1 = 2, C_2 = C_3 = 0$ and suppose that $A(y_0) = \frac{1}{\sqrt{y_0}}$. Then the vector field

$$\overline{S} = \frac{1}{\sqrt{y_0}}\frac{\partial}{\partial y_0}.$$

Translations along this vector field from $t = 0$ to t and the inverse transformation are

$$\Phi_t : (x, y_0) \longmapsto \left(x, \frac{1}{4}\left(8y_0^{3/2} - 12t\right)^{2/3}\right),$$
$$\Phi_t^{-1} : (x, y_0) \longmapsto \left(x, \left(\frac{3}{2}t + y_0^{3/2}\right)^{2/3}\right),$$

and the general solution of the equation

$$y' + \frac{1}{\sqrt{y_0}} = 0$$

is

$$y(x) = \frac{1}{4}\left(\alpha - 12x\right)^{2/3}, \tag{3.3}$$

where α is arbitrary constant. Its graph is shown in Figure 3.1.

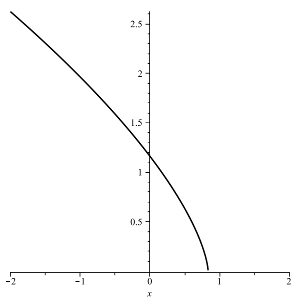

FIGURE 3.1. Graph of solution (3.3) with $\alpha = 10$

Applying the transformation Φ_t^{-1} to this solution, we obtain the following solution of the classical Harry Dym equation:

$$u(t,x) = \frac{1}{4}\left(8\alpha - 12(x+t)\right)^{2/3}. \qquad (3.4)$$

Remark 3.2. Constructed solution (3.4) is similar to the solution obtained by an auto-Backlund transformation [2]:

$$u(t,x) = \left(-3\alpha(x+4\alpha^2 t)\right)^{2/3}.$$

Example 3.3. Let $f(u) = u^{1/3}$ then

$$A(y_0) = \pm\sqrt{C_1 y_0^{5/3} + C_2 y_0 + C_3}.$$

To simplify the calculations, we put $C_1 = 25, C_2 = C_3 = 0$ and suppose that $A(y_0) = y_0^{5/6}$. Then the restriction of the function ϕ to the equation $F = 0$ is $\overline{\phi} = -\frac{5}{9} y_0^{5/6}$ and the corresponding evolutionary vector field is

$$\overline{S} = -\frac{5}{9} y_0^{5/6} \frac{\partial}{\partial y_0}.$$

Then

$$\Phi_t : (x, y_0) \longmapsto \left(x,\ \frac{15625}{24794911296}\left(t - \frac{54}{5} y_0^{1/3}\right)^6\right),$$

$$\Phi_t^{-1} : (x, y_0) \longmapsto \left(x,\ \frac{15625}{24794911296}\left(t + \frac{54}{5} y_0^{1/3}\right)^6\right).$$

The general solution of the equation $y' + y_0^{5/6} = 0$ is

$$y(x) = \frac{1}{46656}(x+\alpha)^6,$$

where α is arbitrary constant. Applying the transformation Φ_t^{-1} to this solution, we obtain the following solution of GHD equation:

$$u_t = u^{1/3} u_{xxx}$$

(see Figure 2.1):

$$u(t,x) = \frac{1}{24794911296}\left(9(x+\alpha) + 5t\right)^6. \qquad (3.5)$$

Example 3.4. Let $f(u) = e^u$ then

$$A(y_0) = \pm\sqrt{C_1 y_0 + C_2 e^{y_0} + C_3}.$$

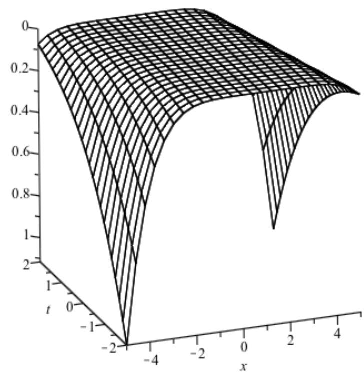

FIGURE 3.2. Graph of solution (3.5)

In order to simplify the calculations, put $C_1 = 0, C_2 = -1, C_3 = 0$. Then $A(y_0) = e^{-\frac{y_0}{2}}$, whence $\overline{\phi} = -\frac{1}{2}e^{-y_0/2}$, and

$$\Phi_t : (x, y_0) \longmapsto \left(x, -\ln \frac{16}{(te^{-\frac{y_0}{2}} - 4)^2} + y_0\right).$$

The general solution of the equation $y' + e^{-y/2} = 0$ is

$$y(x) = -\ln \frac{4}{(x+\alpha)^2},$$

where α is arbitrary constant. Applying the transformation Φ_t^{-1} to this solution, we obtain the following solution of the GHD equation $u_t = e^u u_{xxx}$:

$$u(t, x) = -2\ln \frac{4}{(2(x+\alpha)+t)^2}. \tag{3.6}$$

(see Figure (2.2)).

4. Second order dynamics

In this section we will describe the second order dynamics of the equation (1.2) in the following form:

$$F = y_2 + A(y_0)y_1 + B(y_0), \tag{4.1}$$

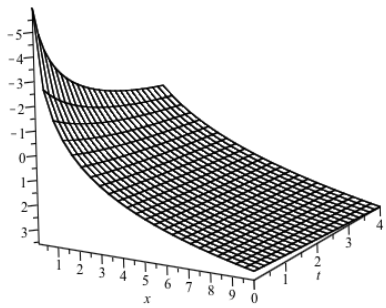

FIGURE 3.3. Graph of solution (3.6)

where A, B are some functions. Notice that the equation 2.3 can be written as the following overdetermined system:

$$\begin{cases} B^2(3fA' + Af') = 0, \\ 3B\left(-fB'' + (A^2 - B')f' + 3fAA'\right) = 0, \\ fA''' + 2f'A'' + A'f'' = 0, \\ -6fAA'' + fB''' + 2f'B'' - 8AA'f' + (B' - A^2)f'' - 3f(A')^2 = 0, \\ -6fBA'' - 3fAB'' - ABf'' + (-7f'B + (6(A^2 - \frac{1}{2}B'))f)A' + \\ \hfill +2Af'(A^2 - 2B') = 0. \end{cases}$$

Solving this system for nontrivial f (i.e. $f \neq 0$) we get three solutions:

(1) $A = B = 0$, f is arbitrary;

(2) $A = 0$, $B = C_1 + C_2 \int \dfrac{dy_0}{f(y_0)}$, f is arbitrary;

(3) $A = \pm \dfrac{1}{\sqrt{C_1 y_0 + C_2}}$, $B = C_3 + C_4 \int A^3 dy_0$, $f = \dfrac{C_5}{A^3}$,

where $C_1, \ldots, C_5$ are arbitrary constants.

Consider these cases sequentially.

Case 1. Equation (4.1) has the form $y'' = 0$. But since $\overline{\phi} = 0$, the vector field $\overline{S} = 0$ vanishes and we cannot construct solutions of equation (1.2).

Case 2. Equation (4.1) has the form

$$y'' + C_1 + C_2 \int \frac{dy_0}{f(y_0)} = 0$$

and $\overline{\phi} = -C_2 y_1$. Therefore

$$\overline{S} = -C_2 y_1 \frac{\partial}{\partial y_0} - C_2 \left(C_1 + C_2 \int \frac{dy_0}{f(y_0)} \right) \frac{\partial}{\partial y_1}.$$

Case 3. Equation (4.1) has the form

$$y'' + \frac{y_1}{\sqrt{C_4 y_0 + C_2}} - \frac{2C_2}{C_4 \sqrt{C_4 y_0 + C_2}} + C_1.$$

For simplicity, put $C_1 = a, C_2 = b$, $C_3 = C_4 = 0$, $C_5 = c$ and suppose that the function A is positive, i.e.

$$A(y_0) = \frac{1}{\sqrt{a y_0 + b}}, \quad B(y_0) = 0, \quad f(y_0) = c(a y_0 + b)^{3/2}.$$

Then the equation (4.1) will have the form

$$y'' + \frac{1}{\sqrt{ay + b}} = 0$$

and its solution can be found in implicit form:

$$\begin{aligned} &-4C_1\sqrt{ay+b} - 2\operatorname{arctanh}(2C_1\sqrt{ay+b}), \\ &\pm \ln(4C_1^2 ay + 4bC_1^2 - 1) \mp 2\ln C_1 + 4C_1 x + C_2 = 0, \end{aligned} \tag{4.2}$$

where C_1, C_2 are constants. Note that the function y is 2-valued. Since the vector field (2.4) has the form

$$\mathcal{D} = \frac{\partial}{\partial x} + y_1 \frac{\partial}{\partial y_0} + \frac{y_1}{\sqrt{a y_0 + b}} \frac{\partial}{\partial y_1},$$

the restriction of the function ϕ to the equation $F = 0$ is

$$\overline{\phi} = \frac{c}{2} y_1 (2\sqrt{a y_0 + b} + a y_1).$$

and the corresponding evolutionary vector field is

$$\overline{S} = \frac{c y_1}{2} (2\sqrt{a y_0 + b} + a y_1) \frac{\partial}{\partial y_0} + \frac{c y_1 \left(3 a y_1 + 2\sqrt{a y_0 + b}\right)}{2\sqrt{a y_0 + b}} \frac{\partial}{\partial y_1}.$$

This vector field generates the shift transformation Φ_t. Applying the transformation Φ_t^{-1} to the expression, we get an implicit representation of the solution to GHD equation

$$u_t = c(au + b)^{3/2} u_{xxx}.$$

Unfortunately, shift transforms cannot be found explicitly, but one can use the method of numerical integration.

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(In)homogeneous invariant compact convex sets of probability measures

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Abstract. It is proved that for any iterated function system of contractions on a complete metric space there exists an invariant compact convex sets of probability measures of compact support on this space. A similar result is proved for the inhomogeneous compact convex sets of probability measures of compact support.

Анотація. Математичні підвалини теорії фракталів запропонував Дж. Гатчінсон у 80-х роках минулого століття. Зокрема, він означив поняття атрактора (або інваріантного об'єкта) для ітерованої системи стискуючих відображень (скорочено IFS) на повному метричному просторі і довів існування таких атракторів у гіперпросторі (просторові непорожніх компактних підмножин) та просторі ймовірнісних мір з компактними носіями на повному метричному просторі. Доведення Гатчінсона використовують принцип стискуючих відображень і, зокрема, потребують відповідної метризації простору ймовірнісних мір.

Незабаром аналогічні результати було отримано і для неоднорідних атракторів (тобто атракторів з приєднаними ущільнюючими множинами), які є природними узагальненнями інваріантних множин та інваріантних мір.

У цій статті ми запроваджуємо поняття інваріантного об'єкта для IFS у просторі компактних опуклих множин ймовірнісних мір з компактними носіями у повному метричному просторі. Такі компактні опуклі множини мір мають численні застосування у теорії очікуваної корисності. На відміну від гіперпростору компактних опуклих підмножин повного метричного простору, інваріантні об'єкти в якому виглядають регулярними, атрактори IFS у просторі компактних опуклих множин мір зберігають іррегулярність, притаманну фрактальним множинам.

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Одним з основних результатів є теорема існування та єдиності інваріантної компактної опуклої множини ймовірнісних мір з компактними носіями у повному метричному просторі. Окрім традиційного підходу до доведення такого типу результатів, що використовує принцип стискуючих відображень, ми пропонуємо також і функціональний підхід, який не опирається на метризацію, а натомість використовує функціональне зображення компактних опуклих підмножин у просторах ймовірнісних мір.

Аналогічні результати отримано і для випадку неоднорідних інваріантних опуклих множин ймовірнісних мір.

1. Introduction

Hutchinson [7] proved the existence of invariant sets and invariant probability measures for the iterated function systems in the complete metric spaces. The structure of these two proofs is similar and it exploits, in particular, the functoriality of the constructions involved (i.e., the hyperspaces and spaces of probability measures) as well as existence of special metrizations. This led to several generalizations of the existence results, in particular, to the cases of inclusion hyperspaces (i.e., two-valued measures) [11] and idempotent measures on ultrametric spaces [9].

Another approach is applied in [10] and it is proved therein that there exists an invariant idempotent measure (see [18] for topological aspects of the theory of idempotent measures) for an iterated function system on a complete metric space.

Recently, a related notion of inhomogeneous invariant set and measure was introduced in [15]. The properties of these sets and measures were studied in various publications (see, e.g., [5, 1, 13]).

The compact convex sets of probability measures are used in the maxmin expected utility (MEU) theory [6].

2. Preliminaries

2.1. **Hyperspaces.** Let $\exp X$ denote the set of all nonempty compact subsets of a Tychonov space X. A base of the Vietoris topology on $\exp X$ consists of the sets of the form

$$\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \varnothing \text{ for all } i\},$$

where $n \in \mathbb{N}$ and $U_1, \dots, U_n$ are open sets in X. The obtained space is called the hyperspace of X.

Actually, exp is a functor in the category of Tychonov spaces and continuous maps. Given a map $f\colon X \to Y$, the map $\exp f\colon \exp X \to \exp Y$ acts as follows: $\exp f(A) = f(A)$, $A \in \exp X$.

If (X, d) is a metric space, then the Vietoris topology on $\exp X$ is induced by the Hausdorff metric d_H,

$$d_H(A, B) = \inf\{r > 0 \mid A \subset O_r(B),\ B \subset O_r(A)\},$$

where $O_r(C)$ stands for the open r-neighborhood of a subset C.

By $u_X\colon \exp\exp X = \exp^2 X \to \exp X$ we denote the union map. This map is known to be well defined and, in the case of metric space, nonexpanding.

2.2. **Kantorovich metric.** By $P(X)$ we denote the space of probability measures on a compact Hausdorff space X. We regard the set of probability measures on X also as a set of normed linear functionals on the Banach space $C(X)$ of continuous real-valued functions on X. Given $\mu \in P(X)$, we let $\mu(\varphi) = \int_X \varphi d\mu$, $\varphi \in C(X)$.

The set $P(X)$ is endowed with the weak* topology. A base of this topology is comprised by the sets of the form

$$O\langle\mu_0; \varphi_1, \dots, \varphi_n; \varepsilon\rangle = \{\mu \in P(X) \mid |\mu(\varphi_i) - \mu_0(\varphi_i)| < \varepsilon,\ i = 1, \dots, n\},$$

where $\mu_0 \in P(X)$, $\varphi_1, \dots, \varphi_n \in C(X)$, $\varepsilon > 0$.

Let (X, d) be a compact metric space. By 1-LIP(X) we denote the set of all nonexpanding functions on X, i.e. functions $\varphi\colon X \to \mathbb{R}$ satisfying

$$|\varphi(x) - \varphi(y)| \leqslant d(x, y)$$

for all $x, y \in X$. The Kantorovich metric d_K on the space of probability measures $P(X)$ is defined as follows:

$$d_K(\mu, \nu) = \sup\{|\mu(\varphi) - \nu(\varphi)| \mid \varphi \in \text{1-LIP}(X)\}.$$

Every continuous map $f\colon X \to Y$ between compact spaces induces the map

$$P(f)\colon P(X) \to P(Y)$$

defined by $P(f)(\mu)(A) = \mu(f^{-1}(A))$ for any $\mu \in P(X)$ and any measurable subset $A \subset Y$. In terms of functionals, $P(f)(\mu)(\varphi) = \mu(\varphi f)$ for all $\mu \in P(X)$ and $\varphi \in C(Y)$.

Actually, P is a functor in the category **Comp** of compact Hausdorff spaces.

There is a procedure of extensions of functors from the category **Comp** to the category of Tychonov spaces [4]. In the case of the functor P, this procedure gives the space of probability measures of compact support. Recall that the *support* of $\mu \in P(X)$ is the minimal closed set $A \subset X$ such that $\mu(X \backslash A) = 0$. Alternatively, the support of μ is the minimal closed set $A \subset X$ with the property that, for all $\varphi, \psi \in C(X)$, $\varphi|_A = \psi|_A$ implies $\mu(\varphi) = \mu(\psi)$.

2.3. **Convex sets of probability measures.** Let X be a compact Hausdorff space. Denote by $\mathrm{ccP}(X)$ the hyperspace of closed convex subsets of the space $P(X)$. Given a continuous map $f\colon X \to Y$ between compact spaces, we define the map $\mathrm{ccP}(f) : \mathrm{ccP}(X) \to \mathrm{ccP}(Y)$ as follows:

$$\mathrm{ccP}(f)(A) = \{P(f)(\mu) \mid \mu \in A\}, \quad A \in \mathrm{ccP}(X).$$

It is known that ccP is a functor on the category **Comp** (see, e.g. [16]). Given $A \in \mathrm{ccP}(X)$, we say that the set $\overline{\cup\{\mathrm{supp}(\mu) \mid \mu \in A\}}$ is the *support of* A (denoted $\mathrm{supp}(A)$). (Hereafter, for any set Y in a topological space, we denote by $\overline{Y}$ its closure). Again, applying construction from [4] we extend the functor ccP onto the category of Tychonov spaces. We preserve the notation ccP for this extension.

For any metrizable space X, the space $\mathrm{ccP}(X)$ is exactly the hyperspace of closed convex subsets A of $P(X)$ such that $\mathrm{supp}(A)$ is compact.

Now, assume that X is compact and define a map

$$\theta_X\colon \mathrm{ccP}^2(X) \to \mathrm{ccP}(X),$$

as follows, see [12]. First, for any compact convex subset K of a locally convex space, denote by $b_K\colon P(K) \to K$ the barycenter map. Since $P(X)$ is a subset of the dual space $C(X)'$ endowed with the weak* topology, the hyperspace $\mathrm{ccP}(X)$ can be regarded as a compact convex subset of a locally convex space [14] and therefore one can consider the barycenter map

$$b_{\mathrm{ccP}(X)}\colon P(\mathrm{ccP}(X)) \to \mathrm{ccP}(X).$$

Finally, define θ_X by the formula

$$\theta_X(\mathfrak{A}) = \bigcup_{M\in\mathfrak{A}} b_{\mathrm{ccP}(X)}(M), \quad \mathfrak{A} \in \mathrm{ccP}^2(X).$$

Note that the continuity of θ_X is a consequence of the continuity of the barycenter map [3, Chapt. III, §3, Corollary of Proposition 9] and the union map [17, Proposition 5.2].

In the case when $\mathfrak{B}$ is a compact convex subset of the convex hull of a set $\{M_1, \ldots, M_n\}$, where $M_1, \ldots, M_n \in P(\mathrm{ccP}(X))$, we have

$$\theta_X(\mathfrak{B}) = \left\{ \sum_{i=1}^{n} \alpha_i b_{\mathrm{ccP}(X)}(M_i) \mid \alpha_1, \ldots, \alpha_n \geqslant 0,\ \sum_{i=1}^{n} \alpha_i = 1,\ \sum_{i=1}^{n} \alpha_i M_i \in \mathfrak{B} \right\}.$$

Now, let (X, d) be a metric space. We endow $\mathrm{ccP}(X)$ with the Hausdorff metric induced by the Kantorovich metric on $P(X)$. By [8, Proposition 3.2], the map $\theta_X\colon \mathrm{ccP}^2(X) \to \mathrm{ccP}(X)$ is nonexpanding.

Let $c > 0$. A map $f\colon X \to Y$ from a metric space (X, d) to a metric space (Y, ϱ) is called *c-Lipschitz* if $\varrho(f(x), f(y)) \leqslant c \cdot d(x, y)$ for all $x, y \in X$. As mentioned above, the 1-Lipschitz maps are also called *nonexpanding.*

Proposition 2.4. *Let $f\colon X \to Y$ be a c-Lipschitz map. Then* $\mathrm{ccP}(f)$ *is also a c-Lipschitz map.*

Proof. The proof is a consequence of known results on the estimations of constants for the maps of hyperspaces [7, 2.4 (i)] and of spaces of probability measures [7, Theorem 4.4 (1)(i)]. □

3. Results

Let (X, d) be a complete metric space and $\{f_1, f_2, \dots, f_n\}$ be a finite family of contractions on X (that is, an iterated function system, IFS). Let us consider the discrete topology on the set $\{1, 2, \dots, n\}$. Then the space $P(\{1, 2, \dots, n\})$ can be regarded as the standard $(n-1)$-dimensional simplex Δ^{n-1} in $\mathbb{R}^n$,

$$\Delta^{n-1} = \Big\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0,\ \sum_{i=1}^{n} x_i = 1\Big\}$$

by identifying $\sum\limits_{i=1}^{n} \alpha_i \delta_i \in P(\{1, 2, \dots, n\})$ with $(\alpha_1, \dots, \alpha_n) \in \Delta^{n-1}$.

For $B \in \mathrm{ccP}(\{1, 2, \dots, n\})$ define the map $\Phi_B\colon \mathrm{ccP}(X) \to \mathrm{ccP}(X)$ as follows. Let $A \in \mathrm{ccP}(X)$ and $g_A\colon \{1, 2, \dots, n\} \to \mathrm{ccP}(X)$ be the map sending i to $\mathrm{ccP}(f_i)(A)$. Then we set

$$\Phi_B(A) = \theta_X(\mathrm{ccP}(g_A)(B)).$$

We say that $A \in \mathrm{ccP}(X)$ is an *invariant set of probability measures* for $\{f_1, f_2, \dots, f_n\}$ and B whenever $A = \Phi_B(A)$.

Theorem 3.1. *For any* IFS $\{f_1, f_2, \dots, f_n\}$ *and* $B \in \mathrm{ccP}(\{1, 2, \dots, n\})$ *there exists a unique invariant closed convex set of probability measures.*

Proof. We first consider the case of compact space X. Note that the map Φ_B is a contraction. This follows from the fact that the functor ccP preserves c-maps and the map θ_X is nonexpanding. By the Banach Contraction Principle, there exists a unique $A \in \mathrm{ccP}(X)$ such that $A = \Phi_B(A)$.

In the case of noncompact space X, consider the map $\Psi\colon \exp X \to \exp X$ defined as follows: $\Psi(D) = \bigcup\limits_{i=1}^{n} f_i(D)$. It follows from [7, 3.1 (3)(viii)] that the set $\overline{\bigcup\limits_{i=1}^{\infty} \Psi^i(D)}$ is compact for any $D \in \exp X$.

Now, consider an arbitrary $C \in \mathrm{ccP}(X)$ and let $K = \mathrm{supp}(C)$. Then the set $Y = \overline{\bigcup\limits_{i=1}^{\infty} \Psi^i(K)}$ is compact. Note that $f_i(Y) \subset Y$, $i = 1, \dots, n$. Since

$$C \in \mathrm{ccP}(Y) \subset \mathrm{ccP}(X),$$

the above arguments show that there exists an invariant closed convex set of probability measures $A_0 \in \mathrm{ccP}(Y) \subset \mathrm{ccP}(X)$. □

Suppose that we are given an IFS $\{f_1, f_2, \dots, f_n\}$ on X, B is an element of $\mathrm{ccP}(\{0, 1, \dots, n\})$, and $C \in \mathrm{ccP}(X)$. For any $A \in \mathrm{ccP}(X)$ let

$$g'_{A,C} \colon \{0, 1, 2, \dots, n\} \to \mathrm{ccP}(X)$$

be defined by the formulas:

$$g'_{A,C}(0) = C, \qquad g'_{A,C}(i) = \mathrm{ccP}(f_i)(A), \ (i = 1, \dots, n).$$

Define $\Phi'_{B,C} \colon \mathrm{ccP}(X) \to \mathrm{ccP}(X)$ by

$$\Phi'_{B,C}(A) = \theta_X(\mathrm{ccP}(g'_{A,C})(B)).$$

Then the set A satisfying $A = \Phi'(A)$ is called an *inhomogeneous invariant* convex set of probability measures.

Theorem 3.2. *For any* IFS $\{f_1, f_2, \dots, f_n\}$, $B \in \mathrm{ccP}(\{0, 1, \dots, n\})$ *and* $C \in \mathrm{ccP}(X)$ *there exists a unique inhomogeneous invariant convex set of probability measures.*

Proof. Similarly to the proof of the previous theorem, in the compact case we apply the Banach Contraction Principle to the map Φ'. The non-compact case can be reduced to the compact one similarly as in the proof of Theorem 3.1. □

Proposition 3.3. *If the set* $B \in \mathrm{ccP}(\{1, 2, \dots, n\})$ *from the definition of invariant convex set of probability measures is a singleton, then the obtained invariant convex set of probability measures is a singleton as well.*

Proof. Let $B = \{\mu\} \in \mathrm{ccP}(\{1, 2, \dots, n\})$, for some $\mu \in P(\{1, 2, \dots, n\})$, where $\mu = \sum_{i=1}^{n} \alpha_i \delta_i$. We start with $A_0 = \{\nu_0\} \in \mathrm{ccP}(X)$. Then clearly

$$A_1 = \Phi(A_0) = \Big\{\sum_{i=1}^{n} \alpha_i P(f_i)(\nu_0)\Big\} = \{\nu_1\}$$

and this easily implies that the invariant set of probability measures A_∞ in this case is $\{\nu_\infty\}$, where ν_∞ is the invariant measure in the sense of [7] corresponding to the IFS $\{f_1, \dots, f_n\}$ and $\mu = \sum_{i=1}^{n} \alpha_i \delta_i$. □

A similar statement can be formulated and proved in the inhomogeneous case. Therefore our considerations are in some sense extensions of known results from [7] and [15] on probability measures.

4. Functional approach

Let X be a compact Hausdorff space. Every $A \in \mathrm{ccP}(X)$ determines a functional $F_A\colon C(X) \to \mathbb{R}$ defined as follows:

$$F_A(\varphi) = \sup_{\mu\in A} \mu(\varphi),\ \varphi \in C(X).$$

Proposition 4.1. *If $A, B \in \mathrm{ccP}(X)$ and $A \neq B$, then $F_A \neq F_B$.*

Proof. Denote by

$$\iota\colon P(X) \to \prod_{\varphi\in C(X)} \mathbb{R}_\varphi$$

the canonical embedding $\iota(\mu) = (\mu(\varphi))_{\varphi\in C(X)}$, where $\mathbb{R}_\varphi$ is a copy of $\mathbb{R}$ for every $\varphi \in C(X)$.

Without loss of generality one may assume that there exists $\mu \in A\backslash B$. Since B is compact, there are $\varphi_1, \dots, \varphi_k \in C(X)$, for some $k \in \mathbb{N}$, such that $p(\mu) \notin p(B)$, where $p\colon \prod\limits_{\varphi\in C(X)} \mathbb{R}_\varphi \to \prod\limits_{i=1}^{k} \mathbb{R}_{\varphi_i}$ is the canonical projection.

Since $p(B)$ is compact and convex, it follows from the hyperplane separation theorem that there exists a linear functional $l\colon \prod\limits_{i=1}^{k} \mathbb{R}_{\varphi_i} \to \mathbb{R}$ such that $\sup_{\nu\in B} l(p(\nu)) < l(p(\mu))$.

Then there exists $(l_1, \dots, l_k) \in \mathbb{R}^k$ such that

$$l(x_1, \dots, x_k) = \sum_{i=1}^{k} l_i x_i,\ (x_1, \dots, x_k) \in \mathbb{R}^k.$$

Now, let $\psi = \sum\limits_{i=1}^{k} l_i\varphi_i$. Then for each $\nu \in P(X)$

$$\nu(\psi) = \nu\left(\sum_{i=1}^{k} l_i\varphi_i\right) = \sum_{i=1}^{k} l_i\nu(\varphi_i) = l(p(\iota(\nu))).$$

Therefore, $\mu(\psi) > \sup\limits_{\nu\in B} \nu(\psi) = F_B(\psi)$. Hence $F_A(\psi) \geqslant \mu(\psi) > F_B(\psi)$, i.e. $F_A \neq F_B$. □

Let τ^* be the weak* topology on the set $\mathcal{F} = \{F_A \mid A \in \mathrm{ccP}(X)\}$, i.e., the topology induced from the product topology on $\mathbb{R}^{C(X)}$. A base of this topology is comprised by the sets of the form

$$O'\langle F_{A_0}; \varphi_1, \dots, \varphi_n; \varepsilon\rangle = \\ = \left\{F_A \;\middle|\; A \in \mathrm{ccP}(X),\ |F_A(\varphi_i) - F_{A_0}(\varphi_i)| < \varepsilon,\ i = \overline{1, n}\right\},$$

where $A_0 \in \mathrm{ccP}(X)$, $\varphi_1, \dots, \varphi_n \in C(X)$, $\varepsilon > 0$.

Proposition 4.2. *The map* $A \mapsto F_A \colon \mathrm{ccP}(X) \to \mathcal{F}$ *is continuous.*

Proof. Note that the sets $O'\langle F_{A_0}; \varphi_0; \varepsilon\rangle$, where $A_0 \in \mathrm{ccP}(X)$, $\varphi_0 \in C(X)$, and $\varepsilon > 0$, comprise a subbase for the topology τ^*.

Given such a set $O'\langle F_{A_0}; \varphi_0; \varepsilon\rangle$ and $\mu \in A_0$, consider the set $O\langle \mu; \varphi_0; \varepsilon\rangle$. Then the family $\{O\langle \mu; \varphi_0; \varepsilon\rangle \mid \mu \in A_0\}$ is an open cover of A_0. Let

$$\{O\langle \mu_i; \varphi_0; \varepsilon\rangle \mid i = 1, \dots, n\}$$

be a finite subcover of this cover. Then $\langle O\langle \mu_1; \varphi_0; \varepsilon\rangle, \dots, O\langle \mu_n; \varphi_0; \varepsilon\rangle\rangle$ is a neighborhood of A_0 in $\mathrm{ccP}(X)$.

We are going to show that for each $A \in \langle O\langle \mu_1; \varphi_0; \varepsilon\rangle, \dots, O\langle \mu_n; \varphi_0; \varepsilon\rangle\rangle$ the functional $F_A \in O'\langle F_{A_0}; \varphi_0; \varepsilon\rangle$. This will finish the proof.

If $\max\{\mu(\varphi_0) \mid \mu \in A\} = \mu_0(\varphi)$ for some $\mu_0 \in A$, then there exists $i \in \{1, \dots, n\}$ such that $\mu_0 \in O\langle \mu_i; \varphi_0; \varepsilon\rangle$. Then

$$F_A(\varphi_0) = \mu_0(\varphi_0) < \mu_i(\varphi_0) + \varepsilon \leqslant F_{A_0}(\varphi_0) + \varepsilon.$$

One can similarly prove that $F_{A_0}(\varphi_0) \leqslant F_A(\varphi_0) + \varepsilon$. □

Corollary 4.3. *The map* $A \mapsto F_A \colon \mathrm{ccP}(X) \to \mathcal{F}$ *is a homeomorphism.*

Proof. Due to compactness of X, the space $\mathrm{ccP}(X)$ is compact, and the assertion follows from the hausdorffness of $\mathcal{F}$ and Proposition 4.1. □

Now the mentioned functional representation $A \mapsto F_A$ of compact convex sets of probability measures allows us to obtain a purely functional proof of the main results of this paper in the spirit of [10, Theorem 1].

5. Remarks

In the case when $X = \mathbb{R}^n$ and the maps $f_1, \dots, f_n$ are similarities, one can find many pictures of invariant and inhomogeneous sets in the literature.

The invariant probability measures can be visualized in a gray scale by using the random iteration algorithm (see [2, Chapt. IX] for details). An open problem is that of visualization of invariant convex sets of probability measures.

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On semiconvexity of open sets with smooth boundary in the plane

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Abstract. We study properties of classes of generalized convex sets in the plane known as 1-*semiconvex* and *weakly* 1-*semiconvex*. It is proved that an open, weakly 1-semiconvex but not 1-semiconvex set with smooth boundary in the plane consists of at least four connected components.

Анотація. В даній роботі вивчаються властивості класів узагальнено опуклих множин на площині, які називаються 1-напівопуклими та слабко 1-напівопуклими.

Відкрита множина багатовимірного дійсного евклідового простору $\mathbb{R}^n$ називається 1-*напівопуклою*, якщо для кожної точки із доповнення цієї множини до всього простору існує промінь, який починається в цій точці і не перетинає задану множину. Відкрита множина простору $\mathbb{R}^n$ називається *слабко* 1-*напівопуклою*, якщо для кожної точки межі множини існує промінь, який починається в цій точці та не перетинає задану множину. Ці поняття були введені Юрієм Борисовичем Зелінським. Усяка відкрита 1-напівопукла множина очевидно є слабко 1-напівопуклою. Ю. Б. Зелінський показав, що зворотнє твердження є невірним. Виявилось, що клас відкритих множин на площині, які є слабко 1-напівопуклими але не 1-напівопуклими, є досить широким і кожна множина з цього класу є незв'язною і складається не менше ніж з трьох компонент зв'язності.

Дана робота присвячена переважно дослідженню нових властивостей слабко 1-напівопуклих але не 1-напівопуклих множин на площині. Тут вони, для зручності, називаються множинами, що мають Z-*властивість*.

The work is dedicated to the memory of Yu. B. Zelinskii. The author expresses her gratitude to Professors Sergiy Maksymenko, Sergiy Plaksa, Oleksandr Bakhtin, Serhii Gryshchuk, Ruslan Salimov, Vitalii Shpakivskyi for valuable advice in planning the study and recommendations for the design of the paper. The author also is grateful to translators Julia Sheichenko and Anna Sheichenko who assisted in the preparation of the manuscript before being sent to the publisher. The author expresses her gratitude to all anonymous Referees of the paper for their comments.

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Основний результат роботи наступний: доведено, що відкрита множина з гладкою межею на площині, яка має Z-властивість, складається щонайменше з чотирьох компонент зв'язності.

Аннотация. В данной работе изучаются свойства классов обобщенно выпуклых множеств на плоскости, которые называются 1-*полувыпуклыми* и *слабо* 1-*полувыпуклыми*. В частности, доказано, что открытое слабо 1-полувыпуклое, но не 1-полувыпуклое множество с гладкой границей на плоскости состоит не менее чем из четырёх компонент связности.

1. Introduction

A class of m-semiconvex sets is one of the classes of generalized convex sets. A semiconvexity notion was proposed by Yu. Zelinskii [9] and it was used in the formulation of a generalization of shadow problem. The *shadow problem* was proposed by G. Khudaiberganov [4] in 1982. It requires *to find the minimal number of open (closed) balls in the real Euclidean space* $\mathbb{R}^n$ *that are pairwise disjoint, centered on a sphere* S^{n-1} *(see* [6]*), do not contain the sphere center, and such that any straight line passing through the sphere center intersects at least one of the balls*.

To formulate the *generalized shadow problem*, let us give at first the following definitions which we also use in our investigation.

Each m-dimensional affine subspace of the space $\mathbb{R}^n$, $0 \leqslant m < n$, is called an *m-dimensional plane*.

Definition 1.1. One of two parts of an m-dimensional plane, $m \geqslant 1$, of the space $\mathbb{R}^n$, $n \geqslant 2$, into which it is divided by its any $(m-1)$-dimensional plane (herewith, the points of the $(m-1)$-dimensional plane are included) is said to be an *m-dimensional half-plane*.

For instance, the 1-dimensional half-plane is a ray, the 2-dimensional half-plane is a half-plane, etc.

Definition 1.2. ([7]) A subset $E \subset \mathbb{R}^n$ is called *m-semiconvex with respect to a point* $x \in \mathbb{R}^n \backslash E$, $1 \leqslant m < n$, if there exists an m-dimensional half-plane L such that $x \in L$ and $L \cap E = \varnothing$.

A subset $E \subset \mathbb{R}^n$ is called *m-semiconvex*, $1 \leqslant m < n$, if it is m-semiconvex with respect to every point $x \in \mathbb{R}^n \backslash E$.

One can easily see that both definitions satisfy the axiom of convexity: the intersection of each subfamily of these sets also satisfies the definition. Thus, for any subset $E \subset \mathbb{R}^n$ we can consider the minimal m-semiconvex set containing E. This set is called the *m-semiconvex hull of E*.

The generalized shadow problem requires *to find the minimum number of pairwise disjoint closed (open) balls in* $\mathbb{R}^n$ *(centered on a sphere* S^{n-1} *and whose radii are smaller than the radius of the sphere) such that any ray starting at the center of the sphere necessarily intersects at least one of these balls.*

In the terms of m-semiconvexity this problem can be reformulated as follows: *what is the minimum number of pairwise disjoint closed (open) balls in* $\mathbb{R}^n$ *whose centers are located on a sphere* S^{n-1} *and the radii are smaller than the radius of this sphere such that the center of the sphere belongs to the* 1*-semiconvex hull of the family of these balls?*

In [9] the generalized shadow problem is solved as $n = 2$ and only the sufficient number of balls is indicated for $n = 3$.

In the 60's L. Aizenberg and A. Martineau proposed their notions of a linearly convex set in the multi-dimensional complex space $\mathbb{C}^n$. The first author considered domains and their closures and used boundary points of the domains in his definition [1, 2]. The second author used all points of the complement to a subset of the space $\mathbb{C}^n$, [5]. If one uses these definitions not only for domains and compact sets, then Aizenberg's definition isolates one connected component of a set which is linearly convex in the sense of Martineau.

Guided by similar ideas Yu. Zeliskii suggested to distinguish m-semiconvex and weakly m-semiconvex sets.

We will use the following standard notations. For a subset $G \subset \mathbb{R}^n$ let $\overline{G}$ be its closure, $\operatorname{Int} G$ be its interior, and $\partial G = \overline{G} \setminus \operatorname{Int} G$ be its boundary.

Say that a set A *is approximated from the outside* by a family of open sets A_k, $k = 1, 2, \ldots$, if $\overline{A}_{k+1}$ is contained in A_k, and $A = \bigcap_k A_k$ (see [2]).

Definition 1.3. ([8]) An open subset $G \subset \mathbb{R}^n$ is called *weakly* m*-semiconvex*, $1 \leqslant m < n$, if it is m-semiconvex with respect to any point $x \in \partial G$. A subset $E \subset \mathbb{R}^n$ is called *weakly* m*-semiconvex* if it can be approximated from the outside by a family of open weakly m-semiconvex sets.

Thus, each weakly m-semiconvex set A is either open or closed. Among closed weakly m-semiconvex sets there also are sets with empty interior:

$$A = \overline{A} = \overline{A} \setminus \operatorname{Int} A = \partial A.$$

Theorem 1.4. ([8]) *Let* $E \subset \mathbb{R}^2$ *be an open, weakly* 1*-semiconvex and not* 1*-semiconvex subset. Then* E *is disconnected.*

The formulation of the following theorem is equivalent to Theorem 1.4 but will be also in use.

Theorem 1.5. *Let $E \subset \mathbb{R}^2$ be open, connected and weekly 1-semiconvex. Then E is 1-semiconvex.*

The maximal connected subsets of a topological space A are called *connected components (components)* of A (see [2]).

In [8] it was constructed the example of an open, weakly 1-semiconvex, and not 1-semiconvex set (see Figure 2.3b). It was also conjectured that every open, weakly 1-semiconvex, and not 1-semiconvex set consists of at least three components. The latter statement was proved in [3].

Theorem 1.6. ([3]) *Let $E \subset \mathbb{R}^2$ be an open, weakly 1-semiconvex, and not 1-semiconvex subset. Then E consists of at least three connected components.*

We say that a component A of an open, bounded subset of the plane has *smooth boundary* if ∂A is the image of a C^1-embedding of the unit circle. We say that an open, bounded subset of the plane has smooth boundary if each of its components has smooth boundary.

The present paper continues the research of Yu. Zelinskii by investigating properties of 1-semiconvex and weakly 1-semiconvex open sets with smooth boundary in the plane.

For the convenience, we give the following

Definition 1.7. We say that a set in the plane has *Z-property* if the set is weakly 1-semiconvex and not 1-semiconvex.

The main result of thee present paper is the following

Theorem 1.8. *Suppose that an open, bounded subset $E \subset \mathbb{R}^2$ with smooth boundary has Z-property. Then E consists of at least four components.*

2. Auxiliary results

In what follows, unless otherwise is specified, a point of the space $\mathbb{R}^2$ will be denoted by a small or capital Latin letter and the ray starting at that point will be denoted by a small Greek letter with the index denoting this point. The straight segment between points $x, y \in \mathbb{R}^n$ will be denoted by xy and the distance between them will be denoted by $|x - y|$.

Let us provide several auxiliary statements.

Proposition 2.1. *If an open subset $E \subset \mathbb{R}^n$ is m-semiconvex, then it is weakly m-semiconvex.*

Proposition 2.2. *There exist sets in the plane having Z-property.*

In the following example we present an open subset of $\mathbb{R}^2$ with Z-property.

Example 2.3. Figure 2.1a) shows a set E consisting of four open rectangles with a common rays passing through their boundaries. By the construction, for any boundary point of E there exists a ray that does not intersect E, while for any point of the interior of the rhombus $abcd$ such a ray can not be found.

Similarly, in Figure 2.1b), the set consisting of three open components is weakly 1-semiconvex. On the other hand it is not 1-semiconvex, since each ray η_x starting at each point x of the interior of the triangle abc intersects that set.

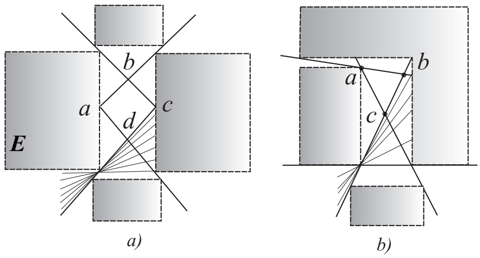

FIGURE 2.1.

Example 2.4. An example of open set with smooth boundary having Z-property can be obtained by replacing the rectangles from Figure 2.1a) with open sets having smooth boundaries tangent to the same rays and such that their union is weakly 1-semiconvex. For instance, this property holds for a system of four open disks shown in Figure 2.2a).

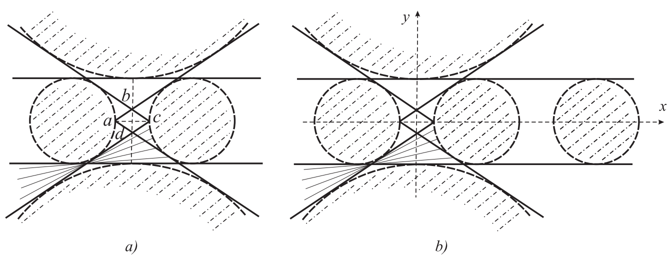

FIGURE 2.2.

Suppose that two disks of that system have the same radii r and their centers belong to the axis Ox and are symmetric with respect to the origin O, see Figure 2.2b). Assume that the other two disks have radii R and their centers belong to the axis Oy. If we place the center of the fifth open disk with radius r at the point $(R+r,0)$, then the union of those five open disks will have Z-property. Adding more disks of radius r and centers on the axis Ox in the positive direction, we will get an open set with smooth boundary having Z-property and consisting of any finite or even countable number of components.

Further it will be shown that in the example of a set having Z-property and consisting of three components it is not possible to replace the components with those that have smooth boundary while maintaining Z-property.

The set of all points of rays starting at a point $x \in \mathbb{R}^n \backslash A$ and passing through a subset $A \subset \mathbb{R}^n$ is called the *cone* over the set A with respect to the point x and is denoted by $\mathrm{C}_x A$. The boundary of $\mathrm{C}_x A$ consists of rays which are called *boundary rays*. The boundary rays of $\mathrm{C}_x A$ are called *supporting* for the set A (with respect to the point x). We suppose that $x \notin \mathrm{C}_x A$ whenever A is open and $x \in \mathrm{C}_x A$ otherwise.

It can be proved that if A is bounded and open (closed), then $\mathrm{C}_x A$ is open (closed) as well and $\mathrm{C}_x \overline{A} \equiv \overline{\mathrm{C}_x A}$.

Lemma 2.5. *A ray γ_x is supporting for an open bounded subset $E \subset \mathbb{R}^n$ iff the following conditions are satisfied:*

1) $\gamma_x \cap \partial E \neq \varnothing$,
2) $\gamma_x \cap E = \varnothing$.

Proof. *Necessity.* Suppose $a' \in \gamma_x \cap E \neq \varnothing$, then there exists a neighborhood $U(a') \subset E$ of a' such that $\gamma_x \subset \mathrm{C}_x(U(a')) \subset \mathrm{C}_x E$ and the ray γ_x is not the boundary one. If $\gamma_x \cap \partial E = \varnothing$, then $\gamma_x \cap \overline{E} = \varnothing$, i.e $\gamma_x \subset \mathbb{R}^n \backslash \mathrm{C}_x \overline{E} \equiv \mathbb{R}^n \backslash \overline{\mathrm{C}_x E}$ which implies that the ray γ_x is also not the boundary one.

Sufficiency. Consider the cone $\mathrm{C}_x E$. Since E is open, $\mathrm{C}_x E$ is open as well. We need to show that the ray γ_x satisfying conditions 1) and 2) coincides with one of boundary rays of $\mathrm{C}_x E$. If $\gamma_x \subset \mathbb{R}^n \backslash \overline{\mathrm{C}_x E}$, then condition 1) of the lemma is not true. On the other hand, if $\gamma_x \subset \mathrm{C}_x E$, then $\gamma_x \cap E \neq \varnothing$, by the definition of $\mathrm{C}_x E$, which contradicts to condition 2). Thus, $\gamma_x \subset \partial \mathrm{C}_x E$. □

Lemma 2.6. *For an open connected weakly* 1*-semiconvex subset $E \subset \mathbb{R}^2$ there exists at least one but no more than two supporting rays starting at some point $x \in \mathbb{R}^2 \backslash E$.*

Proof. By Theorem 1.5 the set E is 1-semiconvex. Thus for any point $x \in \mathbb{R}^2 \backslash E$ there exists a ray γ_x such that $\gamma_x \cap E = \varnothing$. Since E is not empty, open and connected, $\mathrm{C}_x E$ is a plane angle $\angle \alpha > 0$ not containing its sides.

If γ_x is unique, i.e. $\gamma_x \equiv \mathbb{R}^2 \backslash \mathrm{C}_x E$, then $\angle\alpha = 2\pi$ and γ_x is its unique side. On the other hand, if $\gamma_x \subsetneq \mathbb{R}^2 \backslash \mathrm{C}_x E$, then $0 < \angle\alpha < 2\pi$ with two sides. $\square$

An example of open, connected, and weakly 1-semiconvex set E for which there exists a point $x \in \mathbb{R}^2 \backslash E$ admitting a unique supporting ray ξ_x is shown in Figure 2.3a).

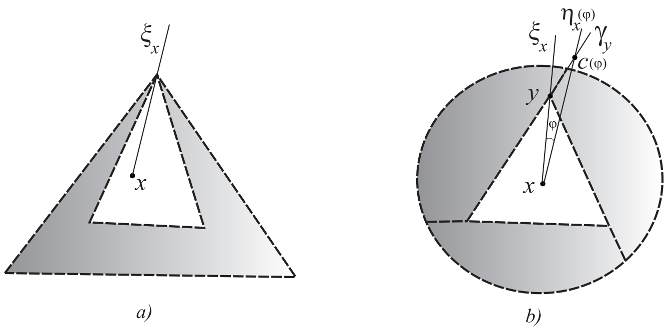

FIGURE 2.3.

Definition 2.7. A point $x \in \mathbb{R}^n \backslash A$ is called *a point of m-nonsemiconvexity of a subset $A \subset \mathbb{R}^n$* if there is no m-dimensional half-plane that has x on its boundary and does not intersect A.

Lemma 2.8. *Suppose that an open subset $E = \bigcup_{j=1}^{k} E_j \subset \mathbb{R}^2$, $k \geqslant 3$, $k \in \mathbb{N}$, where E_j are its components, has Z-property. Let also x be a point of 1-nonsemiconvexity of E. Then each component E_j has exactly two supporting rays starting at x.*

Proof. Since E has Z-property, it follows that each of its 1-nonsemiconvexity point x belongs to $\mathbb{R}^2 \backslash \overline{E}$.

By Lemma 2.8 each component E_j has either one or two supporting rays starting at x. Suppose some component E_{j^1}, $j^1 \in \{1, 2, \ldots, k\}$, has a unique supporting ray ξ_x. Since E is not 1-semiconvex, it follows that $\xi_x \cap (E \backslash E_{j^1}) \neq \varnothing$. Let ξ'_x be the ray complementary to ξ_x and y be the nearest point of ∂E_{j^1} to x along the ray ξ'_x.

Notice that any ray starting at y and not containing the ray ξ_x intersects E_{j^1}. Indeed, suppose there exists a ray γ_y not containing the ray ξ_x and such that $\gamma_y \cap E_{j^1} = \varnothing$. Let ξ_y be the ray starting at y and containing ξ_x. Then the polygonal chain $\gamma_y \cup \{y\} \cup \xi_y$ cuts the plane into two open components (parts). Since $E_{j^1} \cap (\gamma_y \cup \{y\} \cup \xi_y) = \varnothing$ and E_{j^1} is connected, it is completely contained in one of those parts. On the other hand, there exists a ray η_x contained in the other part, which gives $\eta_x \cap E_{j^1} = \varnothing$. Since η_x differs from ξ_x, it is clear that ξ_x is not a unique supporting ray for E_{j^1} with respect to the point x. This contradicts to our assumption that E_{j^1} has a unique supporting ray.

Furthermore, since ξ_x intersects $E \backslash E_{j^1}$, the ray $\xi_y \supset \xi_x$ also intersects $E \backslash E_{j^1}$. Thus, $y \in \partial E$ is a point of 1-nonsemiconvexity of E, which contradicts to weak 1-semiconvexity of E and none of the components of E have a unique supporting ray. □

Definition 2.9. Let $A \subset \mathbb{R}^n$, A_1 be a component of A, and $x \in \mathbb{R}^n \backslash (\overline{A})$. A ray ξ_x is called *inner supporting for the set A with respect to A_1*, whenever ξ_x is supporting for A_1, $\xi_x \cap (A \backslash A_1) \neq \varnothing$, and there exists a point $a \in \xi_x \cap \partial A_1$ such that $|a - x| < |b - x|$ for any point $b \in \xi_x \cap (A \backslash A_1)$.

Definition 2.10. Let $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n \backslash (\overline{A})$. A ray ξ_x, is called *inner supporting for A* if there exists a component A_1 of A such that ξ_x is inner supporting for A with respect to A_1.

Lemma 2.11. *Let E_1, E_2 be two components of a subset $E \subset \mathbb{R}^n$ and let $x \in \mathbb{R}^n \backslash \overline{E}$. If a ray ξ_x is inner supporting for $E_1 \cup E_2$ and*

$$\xi_x \cap (E \backslash (E_1 \cup E_2)) = \varnothing,$$

then ξ_x is inner supporting for E.

Proof. Without loss of generality, suppose that ξ_x is inner supporting for the set $E_1 \cup E_2$ with respect to E_1. Then ξ_x is supporting for E_1 and has the following properties:

a) $\xi_x \cap E_2 \neq \varnothing$,
b) there exists a point $a \in \xi_x \cap \partial E_1$ such that $|a - x| < |b - x|$ for every $b \in \xi_x \cap E_2$ by Definition 2.9.

Condition a) together with condition $E_2 \subset E \backslash E_1$ gives $\xi_x \cap (E \backslash E_1) \neq \varnothing$. Since $\xi_x \cap (E \backslash (E_1 \cup E_2)) = \varnothing$, we have

$$\xi_x \cap E_2 \equiv \xi_x \cap ((E \backslash (E_1 \cup E_2)) \cup E_2) \equiv \xi_x \cap (E \backslash E_1),$$

which together with condition b) gives $|a - x| < |b - x|$ for the point $a \in \xi_x \cap \partial E_1$ and for any point $b \in \xi_x \cap (E \backslash E_1)$. Thus, E_1 is a component of E such that ξ_x is an inner supporting ray for E with respect to E_1. □

Lemma 2.12. *Let $E = \bigcup_{j=1}^{k} E_j \subset \mathbb{R}^2$, $3 \leqslant k < \infty$, $k \in \mathbb{N}$, be an open bounded set having Z-property, where E_j are its components. Let $x \in \mathbb{R}^2 \backslash \overline{E}$ be a point of 1-nonsemiconvexity of E, $y \in \partial E$ the nearest point of ∂E to x along some ray η_x, and γ_y any ray that does not intersect E.*

Then there exists an inner supporting ray ξ_x for E with respect to some component E_{j^0}, $j^0 \in \{\overline{1,k}\}$, such that

1) $\xi_x \cap \gamma_y = c \neq \varnothing$;
2) $|a - x| \leqslant |c - x|$ *for any point* $a \in \xi_x \cap \partial E_{j^0}$;
3) $E \cap \operatorname{Int}(\triangle xyc) = \varnothing$.

Proof. Since x is a point of 1-nonsemiconvexity of E, γ_y does not lie on the straight line that contains η_x.

Choose the polar coordinate system (φ, ρ) in $\mathbb{R}^2$ in which x is the pole, the ray $\eta_x = \eta_x(0)$ is the polar axis, $\eta_x(\varphi)$ is a ray starting at x and constituting an angle φ with ray η_x, and a positive angular coordinate φ is determined by a ray starting at x and intersecting ray γ_y.

Let $\eta_x(\phi)$, $0 < \phi < \pi$, be the ray that is parallel to γ_y. Then rays $\eta_x(\varphi)$, $0 < \varphi < \phi$, intersect γ_y. We will also use the following notations: $c(\varphi) = \eta_x(\varphi) \cap \gamma_y$ and $xc(\varphi)$ is the interval between points x and $c(\varphi)$.

Let Φ be the set of all $\varphi \in (0, \phi)$ such that $xc(\varphi) \cap E \neq \varnothing$. Then Φ is non-empty. Indeed, since $\eta_x(\phi) \cap E \neq \varnothing$ and E is open, there exists $\varepsilon > 0$ small enough and such that $xc(\phi - \varepsilon) \cap E \neq \varnothing$.

Let $J \subseteq \{1, 2, \dots, k\}$ be the set of all indexes $j \in \{1, 2, \dots, k\}$ such that there exists $\varphi \in \Phi$ for which $xc(\varphi) \cap E_j \neq \varnothing$. Let also $\Phi_j \subseteq \Phi$, $j \in J$, be the set of all $\varphi \in \Phi$ with $xc(\varphi) \cap E_j \neq \varnothing$.

Put $\varphi_j = \inf \Phi_j$, $j \in J$. Then it is clear that $\varphi_j \in [0, \phi)$ and

a) $xc(\varphi_j) \cap \partial E_j \neq \varnothing$,
b) $xc(\varphi_j) \cap E_j = \varnothing$,
c) $xc(\varphi_j + \varepsilon) \cap E_j \neq \varnothing$, for $\varepsilon > 0$ small enough.

This implies that $E_j \cap \operatorname{Int}(\triangle xyc(\varphi_j)) = \varnothing$, $j \in J$.

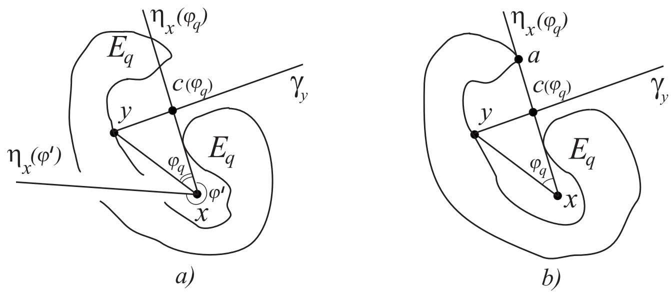

FIGURE 2.4.

We claim that if $\eta_x(\varphi_q) \cap E_q \neq \varnothing$ for some $q \in J$, then E_q does not have supporting rays starting at x, see Figure 2.4a). Indeed, suppose $\eta_x(\varphi')$, $\varphi' \neq \varphi_q$, is supporting for E_q. Then $\eta_x(\varphi') \cap E_q = \varnothing$. Let $\gamma_{c(\varphi_q)}$ be the ray starting at point $c(\varphi_q)$ and lying on ray γ_y. If $\eta_x(\varphi') \cap \gamma_{c(\varphi_q)} = \varnothing$, then the polygonal chain

$$L := \eta_x(\varphi') \cup \{x\} \cup xc(\varphi_q) \cup \{c(\varphi_q)\} \cup \gamma_{c(\varphi_q)}$$

cuts the plane into two open components. Herewith, $L \cap E_q = \varnothing$. One of the components of $\mathbb{R}^2 \backslash L$ contains a part of the ray $\eta_x(\varphi_q)$ intersecting E_q. Since E_q is connected, it is entirely contained in this component. On the other hand, since the interval $xc(\varphi)$, $\varphi_q < \varphi < \phi$, is contained in the second component of $\mathbb{R}^2 \backslash L$ and $xc(\varphi_q + \varepsilon) \cap E_q \neq \varnothing$, for $\varepsilon > 0$ small enough, E_q is contained in the second component. We have reached a contradiction.

If $\eta_x(\varphi') \cap \gamma_{c(\varphi_q)} \neq \varnothing$, then the polygonal chain L cuts the plane into three open components. The component of $\mathbb{R}^2 \backslash L$ containing the part of the ray $\eta_x(\varphi_q)$ intersecting E_q contains E_q as well. Moreover, the component of $\mathbb{R}^2 \backslash L$ bounded by the triangle generated by the intersection of rays $\eta_x(\varphi')$, $\gamma_{c(\varphi_q)}$, and $\eta_x(\varphi_q)$ also contains E_q, since $xc(\varphi_q + \varepsilon) \cap E_q \neq \varnothing$, for $\varepsilon > 0$ small enough. This contradicts to the fact that E_q is connected. Thus, our assumption is incorrect and E_q does not have supporting rays starting at x.

Similarly, it can be proved that if $\eta_x(\varphi_q) \cap E_q = \varnothing$ but there exists a point $a \in \eta_x(\varphi_q) \cap \partial E_q$ such that $|a - x| > |c(\varphi_q) - x|$ for some $q \in J$, then E_q has a unique supporting ray starting at x coinciding with $\eta_x(\varphi_q)$ (see Figure 2.4 b).

The cases when component E_q does not have supporting rays or has a unique supporting ray contradict to Lemma 2.8. Then by Lemma 2.5, for any $j \in J$ the ray $\eta_x(\varphi_j)$ is supporting for E_j and has the following properties:

a) $\eta_x(\varphi_j) \cap \gamma_y = c(\varphi_j)$;

b) $|a - x| \leqslant |c(\varphi_j) - x|$ for any point $a \in \eta_x(\varphi_j) \cap \partial E_j$;

c) $E_j \cap \operatorname{Int}(\triangle xyc(\varphi_j)) = \varnothing$.

Thus, conditions 1) and 2) of our lemma are fulfilled for any ray $\eta_x(\varphi_j)$, $j \in J$.

To finish the proof, we need only to show that among rays $\eta_x(\varphi_j)$, $j \in J$, there is the one that is inner supporting for E and satisfying the lemma condition 3). Consider the angle $\varphi_{j^0} := \min_j \varphi_j$, $j^0 \in J$, and note that by the constructions $E \cap \triangle\operatorname{Int}(xyc(\varphi_{j^0})) = \varnothing$. Thus, condition 3) holds for the ray $\eta_x(\varphi_{j^0})$. Since x is a point of 1-nonsemiconvexity of E, it follows that $\eta_x(\varphi_{j^0}) \cap (E \backslash E_{j^0}) \neq \varnothing$. It then follows from properties 1) and 2), that $|a - x| < |b - x|$ for any point $a \in \eta_x(\varphi_{j^0}) \cap \partial E_{j^0}$ and any point $b \in \eta_x(\varphi_{j^0}) \cap (E \backslash E_{j^0})$. Thus, $\xi_x := \eta_x(\varphi_{j^0})$ is an inner supporting ray of E with respect to E_{j^0} satisfying the conditions 1)-3) of the lemma. $\square$

Remark 2.13. Notice that if ∂E is not smooth at y, then the inner supporting ray ξ_x can coincide with the ray η_x. In this case the points c and y coincide and $\triangle xyc$ degenerates into the interval xy, see Figure 2.3b). Then all rays $\eta_x(\varphi)$ which are close enough to $\xi_x \equiv \eta_x$ and intersect ray γ_y must also intersect the component E_{j^0}.

If ∂E is smooth, then $\triangle xyc$ is non-degenerate, due to the fact that y is the nearest to x along η_x.

Definition 2.14. We say that *a subset $A \subset \mathbb{R}^n$ is projected from a point $x \in \mathbb{R}^n$ on a subset $B \subset \mathbb{R}^n$* if any ray starting at point x and intersecting A intersects B as well.

Lemma 2.15. *Suppose an open subset $E \subset \mathbb{R}^2$ has Z-property and consists of three components. Then none of its components is projected on the union of the others from a point of 1-nonsemiconvexity of E.*

Proof. Let E_1, E_2, E_3 be connected components of E and $x \in \mathbb{R}^2 \backslash \overline{E}$ be a point of 1-nonsemiconvexity of E. Without loss of generality, suppose that E_1 is projected from x on $E_2 \cup E_3$, see Figure 2.5. Then the set, consisting only of components E_2, E_3, has Z-property, which contradicts to Theorem 1.6. □

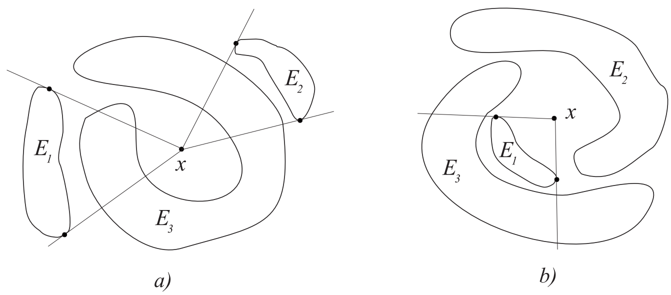

FIGURE 2.5.

Let $E \subset \mathbb{R}^2$ be an open subset and γ a straight line passing through some boundary point x of E and does not intersect set E in some neighborhood U of x, i.e. $\gamma \cap E \cap U = \varnothing$. Notice that γ cuts the plane into two half-planes. Let P be the half-plane such that $P \cap U \cap E = \varnothing$. We say that a ray η_x starting at x *lies above the straight line γ* if $\eta_x \subset P$.

Lemma 2.16. *Suppose an open bounded set $E \subset \mathbb{R}^2$ has Z-property and consists of three components. If ∂E is smooth, then E has three distinct inner supporting rays starting at a point of 1-nonsemiconvexity of E and* *each of them is inner supporting for E with respect to a unique component of the set.*

Proof. Let E_j, $j = 1, 2, 3$, be components of E and let $x \in \mathbb{R}^2 \backslash \overline{E}$ be a point of 1-nonsemiconvexity of the set. By Lemma 2.8 each E_j, $j = 1, 2, 3$, has exactly two supporting rays which we denote by $\xi_x^{j,1}$, $\xi_x^{j,2}$, respectively. Notice that in general some of $\xi_x^{j,k}$, $j = 1, 2, 3$, $k = 1, 2$, can coincide, see Figure 2.7b).

By Lemma 2.15, none of the components of E is projected on the union of the others from x. Then for $i = 1, 2, 3$ there exists a ray τ_x^i starting at x and intersecting a unique component E_i. In other words,

$$\tau_x^1 \cap E \equiv \tau_x^1 \cap E_1, \qquad \tau_x^2 \cap E \equiv \tau_x^2 \cap E_2, \qquad \tau_x^3 \cap E \equiv \tau_x^3 \cap E_3.$$

Then the rays τ_x^1, τ_x^2 and τ_x^3 cut the plane into three open components G_1, G_2, G_3 such that

$$\partial G_1 = \tau_x^2 \cup \tau_x^3, \qquad \partial G_2 = \tau_x^3 \cup \tau_x^1, \qquad \partial G_3 = \tau_x^1 \cup \tau_x^2.$$

Consider the closure of the domain G_3 between the rays τ_x^1, τ_x^2. We claim that $\overline{G}_3$ does not contain points of E_3. Indeed, $\partial G_3 \cap E_3 = \varnothing$ and ∂G_3 cuts $\mathbb{R}^2 \backslash \{x\}$ into two components: G_3 and $G_2 \cup G_1 \cup \tau_x^3$. Since E_3 is connected, it must be completely contained in one of those components. Hence $E_3 \cap \tau_x^3 \neq \varnothing$ implies that $E_3 \subset G_2 \cup G_1 \cup \tau_x^3$.

Thus, the union of rays starting at x and intersecting both E_1, E_2, is open and connected in $\overline{G}_3$ and its boundary consists of one ray supporting for E_1 and one ray supporting for E_2. Without loss of generality, suppose $\xi_x^{1,1}, \xi_x^{2,2} \subset \overline{G}_3$. Analogically, $\xi_x^{2,1}, \xi_x^{3,2} \subset \overline{G}_1$, $\xi_x^{3,1}, \xi_x^{1,2} \subset \overline{G}_2$.

Now we prove that for a fixed $k = 1, 2, 3$ one and only one of rays $\xi_x^{i,1}, \xi_x^{j,2} \subset \overline{G}_k$, $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, is inner supporting for E. It is sufficient to prove this for $k = 3$. Moreover, due to Lemma 2.11, it suffices to show that one and only one of rays $\xi_x^{1,1}$, $\xi_x^{2,2}$ is inner supporting for the set $E_1 \cup E_2$.

By Lemma 2.5, $\xi_x^{j,j} \cap \partial E_j \neq \varnothing$, $\xi_x^{j,j} \cap E_j = \varnothing$, $j = 1, 2$. Consider the points $y_j \in \xi_x^{j,j} \cap \partial E_j$, $j = 1, 2$. By smoothness of ∂E and since $\xi_x^{j,j} \cap E_j = \varnothing$, $j = 1, 2$, the rays $\xi_x^{1,1}$, $\xi_x^{2,2}$ are tangent to ∂E_1, ∂E_2 at the points y_1, y_2, respectively. Without loss of generality, suppose that there are points $b_1, b_2 \in \xi_x^1 \cap E_2$ such that $|b_1 - x| < |y_1 - x| < |b_2 - x|$, see Figure 2.6a). Since ray $\xi_x^{1,1}$ is tangent to ∂E_1 at y_1, all rays starting at y_1 and not intersecting E should lie above the straight containing $\xi_x^{1,1}$. On the other hand, all these rays intersect any curve in E_2 connecting points b_1, b_2, as E_2 is connected, which gives that these rays intersect E_2 and therefore y_1 is a point of 1-nonsemiconvexity of E. This contradicts weakly 1-semiconvexity of the set.

Thus, $|b-x| < |y_1-x|$ or $|b-x| > |y_1-x|$ for any point $b \in \xi_x^{1,1} \cap E_2$. These statements are also true for the ray $\xi_x^{2,2}$. Furthermore, $|b_1 - x| > |y_2 - x|$ for any point $b_1 \in \xi_x^2 \cap E_1$ if and only if $|b_2 - x| < |y_1 - x|$ for any point $b_2 \in \xi_x^{1,1} \cap E_2$. Swapping indices we will also get that $|b_2 - x| > |y_1 - x|$ for any point $b_2 \in \xi_x^{1,1} \cap E_2$ if and only if $|b_1 - x| < |y_2 - x|$ for any point $b_1 \in \xi_x^{2,2} \cap E_1$, see Figure 2.6b). Otherwise, E_1, E_2 would be overlapping.

Hence, $\xi_x^{j,j}$ is supporting for E_j and $\xi_x^{j,j} \cap ((E_1 \cup E_2)\backslash E_j) \neq \varnothing$, $j = 1, 2$, but $|b - x| > |y_j - x|$ for each $y_j \in \partial E_j$ and each $b \in \xi_x^{j,j} \cap ((E_1 \cup E_2)\backslash E_j)$, if $j = 1$ or $j = 2$, which was necessary to prove.

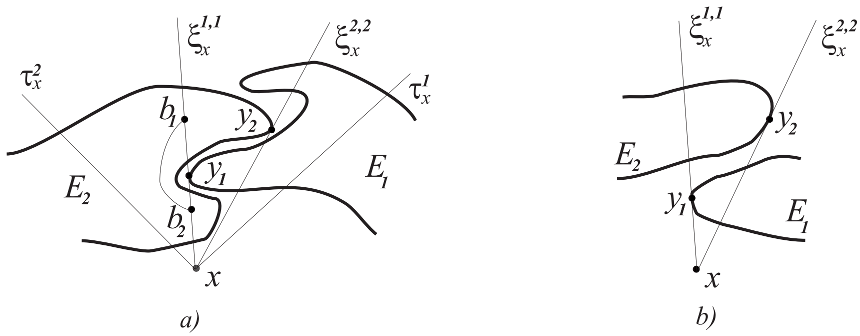

FIGURE 2.6.

Thus, E has no more than three inner supporting rays which we denote by ξ_x^k, $i = 1, 2, 3$, such that $\xi_x^k \subset \overline{G}_k$. Herewith, if $\xi_x^k \subset G_k$, $k = 1, 2, 3$, then the ray ξ_x^k is supporting for a unique of components E_j, $j = 1, 2, 3$, $j \neq k$, and therefore, it is inner supporting for E with respect to a unique component of E.

Since each $\overline{G}_k$ contains only points of E_i, E_j for

$$(i, j, k) = (1, 2, 3),\ (2, 3, 1),\ (3, 1, 2),$$

neither of the rays $\xi_x^k \subset \overline{G}_k$, $k = 1, 2, 3$, can be inner supporting with respect to three components E_j, $j = 1, 2, 3$, at the same time.

Since $\overline{G}_i \cap \overline{G}_j = \tau_x^k$, $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$, a ray $\xi_x^k \subset \overline{G}_k$, $k = 1, 2, 3$, can be inner supporting with respect to two components if and only if it coincides with a neighboring inner supporting ray on the common boundary of the respective sets G_j, $j = 1, 2, 3$. Let us consider this case and prove that it is not possible under the lemma conditions. By this we will show that E has not less than three inner supporting rays starting at

x and ξ_x^k is inner supporting for E with respect to a single component of the set in ∂G_k, $k=1,2,3$, as well.

Without loss of generality, assume that two inner supporting rays ξ_x^1, ξ_x^2 coincide with the boundary ray τ_x^3, see Figure 2.7a). By the constructions, ξ_x^1 can be supporting for E_2 or E_3. But since τ_x^3 intersects component E_3 and ξ_x^1 coincides with τ_x^3, the ray ξ_x^1 is supporting for E_2. Similarly, the ray ξ_x^2 is supporting for E_1.

Let y_j be the nearest to x point of $\tau_x^3 \cap \partial E_j$, $j=1,2$. Since $\tau_x^3 \cap E_j = \varnothing$, $j=1,2$, and by smoothness of ∂E, the ray τ_x^3 is tangent to ∂E_j at the points y_j, $j=1,2$. The points y_1, y_2 do not coincide, since otherwise one can not draw a ray starting at the point $y_1=y_2$ and does not intersect E, which contradicts to weak 1-semiconvexity of E. Assume for definiteness

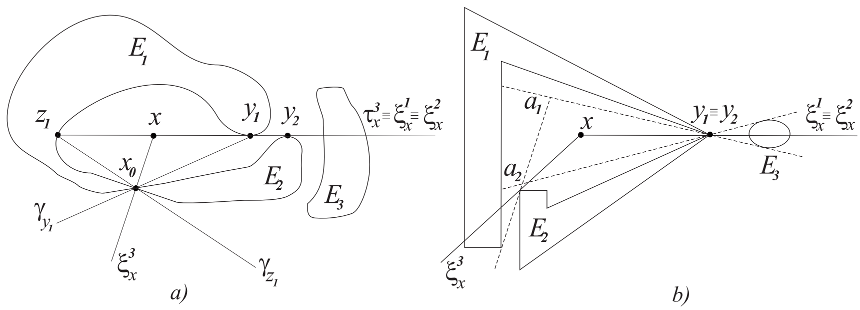

FIGURE 2.7.

that x is closer to y_1 than to y_2, i.e. $|x-y_1|<|x-y_2|$. Then, let us draw any ray γ_{y_1} starting at y_1 and not intersecting E. It lies above the straight line containing the ray τ_x^3, since ∂E is smooth. The ray, complementary to τ_x^3, intersects ∂E at the point z_1 which is nearest to x along this ray.

Let us draw a ray γ_{z_1} starting at z_1 and also not intersecting E. By Lemma 2.12, among all rays starting at point x and crossing rays γ_{y_1} and γ_{z_1} there exist two inner supporting rays different from τ_x^3. If they are distinct, this contradicts to the fact that E has only two inner supporting rays by our assumption. Such an inner supporting ray is unique only if rays γ_{y_1} and γ_{z_1} are intersecting and the inner supporting ray ξ_x^3 passes through the point $x_0 \in \gamma_{y_1} \cap \gamma_{z_1}$.

Since the polygonal chain $\gamma_{y_1} \cup \{y_1\} \cup y_1 z_1 \cup \{z_1\} \cup \gamma_{z_1}$ is self-intersecting, it cuts the plane into three components (parts). The first part contained inside the $\triangle y_1 z_1 x_0$ does not have points of E, by condition 3) of Lemma 2.12. Since the ray $\tau_{y_1}^3 \subset \tau_x^3$ is such that $y_1 \in \tau_{y_1}^3 \cap \partial E_1$, $y_2 \in \tau_{y_1}^3 \cap \partial E_2$, and $\tau_{y_1}^3 \cap E_3 \neq \varnothing$, the second part of the plane that holds $\tau_{y_1}^3$ contains all three components of E. Thus, the third part also does not have points of E, which allows the ray ξ_x^3 to not intersect E. This contradicts to non 1-semiconvexity of E. Lemma is proved. □

Remark 2.17. Lemma 2.16 fails for sets with non-smooth boundary. Figure 2.7b) contains an example of open bounded set having Z-property and consisting of three components which has only two inner supporting rays starting at a point of 1-nonsemiconvexity $x \in \mathrm{Int}(\triangle y_1 a_1 a_2)$.

In the following lemmas we will assume that inner supporting rays of the set start at some point of its 1-nonsemiconvexity.

Lemma 2.18. *There exists no open bounded subset $E \subset \mathbb{R}$ with smooth boundary having Z-property, consisting of three components E_1, E_2, E_3, and such that two of inner supporting rays of E are inner supporting with respect to the same component.*

Proof. Let us prove the lemma by the contradiction. Let $x \in \mathbb{R}^2 \backslash \overline{E}$ be a point of 1-nonsemiconvexity of E. By Lemma 2.16, E has three inner supporting rays ξ_x^i, $i = 1, 2, 3$, and each of them is inner supporting with respect to a unique component of E.

Without loss of generality, suppose that the rays ξ_x^1 and ξ_x^2 are inner supporting for E with respect to the component E_1 and ξ_x^3 is inner supporting with respect to E_3.

Consider the intersections $\xi_x^1 \cap \partial E_1$, $\xi_x^2 \cap \partial E_1$, $\xi_x^3 \cap \partial E_3$. By Lemma 2.5, these intersections are not empty. Let $y_1 \in \xi_x^1 \cap \partial E_1$, $y_2 \in \xi_x^2 \cap \partial E_1$, $y_3 \in \xi_x^3 \cap \partial E_3$ be the nearest points to x. Since ξ_x^i, $i = 1, 2, 3$, does not intersect the component for which it is supporting and by smoothness of ∂E, it follows that ξ_x^i is tangent to the boundary of the correspondent component at the point y_i, $i = 1, 2, 3$. This implies that all rays starting at y_i and not intersecting E lie above the straight line containing the supporting ray ξ_x^i.

Let S be the sector between rays ξ_x^1 and ξ_x^2 containing the component E_1. We claim that then the ray ξ_x^3 is not contained in S. Indeed, otherwise E_3 would be contained in the open component of $\mathbb{R}^2 \backslash E$ bounded by the intervals xy_1, xy_2 and the part of ∂E_1 between points y_1, y_2. Then E_3 would be projected on E_1, which contradicts to Lemma 2.15. Therefore ξ_x^3 is contained in the open sector that is complementary to S.

Let α be the angle of sector S. Consider two cases: a) $\alpha \geqslant \pi$ and b) $\alpha < \pi$.

a) Let $z_3 \in \partial E$ be the point contained in the ray complementary to ξ_x^3 and closest to x. Then $z_3 \in \partial E_1$, see Figure 2.8. Let us draw a ray γ_{z_3} that does not intersect E. It passes through the interval xy_1 or xy_2. It is sufficient to consider the case with interval xy_1. For the interval xy_2, the arguments will be the same.

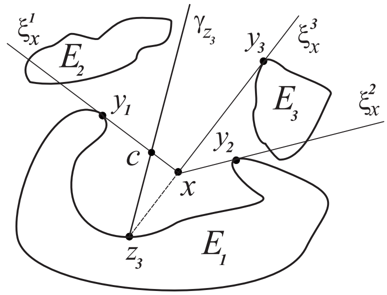

FIGURE 2.8.

By Lemma 2.12, one of the rays ξ^i_x, $i = 1, 2, 3$, satisfies conditions 1)-3) of that lemma for the ray γ_{z_3}. Denote that ray by ξ_x.

Since ξ^2_x lies in the half-plane not containing γ_{z_3}, the ray ξ^2_x does not intersect γ_{z_3} and therefore $\xi_x \neq \xi^2_x$. As $\gamma_{z_3} \not\supseteq \xi^3_x$, the ray ξ^3_x does not intersect ray γ_{z_3} as well and therefore $\xi_x \neq \xi^3_x$.

Let $c = \xi^1_x \cap \gamma_{z_3}$. Since ∂E is smooth and z_3 belongs to the open part of ∂E_1 between points y_1 and y_2, it follows that the point $y_1 \in \partial E_1$ satisfies the inequality $|y_1 - x| > |c - x|$ and therefore condition 2) of Lemma 2.12 is not fulfilled for the ray ξ^1_x. Thus, $\xi_x \neq \xi^1_x$ as well.

We get a contradiction, whence our assumption is incorrect for $\alpha \geqslant \pi$.

b) Suppose $\alpha < \pi$. If the inner supporting ray ξ^3_x is contained in the open sector S' between the rays complementary to ξ^1_x and ξ^2_x, then $z_3 \in \partial E_1$ and the arguments should be as in the case a). Thus, we again get a contradiction.

Let α_1 (resp α_2) be the angle between nearest-neighbor rays ξ^3_x and ξ^1_x (resp. ξ^3_x and ξ^2_x), see Figure 2.9. Suppose ξ^3_x is not contained in the sector S'. Then $\alpha_1 \neq \alpha_2$.

For definiteness assume that $\alpha_1 > \alpha_2$ and $\alpha_1 \geqslant \pi$. By smoothness of ∂E, the ray γ_{y_1} that does not intersect E should lie above the straight that contains ξ^1_x. But all such rays intersect E. Indeed, if one assumes that there exists a ray γ_{y_1} not intersecting E, then by Lemma 2.12, one of the inner supporting rays ξ^j_x, $j = 1, 2, 3$, intersects γ_{y_1}. However the ray ξ^1_x can intersect γ_{y_1} only if $\gamma_{y_1} \subset \xi^1_x$, which is not possible, since ξ^1_x intersects E and γ_{y_1} does not. By the constructions, the rays ξ^3_x and ξ^2_x lie in the half-plane different from the one that contains γ_{y_1}. Then the rays ξ^3_x, ξ^2_x also do not intersect γ_{y_1}.

Thus, y_1 is a point of 1-nonsemiconvexity of E, which contradicts to its weak 1-semiconvexity. We have now reached a contradiction and our assumption is wrong for $\alpha < \pi$ as well. Lemma is proved. □

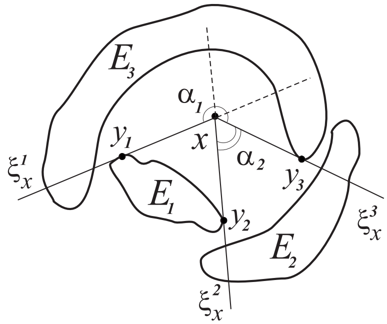

FIGURE 2.9.

Lemma 2.18 fails for sets with non-smooth boundary as $\alpha < \pi$, see Figure 2.1b).

Lemma 2.19. *There exists no open bounded set with smooth boundary in the plane having Z-property, consisting of three components and such that there is a bijection between the inner supporting rays of the set and the components with respect to which they are inner supporting.*

Proof. Let us prove the lemma by the contradiction. Suppose set $E \subset \mathbb{R}^2$ is open, bounded, having Z-property, and consisting of three components E_j, $j = 1, 2, 3$. Let $x \in \mathbb{R}^2 \backslash \overline{E}$ be a point of 1-nonsemiconvexity of E. By Lemma 2.16, E has three inner supporting rays ξ_x^i, $i = 1, 2, 3$. Thus, without loss of generality, suppose ξ_x^i is inner supporting for E with respect to E_i, $i = 1, 2, 3$.

Let $y_i \in \xi_x^i \cap \partial E_i$ be the nearest point of $\xi_x^i \cap \partial E_i \neq \varnothing$ to x, Figure 2.10a). Since neither of the rays ξ_x^i, $i = 1, 2, 3$, intersect the respective component E_i for which it is supporting, it follows from smoothness of ∂E that each ray ξ_x^i is tangent to ∂E_i at the point y_i.

By Lemma 2.8, each component E_i, $i = 1, 2, 3$, is contained in the respective sector between two supporting rays of $\angle \alpha_i > 0$, $i = 1, 2, 3$. Without loss of generality, let us consider the polar coordinate system (φ, ρ), where point x is the pole, inner supporting ray ξ_x^1 is the polar axis, and $\eta_x(\varphi)$ is a ray starting at x and generating an angle φ with ray ξ_x^1, see Figure 2.10. Since E is weakly 1-semiconvex, there exists a ray γ_{y_1} not intersecting E. As ∂E_1 is smooth, the ray γ_{y_1} should lie above the straight that contains ray ξ_x^1. Let us chose a positive angular coordinate determined by a ray starting at x and intersecting ray γ_{y_1}. In this coordinate system, let angle ϕ be such that $\eta_x(\phi) \parallel \gamma_{y_1}$, $0 < \phi < \pi$.

By Lemma 2.12, for the ray γ_{y_1} there exists an inner supporting ray ξ_x satisfying conditions 1)-3) of that lemma. Since E has only three inner supporting rays, one of rays ξ_x^i, $i = 1, 2, 3$, coincides with ξ_x.

Since ∂E is smooth, the triangle xy_1c, where $c = \xi_x \cap \gamma_{y_1}$, does not degenerate into the interval xy_1. Then $\xi_x \neq \xi_x^1$, which gives $\xi_x = \xi_x^2$ or $\xi_x = \xi_x^3$.

Without loss of generality suppose that $\xi_x = \xi_x^2$. Then by Lemma 2.12, $\xi_x^2 = \eta_x(\varphi_2)$, $0 < \varphi_2 < \phi < \pi$, and

1) $\xi_x^2 \cap \gamma_{y_1} = c(\varphi_2)$;

2) $y_2 \in xc(\varphi_2)$;

3) $E \cap \operatorname{Int}(\triangle xy_1c(\varphi_2)) = \varnothing$.

This implies that, in a certain coordinate system, the component E_2 is contained in the angle that corresponds to the interval $(\varphi_2, \varphi_2+\alpha_2)$ between its two supporting rays. Then E_1 is contained in the angle $(2\pi - \alpha_1, 2\pi)$.

If $\varphi_2 + \alpha_2 > 2\pi - \alpha_1$, then $(\varphi_2, \varphi_2 + \alpha_2) \cap (2\pi - \alpha_1, 2\pi) \neq \varnothing$, see Figure 2.10. Hence neither of these intervals is completely contained in the other one, otherwise one of the components E_1, E_2 would be projected on the other. Consider the rays τ_x^1, τ_x^2 from the proof of Lemma 2.16. They are contained in the angle $(\varphi_2, \varphi_2 + \alpha_2) \cup (2\pi - \alpha_1, 2\pi) = (\varphi_2, 2\pi)$.

Let $[\varphi_3, \varphi_4] \subset (\varphi_2, 2\pi)$ be the angle between rays τ_x^1, τ_x^2. The component E_3 is not contained in $[\varphi_3, \varphi_4]$, otherwise E_3 would be projected on $E_1 \cup E_2$. Thus, by the proof of Lemma 2.16, the third inner supporting ray ξ_x^3 is contained in $[\varphi_3, \varphi_4]$ and is inner supporting for E with respect to one of the components E_1, E_2. This contradicts to our assumption that ξ_x^3 is inner supporting with respect to the component E_3.

If $\varphi_2+\alpha_2 \leqslant 2\pi-\alpha_1$, then either angle $[\varphi_2+\alpha_2, 2\pi-\alpha_1]$ or $[0, \varphi_2]$ does not contain points of E_3 and therefore points of E, otherwise, component E_1 would be projected on E_3. But this gives that we can draw a ray starting at x and not intersecting set E, which contradicts to non 1-semiconvexity of E. Thus the assumption is wrong. Lemma is proved. □

Lemma 2.19 also fails for sets with non-smooth boundary, see Figure 2.3b).

3. Proof of Theorem 1.8

By Theorem 1.6, the set E does not consist of one or two components. Due to Example 2.4, it suffices to prove that the set does not consist of three components as well. Let us prove this by contradiction. So, suppose E consists of three connected components E_i, $i = 1, 2, 3$.

Since E is not 1-semiconvex, it follows that there exists a point of 1-nonsemiconvexity $x \in \mathbb{R}^2 \setminus \overline{E}$ of E (since E is a weakly 1-semiconvex set, we do not consider points of ∂E as points of 1-nonsemiconvexity of E).

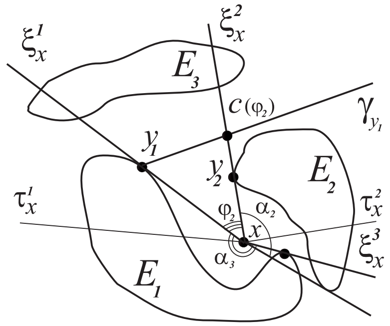

FIGURE 2.10.

By Lemma 2.16, E has three inner supporting rays ξ_x^i, $i = 1, 2, 3$, starting at x, and each of them is inner supporting for E with respect to a single component of the set. In other words, there is a function

$$f\colon \{\xi_x^1, \xi_x^2, \xi_x^3\} \longrightarrow \{E_1, E_2, E_3\}.$$

Since by Lemma 2.8 each component has at most two supporting rays, there possible exactly two cases:

I) f is a bijection, i.e., each ray ξ_x^k, $i = 1, 2, 3$, is inner supporting for E with respect to one and only one component of E, Figure 3.1 a;

II) there exist ξ_x^k, ξ_x^l, $k, l = 1, 2, 3$, $k \neq l$, such that $f(\xi_x^k) = f(\xi_x^l)$, i.e., two rays of ξ_x^k, $i = 1, 2, 3$, are inner supporting for E with respect to the same its component and the third is inner supporting for E with respect to one of the other two components, Figure 3.1b).

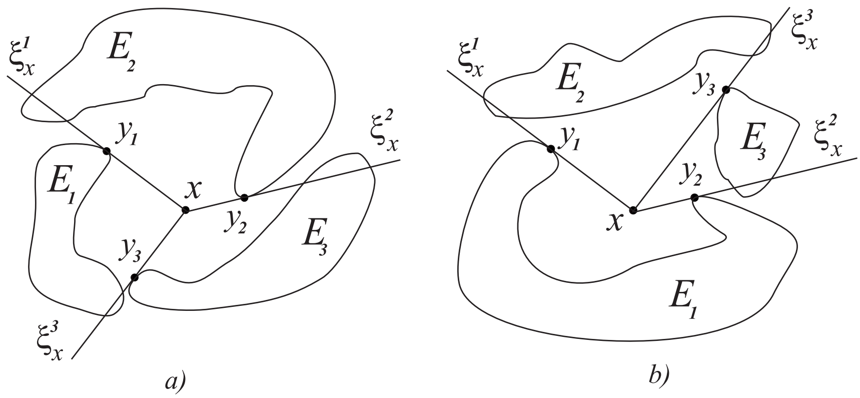

FIGURE 3.1.

Then the case II) is not possible due to Lemma 2.18 while the case I) is not possible by Lemma 2.19. Therefore the assumption that the set consists of three components is incorrect. Example 2.4 completes the proof of Theorem 1.8. □

Corollary 3.1. *Suppose an open bounded subset $E \subset \mathbb{R}^2$ has Z-property and consists of four components with smooth boundary. Then none of its components is projected on the union of the others from a point of 1-nonsemiconvexity of E.*

Proof. The proof is similar to the proof of Lemma 2.15. Let E_i, $i = \overline{1,4}$, be components of E, and $x \in \mathbb{R}^2 \backslash \overline{E}$ be a point of 1-nonsemiconvexity of E. Without loss of generality, suppose that E_1 is projected from x on $\overset{4}{\underset{i=2}{\cup}} E_i$. Then the set, consisting only of components E_i, $i = 2, 3, 4$, has Z-property, which contradicts to Theorem 1.8. □

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