Proceedings of the
International Geometry Center
Vol. 10, no. 2 (2017) pp. 8–23

Warped product semi-slant submanifolds in locally conformal Kaehler manifolds

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Abstract. In 1994, in [13], N. Papaghiuc introduced the notion of semi-slant submanifold in a Hermitian manifold which is a generalization of CR- and slant-submanifolds. In particular, he considered this submanifold in Kaehlerian manifolds, [13]. Then, in 2007, V. A. Khan and M. A. Khan considered this submanifold in a nearly Kaehler manifold and obtained interesting results, [11]. Recently, we considered semi-slant submanifolds in a locally conformal Kaehler manifold and gave a necessary and sufficient conditions for two distributions (holomorphic and slant) to be integrable. Moreover, we considered these submanifolds in a locally conformal Kaehler space form, [4]. In this paper, we define 2-kind warped product semi-slant submanifolds in a locally conformal Kaehler manifold and consider some properties of these submanifolds.

1. Introduction

A Hermitian manifold $\widetilde{M}$ with structure $(J, \widetilde{g})$ is called a *locally conformal Kaehler* (an *l.c.K.-*) *manifold* if each point $x \in \widetilde{M}$ has an open neighbourhood U with differentiable function $\rho : U \to \mathcal{R}$ such that $\widetilde{g}^* = e^{-2\rho}\,\widetilde{g}_{|U}$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is the almost complex structure, $\widetilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\widetilde{g}^*$ and $\mathcal{R}$ is a real number space, [14]. A typical example of an l.c.K.-manifold which is not Kaehlerian is Hopf manifold, [14].

Then we know the following statement, see [10]:

2010 Mathematics Subject Classification: 53C40
Keywords: Locally conformal Kaehler manifold, slant distribution, semi-slant submanifold, warped product semi-slant submanifold
DOI: http://dx.doi.org/10.15673/tmgc.v10i2.650

Proposition 1.1. *A Hermitian manifold $\widetilde{M}(J,\widetilde{g})$ is l.c.K. if and only if there exists a global closed 1-form α which is called Lee form satisfying*

$$(\widetilde{\nabla}_V J)U = -\widetilde{g}(\alpha^\sharp, U)JV + \widetilde{g}(V,U)\beta^\sharp + \widetilde{g}(JV,U)\alpha^\sharp - \widetilde{g}(\beta^\sharp, U)V \tag{1.1}$$

for any $V, U \in T\widetilde{M}$, where $\widetilde{\nabla}$ denotes the covariant differentiation with respect to $\widetilde{g}$, $\alpha^\sharp$ is the dual vector field of α, the 1 form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β and $T\widetilde{M}$ is the tangent bundle of $\widetilde{M}$.

An l.c.K.-manifold $\widetilde{M}(J,\widetilde{g},\alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature. Then, [10], the Riemannian curvature tensor $\widetilde{R}$ with respect to $\widetilde{g}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by the following formulas:

$$\begin{aligned}
4\widetilde{R}(X,Y,Z,W) = c\big\{&\widetilde{g}(X,W)\,\widetilde{g}(Y,Z) - \widetilde{g}(X,Z)\,\widetilde{g}(Y,W)+\\
&+\widetilde{g}(JX,W)\,\widetilde{g}(JY,Z) - \widetilde{g}(JX,Z)\,\widetilde{g}(JY,W)-\\
&\qquad\qquad - 2\widetilde{g}(JX,Y)\,\widetilde{g}(JZ,W)\big\}+\\
&+3\big\{P(X,W)\,\widetilde{g}(Y,Z) - P(X,Z)\widetilde{g}(Y,W)+\\
&\qquad + \widetilde{g}(X,W)\,P(Y,Z) - \widetilde{g}(X,Z)\,P(Y,W)\big\}-\\
&-\widetilde{P}(X,W)\,\widetilde{g}(JY,Z) + \widetilde{P}(X,Z)\,\widetilde{g}(JY,W)-\\
&-\widetilde{g}(JX,W)\,\widetilde{P}(Y,Z) + \widetilde{g}(JX,Z)\,\widetilde{P}(Y,W)+\\
&+2\big\{\widetilde{P}(X,Y)\,\widetilde{g}(JZ,W) + \widetilde{g}(JX,Y)\,\widetilde{P}(Z,W)\big\}
\end{aligned} \tag{1.2}$$

for any $X, Y, Z, W \in T\widetilde{M}$, where P and $\widetilde{P}$ are respectively defined by

$$P(X,Y) = -(\widetilde{\nabla}_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\|\alpha\|^2\,\widetilde{g}(X,Y), \tag{1.3}$$

and

$$\widetilde{P}(X,Y) = P(JX,Y) \tag{1.4}$$

for any $X, Y \in T\widetilde{M}$, where $\|\alpha\|$ is the length of the Lee form α.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. Then we put $M = M_1 \times M_2$ be the product manifold of M_1 and M_2. For a positive differentiable function f on M_2, we define a Riemannian metric tensor g on M as

$$g(U,V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V) \tag{1.5}$$

for any $U, V \in TM$, where π_1 (resp. π_2) denotes the projection operator of M to M_1 (resp. M_2) and π_{1*} (resp π_{2*}) is the differential of π_1 (resp. π_2). Then the Riemannian manifold M is called a *warped product manifold of M_1 and M_2 with the warping function f* and we write it $M_1 \otimes_f M_2$, [12].

Let ∇, ∇_1 and ∇_2 be the covariant differentiation with respect to g, g_1 and g_2, respectively. Then, we have from (1.5)

$$\begin{aligned} \nabla_X Y &= \nabla_{1X} Y - f^2 e^{f^2} g_1(X,Y)(d_2 \log f)^*, \\ \nabla_X Z &= \nabla_Z X = f^2 (Z \log f) X, \\ \nabla_Z W &= \nabla_{2Z} W \end{aligned} \tag{1.6}$$

for any $X, Y \in TM_1$ and $Z, W \in TM_2$, where $d_2 \log f$ means the differential of $\log f$ and $(d_2 \log f)^*$ is the dual vector field of $d_2 \log f$.

By virtue of (1.6), the curvature tensor R with respect to g is written as

$$\begin{aligned} R(X_1,X_2,X_3,X_4) &= e^{f^2}[R_1(X_1,X_2,X_3,X_4) - \\ &\quad - f^4 e^{f^2} \|\nabla_2 \log f\|^2 \{g_1(X_1,X_4) g_1(X_2,X_3) - \\ &\quad - g_1(X_1,X_3) g_1(X_2,X_3)\}], \\ R(X_1,Z_1,Z_2,X_2) &= -f^2 e^{f^2} \big\{(2+f^2)(Z_1 \log f)(Z_2 \log f) + \\ &\quad + \nabla_{2Z_1} \nabla_{2Z_2} \log f \big\} \, g_1(X_1,X_2), \\ R(Z_1,Z_2,Z_3,Z_4) &= R_2(Z_1,Z_2,Z_3,Z_4), \\ Other &= 0, \end{aligned} \tag{1.7}$$

and the Ricci tensor ρ with respect to g is separated as

$$\begin{aligned} &\rho(X_1,X_2) = \rho_1(X_1,X_2) - \\ &\qquad - f^2 e^{f^2} \big\{(2+n_1 f^2)\|\nabla_2 \log f\|^2 + \delta_2 d_2\big\} \, g_1(X_1,X_2), \\ &\rho(X_1,Z_1) = 0, \\ &\rho(Z_1,Z_2) = \rho_2(Z_1,Z_2) - \\ &\qquad - n_1 f^2 \big\{(2+f^2)(\nabla_{2Z_1} \log f)(\nabla_{2Z_2} \log f) + \nabla_{2Z_1} \nabla_{2Z_2} \log f \big\}, \end{aligned} \tag{1.8}$$

for any $X_1, X_2 \in TM_1$ and $Z_1, Z_2 \in TM_2$, where R_1 (resp. R_2) is the Riemannian curvature tensor with respect to g_1 (resp. g_2) and ρ_1 (resp. ρ_2) is the Ricci tensor with respect to g_1 (resp. g_2), d_2 (resp. δ_2) means the differential (resp. codifferential) with respect to g_2, $\|\nabla_2 \log f\|$ is the length of $\nabla_2 \log f$ with respect to g_2 and $n_1 = \dim M_1$.

Finally, if we respectively put τ, τ_1 and τ_2 the scalar curvature with respect to g, g_1 and g_2, then from (1.8), we can easily have

$$\tau = e^{f^2} \tau_1 + \tau_2 - (n_1 - 1) n_1 f^4 \|\nabla_2 \log f\|^2. \tag{1.9}$$

2. Semi-slant-submanifolds in an almost Hermitian manifold

In general, between a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ and its Riemannian submanifold (M, g), we know the Gauss and Weingarten formulas

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \qquad \widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}{}_X N \tag{2.1}$$

for any $X, Y \in TM$ and $N \in T^{\perp}M$, where ∇ is the covariant differentiation with respect to g, σ is the second fundamental form and A_N is the shape operator with respect to N and $\nabla^{\perp}$ is the normal connection, [6]. The second fundamental form σ and the shape operator A are related by $\widetilde{g}(A_N Y, X) = \widetilde{g}(\sigma(Y, X), N)$ for any $Y, X \in TM$ and $N \in T^{\perp}M$.

The Gauss equation is given by

$$\begin{aligned}\widetilde{R}(U, V, W, Z) = R(U, V, W, Z) &+ \widetilde{g}(\sigma(U, Z), \sigma(V, W)\\ &- \widetilde{g}(\sigma(U, W), \sigma(V, Z)),\end{aligned} \tag{2.2}$$

for any $U, V, W, Z \in TM$, [6].

A submanifold M is said to be totally geodesic, if the second fundamental form σ identically vanishes, [6].

We recall a warped product submanifold in a Riemannian manifold.

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold. A submanifold (M, g) is called a *warped product submanifold* of $\widetilde{M}$ if it satisfies

(i) M is a product manifold of 2 submanifolds M_1 and M_2,
(ii) two submanifolds are orthogonal with respect to $\widetilde{g}$,
(iii) for certain Riemannian metric g_1 in M_1, g_2 in M_2 and a certain positive differentiable function f in M_2, the metric tensor g is defined by
$$g(U, V) = e^{f^2} g_1(\pi_{1*}U, \pi_{1*}V) + g_2(\pi_{2*}U, \pi_{2*}V) \tag{2.3}$$
for any $U, V \in TM$ is the induced metric of $\widetilde{g}$, [5].

By virtue of (1.7) and (2.3) the Riemannian curvature $\widetilde{R}$ is separated as

$$\begin{aligned}\widetilde{R}(X_1, X_2, X_3, X_4) = e^{f^2}\{&R_1(X_1, X_2, X_3, X_4)\\ &- f^4 e^{f^2} \|\log f\|^2 g_1(X_1, X_4) g_1(X_2, X_3) - g_1(X_1, X_3) g_1(X_2, X_4)\}\\ &+ \widetilde{g}(\sigma(X_1, X_4), \sigma(X_2, X_3)) - \widetilde{g}(\sigma(X_1, X_3), \sigma(X_2, X_4)),\end{aligned}$$

$$\widetilde{R}(X_1, X_2, X_3, Z_1) = \widetilde{g}(\sigma(X_1, Z_1), \sigma(X_2, X_3)) - \widetilde{g}(\sigma(X_1, X_3), \sigma(X_2, Z_1)),$$

$$\widetilde{R}(X_1, X_2, Z_1, Z_2) = \widetilde{g}(\sigma(X_1, Z_2), \sigma(X_2, Z_1)) - \widetilde{g}(\sigma(X_1, Z_1), \sigma(X_2, Z_2)),$$

$$\begin{aligned}\widetilde{R}(X_1, Z_1, Z_2, X_2) = -f^2 e^{f^2}\{&(2 + f^2)(Z_1 \log f)(Z_2 \log f)\\ &+ \nabla_{2Z_2} \nabla_{2Z_1} \log f\} g_1(X_1, X_2) + \widetilde{g}(\sigma(X_1, X_2), \sigma(Z_1, Z_2))\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&- \widetilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, X_2)),\\
&\widetilde{R}(X_1, Z_1, Z_2, Z_3) = \widetilde{g}(\sigma(X_1, Z_3), \sigma(Z_1, Z_2)) - \widetilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, Z_3)),\\
&\widetilde{R}(Z_1, Z_2, Z_3, Z_4) = R_2(Z_1, Z_2, Z_3, Z_4) +\\
&\qquad\qquad + \widetilde{g}(\sigma(Z_1, Z_4), \sigma(Z_2, Z_3)) - \widetilde{g}(\sigma(Z_1, Z_3), \sigma(Z_2, Z_4)),
\end{aligned}
$$

for any $X_1, X_2, X_3, X_4 \in TM_1$ and $Z_1, Z_2, Z_3, Z_4 \in TM_2$, where R_1 (resp. R_2) is the Riemannian curvature tensor with respect to g_1 (resp. g_2).

For a vector field $U \in TM$, the angle between JU and TM is called the *Wirtingar angle* of U.

A differentiable distribution $\mathcal{D}^\theta : x \to \mathcal{D}^\theta_x$ on M is said to be a *slant* one if for each $U_x \in \mathcal{D}^\theta_x$, the Wirtingar angle of U_x is constant ($= \theta$) for any $x \in M$. In this case, the Wirtingar angle is said to be the *slant angle*. In particular, if TM is slant, then the submanifold is called a *slant* one, [9]. A slant submanifold is *holomorphic* (resp. *totally real*) if its slant angle $\theta = 0$ (resp. $\theta = \frac{\pi}{2}$). A slant submanifold is said to be *proper* if it is neither holomorphic nor totally real.

A submanifold M in $\widetilde{M}$ is called a *semi-slant submanifold* if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_xM$ on M satisfying the following conditions:

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and
(ii) the complementary orthogonal distribution $\mathcal{D}^\theta : x \to \mathcal{D}^\theta_x \subset T_xM$ is slant with slant angle θ, where T_xM is the tangent vector space of M at x, [13].

Remark 2.1. A semi-slant submanifold is a CR-submanifold if the slant angle is equal to $\frac{\pi}{2}$, [1], [2], [3], [7], [8], etc.

A semi-slant submanifold M is said to be *proper* if it is neither CR-, holomorphic, nor totally real.

In a submanifold M of an almost Hermitian manifold $\widetilde{M}(J, \widetilde{g})$, for any $U \in TM$ and $\xi \in T^\perp M$, we write

$$JU = TU + FU, \qquad J\xi = t\xi + h\xi, \tag{2.5}$$

where TU (resp. FU) means the tangential (resp. normal) component of JU and $t\xi$ (resp. $h\xi$) means the tangential (resp. normal) component of $J\xi$.

For a semi-slant submanifold M of an almost Hermitian manifold $\widetilde{M}$, the tangent bundle TM and the normal bundle $T^\perp M$ of M are decomposed as

$$TM = \mathcal{D} \oplus \mathcal{D}^\theta, \qquad T^\perp M7 = F\mathcal{D}^\theta \oplus \nu, \tag{2.6}$$

where ν denotes the orthogonal complementary distribution of $F\mathcal{D}^\theta$ in $T^\perp M$.

Further, in a semi-slant submanifold M we write

$$U = T_1U + T_2U, \tag{2.7}$$

for any $U \in TM$, where T_1U (resp. T_2U) denotes the $\mathcal{D}$ (resp. $\mathcal{D}^\theta$) component of U.

By virtue of (2.7) and (2.7), we can write

$$JU = JT_1U + TT_2U + FT_2U, \tag{2.8}$$

where $JT_1U \in \mathcal{D}$, $TT_2U \in \mathcal{D}^\theta$ and $FT_2U \in F\mathcal{D}^\theta \subset T^\perp M$. Thus if we put

$$QU = JT_1U + TT_2U \tag{2.9}$$

for any $U \in TM$, then Q is an automorphism on TM.

The covariant differentiation $\bar{\nabla}$ of T_1, T_2, T, F, t and h are defined as

$$\begin{aligned}
(\bar{\nabla}_U T_1)V &= \nabla_U(T_1V) - T_1\nabla_U V,\\
(\bar{\nabla}_U T_2)V &= \nabla_U(T_2V) - T_2\nabla_U V,\\
(\bar{\nabla}_U T)V &= \nabla_U(TV) - T\nabla_U V,\\
(\bar{\nabla}_U F)V &= \nabla^\perp_U(FV) - F\nabla_U V,\\
(\bar{\nabla}_U t)\xi &= \nabla_U(t\xi) - t\nabla^\perp_U \xi,\\
(\bar{\nabla}_U h)\xi &= \nabla^\perp_U(h\xi) - h\nabla^\perp_U \xi
\end{aligned} \tag{2.10}$$

for any $U, V \in TM$ and $\xi \in T^\perp M$.

Moreover, if we define the covariant differentiation $\bar{\nabla}$ of Q

$$(\bar{\nabla}_U Q)V = \nabla_U(QV) - Q\nabla_U V \tag{2.11}$$

for any $U, V \in TM$, then using (2.10), we have

$$\begin{aligned}
(\bar{\nabla}_U Q)V = (\widetilde{\nabla}_U J)T_1V + J(\bar{\nabla}_U T_1)V + (\bar{\nabla}_U T)(T_2V)\\
+ T(\bar{\nabla}_U T_2)V + J\sigma(U, T_1V) - \sigma(U, JT_1V)
\end{aligned} \tag{2.12}$$

for any $U, V \in TM$. In particular, for any $X, Y \in \mathcal{D}$, the equation (2.12) is written as

$$(\bar{\nabla}_X Q)Y = (\widetilde{\nabla}_X J)Y + FT_2\nabla_X Y + t\sigma(X, Y) + h\sigma(X, Y) - \sigma(X, TY). \tag{2.13}$$

Now, for $U, V \in TM$, we write

$$(\widetilde{\nabla}_U J)V = \mathcal{P}_U V + \mathcal{Q}_U V, \tag{2.14}$$

where $\mathcal{P}_U V$ (resp. $\mathcal{Q}_U V$) denotes the tangential (resp. normal) part of $(\widetilde{\nabla}_U J)V$.

3. Semi-slant submanifolds in an l.c.K.-manifold

Let M be a semi-slant submanifold of an l.c.K.-manifold $\widetilde{M}(J,\widetilde{g},\alpha)$. Then we have from (1.1) and (2.14)

$$\begin{aligned}\mathcal{P}_U V &= -\widetilde{g}(\alpha_1^\sharp, V)TU + \widetilde{g}(U,V)(T\alpha_1^\sharp + t\alpha_2^\sharp) + \widetilde{g}(TU,V)\alpha_1^\sharp \\ &\quad - \widetilde{g}(T\alpha_1^\sharp + t\alpha_2^\sharp, V)U, \\ \mathcal{Q}_U V &= -\widetilde{g}(\alpha_1^\sharp, V)FU + \widetilde{g}(U,V)(F\alpha_1^\sharp + h\alpha_2^\sharp) + \widetilde{g}(TU,V)\alpha_2^\sharp,\end{aligned} \tag{3.1}$$

where $\alpha_1^\sharp$ (resp. $\alpha_2^\sharp$) means the tangential (resp. normal) component of $\alpha^\sharp$.

In a semi-slant submanifold in an l.c.K.-manifold, we have from $(3.1)_2$

$$\mathcal{Q}_X Y - \mathcal{Q}_Y X = 2\widetilde{g}(TX,Y)\alpha_2^\sharp \tag{3.2}$$

for any $X, Y \in \mathcal{D}$.

Using theorems of V. A. Khan and M. A. Khan on integrability of the distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ of a semi-slant submanifold in an almost Hermitian manifold, in [4], we proved

Proposition 3.1. (I) *The holomorphic distribution $\mathcal{D}$ of a semi-slant submanifold M in an l.c.K.-manifold $\widetilde{M}(J,\widetilde{g},\alpha)$ is integrable if and only if*

$$\sigma(X,TY) - \sigma(Y,TX) = \mathcal{Q}_X Y - \mathcal{Q}_Y X = 2\widetilde{g}(TX,Y)\alpha_2^\sharp \tag{3.3}$$

for any $X, Y \in \mathcal{D}$.

(II) *The slant distribution $\mathcal{D}^\theta$ of a semi-slant submanifold M in an locally conformal Kaehler manifold $\widetilde{M}(J,\widetilde{g},\alpha)$ is integrable if and only if*

$$\begin{aligned}T_1\big(&\nabla_Z TW - \nabla_W TZ + A_{FZ}W - A_{FW}Z + \\ &+ \widetilde{g}({\alpha_1}^\sharp, W)TZ - \widetilde{g}({\alpha_1}^\sharp, Z)TW + 2\widetilde{g}(TW,Z)\alpha_1^\sharp\big) = 0\end{aligned} \tag{3.4}$$

or equivalently

$$\begin{aligned}T_1\big\{&(\bar{\nabla}_Z T)W - (\bar{\nabla}_W T)Z + T[Z,W] + A_{FZ}W - A_{FW}Z + \\ &+ \widetilde{g}({\alpha_1}^\sharp, W)TZ - \widetilde{g}({\alpha_1}^\sharp, Z)TW + 2\widetilde{g}(TW,Z)\alpha_1^\sharp\big\} = 0\end{aligned} \tag{3.5}$$

for any $Z, W \in \mathcal{D}^\theta$.

4. Warped product semi-slant submanifolds in l.c.K.-manifolds

Let $\mathcal{D}$ and $\mathcal{D}^\theta$ be two integrable distributions on a semi-slant submanifold M of an l.c.K.-manifold $\widetilde{M}(J,\widetilde{g},\alpha)$. Then (3.3) and (3.4) hold true. Let also $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$) be the maximal integral submanifold of $\mathcal{D}$ (resp. $\mathcal{D}^\theta$). Then M is a product manifold of $M_{\mathcal{D}}$ and $M_{\mathcal{D}^\theta}$, that is,

$$M = M_{\mathcal{D}} \otimes M_{\mathcal{D}^\theta}. \tag{4.1}$$

We call the submanifold $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$) the *holomorphic* (resp. *slant*) *component* of M.

We define the following two type warped product submanifolds

$$M_1 := M_{\mathcal{D}} \otimes_{f_1} M_{\mathcal{D}^\theta} \tag{4.2}$$

for a certain differentiable function f_1 on $M_{\mathcal{D}^\theta}$ and

$$M_2 := M_{\mathcal{D}^\theta} \otimes_{f_2} M_{\mathcal{D}} \tag{4.3}$$

for a certain differentiable function f_2 on $M_{\mathcal{D}}$. We say that M_1 (resp. M_2) *the first* (resp. *second*) *type warped product semi-slant submanifold* of an l.c.K.-manifold.

In this paper, we mainly consider the first type warped product semi-slant submanifold.

Let M be the first type warped product semi-slant submanifold in an l.c.K.-manifold $\widetilde{M}$. Then the induced metric tensor g on M from $\widetilde{M}$ is given by

$$g(U,V) = e^{f_1^2} g_{\mathcal{D}}(\pi_{\mathcal{D}} * U, \pi_{\mathcal{D}} * V) + g_{\mathcal{D}^\theta}(\pi_{\mathcal{D}^\theta} * U, \pi_{\mathcal{D}^\theta} * V) \tag{4.4}$$

for any $U, V \in TM$, where $g_{\mathcal{D}}$ (resp. $g_{\mathcal{D}^\theta}$) denotes the Riemannian metric on $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$), $\pi_{\mathcal{D}}$ (resp. $\pi_{\mathcal{D}^\theta}$) is the projection operator of M to $M_{\mathcal{D}}$ (resp. $M_{\mathcal{D}^\theta}$) and f_1 is a certain positive differentiable function on $M_{\mathcal{D}^\theta}$. Now, we denote by $\widetilde{\nabla}$, ∇, $\nabla^{\mathcal{D}}$ and $\nabla^{\mathcal{D}^\theta}$ the covariant differentiations with respect to $\widetilde{g}$, g, $g_{\mathcal{D}}$ and $g_{\mathcal{D}^\theta}$, respectively. Since we have from (1.6)

$$\begin{aligned}
&\nabla_X Y = {\nabla^{\mathcal{D}}}_X Y - f_1^2 e^{f_1^2} (d_1 \log f_1)^* g_{\mathcal{D}}(X,Y),\\
&\nabla_X Z = \nabla_Z X = f_1^2 (Z \log f_1) X,\\
&\nabla_Z W = {\nabla^{\mathcal{D}^\theta}}_Z W
\end{aligned} \tag{4.5}$$

for any $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$, where we put $(d_1 \log f_1)$ is the differential of $\log f_1$ with respect to $g_{\mathcal{D}^\theta}$.

Using Gauss formula and the above equation, we obtain

$$\begin{aligned}
&\widetilde{\nabla}_X Y = {\nabla^{\mathcal{D}}}_Y X - f_1^2 e^{f_1^2} (d_1 \log f_1)^* g_{\mathcal{D}}(X,Y) + \sigma(X,Y),\\
&\widetilde{\nabla}_X Z = \widetilde{\nabla}_Z X = f_1^2 (Z \log f_1) X + \sigma(X,Z),\\
&\widetilde{\nabla}_Z W = {\nabla^{\mathcal{D}^\theta}}_Z W + \sigma(Z,W)
\end{aligned} \tag{4.6}$$

for any $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\theta$.

Due to (4.6) between the Riemannian curvature tensors

- $R(U_1, U_2, U_3, U_4)$ with respect to g,
- $R^{\mathcal{D}}(X_1, X_2, X_3, X_4)$ with respect to $g_{\mathcal{D}}$, and
- $R^{\mathcal{D}^\theta}(Z_1, Z_2, Z_3, Z_4)$ with respect to $g_{\mathcal{D}^\theta}$,

we know the following relations:

$$
\begin{aligned}
R(X_1,X_2,X_3,X_4) = e^{f_1^2}[R^{\mathcal{D}}&(X_1,X_2,X_3,X_4) \\
&- f_1^4 e^{f_1^2}\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2\{g_{\mathcal{D}}(X_1,X_4)g_{\mathcal{D}}(X_2,X_3) \\
&- g_{\mathcal{D}}(X_1,X_3)g_{\mathcal{D}}(X_2,X_4)\}], \\
R(X_1,Z_1,Z_2,X_2) = -f_1^2 e^{f_1^2}&\big\{(2+f_1^2)(Z_1\log f_1)(Z_2\log f_1) \\
&+ \nabla^{\mathcal{D}^\theta}{}_{Z_1}\nabla^{\mathcal{D}^\theta}{}_{Z_2}\log f_1\big\}g_{\mathcal{D}}(X_1,X_2), \\
R(Z_1,Z_2,Z_3,Z_4) = R^{\mathcal{D}^\theta}&(Z_1,Z_2,Z_3,Z_4), \\
\text{Others} = 0&,
\end{aligned}
\tag{4.7}
$$

for any $X_1,X_2,X_3,X_4\in\mathcal{D}$ and $Z_1,Z_2,Z_3,Z_4\in\mathcal{D}^\theta$.

By virtue of the above equation and the Gauss equation, we have the following

$$
\begin{aligned}
\widetilde{R}(X_1,X_2,X_3,X_4) &= e^{f_1^2}[R^{\mathcal{D}}(X_1,X_2,X_3,X_4) \\
&\quad - f_1^4 e^{f_1^2}\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2\{g_{\mathcal{D}}(X_1,X_4)g_{\mathcal{D}}(X_2,X_3) \\
&\quad - g_{\mathcal{D}}(X_1,X_3)g_{\mathcal{D}}(X_2,X_4)\}] + \widetilde{g}(\sigma(X_1,X_4),\sigma(X_2,X_3)) \\
&\quad - \widetilde{g}(\sigma(X_1,X_3),\sigma(X_2,X_4)), \\
\widetilde{R}(X_1,X_2,X_3,Z_1) &= \widetilde{g}(\sigma(X_1,Z_1),\sigma(X_2,X_3)) - \\
&\qquad - \widetilde{g}(\sigma(X_1,X_3),\sigma(X_2,Z_1)), \\
\widetilde{R}(X_1,X_2,Z_1,Z_2) &= \widetilde{g}(\sigma(X_1,Z_2),\sigma(X_2,Z_1)) - \\
&\qquad - \widetilde{g}(\sigma(X_1,Z_1),\sigma(X_2,Z_2)), \\
\widetilde{R}(X_1,Z_1,Z_2,Z_3) &= \widetilde{g}(\sigma(X_1,Z_3),\sigma(Z_1,Z_2)) - \\
&\qquad - \widetilde{g}(\sigma(X_1,Z_2),\sigma(Z_1,Z_3)), \\
\widetilde{R}(X_1,Z_1,Z_2,X_2) &= -f_1^2 e^{f_1^2}\{(2+f_1^2)(Z_1\log f_1)(Z_2\log f_1) \\
&\quad + \nabla^{\mathcal{D}^\theta}{}_{Z_1}\nabla^{\mathcal{D}^\theta}{}_{Z_2}\log f_1\}g_{\mathcal{D}}(X_1,X_2) \\
&\quad + \widetilde{g}(\sigma(X_1,X_2),\sigma(Z_1,Z_2)) - \widetilde{g}(\sigma(X_1,Z_2),\sigma(Z_1,X_2)), \\
\widetilde{R}(Z_1,Z_2,Z_3,Z_4) &= R^{\mathcal{D}^\theta}(Z_1,Z_2,Z_3,Z_4) + \widetilde{g}(\sigma(Z_1,Z_4),\sigma(Z_2,Z_3)) \\
&\qquad - \widetilde{g}(\sigma(Z_1,Z_3),\sigma(Z_2,Z_4)),
\end{aligned}
\tag{4.8}
$$

for any $X_1,X_2,X_3,X_4\in\mathcal{D}$ and $Z_1,Z_2,Z_3,Z_4\in\mathcal{D}^\theta$.

Next, we assume that our ambient manifold is an l.c.K.-space form. Then the curvature tensor $\widetilde{R}$ satisfies (1.2). Using this, we can separate the curvature tensor $\widetilde{R}$ as

$$\begin{aligned}
4\widetilde{R}(X_1,X_2,X_3,X_4) = c\big\{&\widetilde{g}(X_1,X_4)\,\widetilde{g}(X_2,X_3)-\widetilde{g}(X_1,X_3)\,\widetilde{g}(X_2,X_4)+\\
&+\widetilde{g}(TX_1,X_4)\,\widetilde{g}(TX_2,X_3)-\widetilde{g}(TX_1,X_3)\,\widetilde{g}(TX_2,X_4)-\\
&\qquad\qquad -2\widetilde{g}(TX_1,X_2)\,\widetilde{g}(TX_3,X_4)\big\}+\\
&+3\big\{P(X_1,X_4)\,\widetilde{g}(X_2,X_3)-P(X_1,X_3)\,\widetilde{g}(X_2,X_4)+\\
&+P(X_2,X_3)\,\widetilde{g}(X_1,X_4)-P(X_2,X_4)\,\widetilde{g}(X_1,X)\big\}-\\
&-\widetilde{P}(X_1,X_4)\,\widetilde{g}(TX_2,X_3)+\widetilde{P}(X_1,X_3)\,\widetilde{g}(TX_2,X_4)-\\
&-\widetilde{P}(X_2,X_3)\,\widetilde{g}(TX_1,X_4)+\widetilde{P}(X_2,X_4)\,\widetilde{g}(TX_1,X_3)+\\
&+2\big\{\widetilde{P}(X_1,X_2)\,\widetilde{g}(TX_3,X_4)+\widetilde{P}(X_3,X_4)\,\widetilde{g}(TX_1,X_2)\big\},
\end{aligned}$$

$$\begin{aligned}
4\widetilde{R}(X_1,X_2,X_3,Z_1) = 3\big\{&P(X_1,Z_1)\,\widetilde{g}(X_2,X_3)-P(X_2,Z_1)\,\widetilde{g}(X_1,X_3)\big\}\\
&-\widetilde{P}(X_1,Z_1)\,\widetilde{g}(TX_2,X_3)+\widetilde{P}(X_2,Z_1)\,\widetilde{g}(TX_1,X_3)\\
&\qquad\qquad +2\widetilde{P}(X_3,Z_1)\,\widetilde{g}(TX_1,X_2),
\end{aligned}$$

$$\begin{aligned}
2\widetilde{R}(X_1,X_2,Z_1,Z_2) = &-c\widetilde{g}(TX_1,X_2)\,\widetilde{g}(TZ_1,Z_2)\\
&+\widetilde{P}(X_1,X_2)\,\widetilde{g}(TZ_1,Z_2)+\widetilde{P}(Z_1,Z_2)\,\widetilde{g}(TX_1,X_2),
\end{aligned}\tag{4.9}$$

$$\begin{aligned}
4\widetilde{R}(X_1,Z_1,Z_2,X_2) = c\big\{&\widetilde{g}(X_1,X_2)\,\widetilde{g}(Z_1,Z_2)+\widetilde{g}(TX_1,X_2)\widetilde{g}\,(TZ_1,Z_2)\big\}\\
&+3\big\{P(X_1,X_2)\,\widetilde{g}(Z_1,Z_2)+P(Z_1,Z_2)\,\widetilde{g}(X_1,X_2)\big\}\\
&-\widetilde{P}(X_1,X_2)\,\widetilde{g}(TZ_1,Z_2)-\widetilde{P}(Z_1,Z_2)\,\widetilde{g}(TX_1,X_2),
\end{aligned}$$

$$\begin{aligned}
4\widetilde{R}(X_1,Z_1,Z_2,Z_3) = 3\{&P(X_1,Z_3)\,\widetilde{g}(Z_1,Z_2)-P(X_1,Z_2)\,\widetilde{g}(Z_1,Z_3)\}\\
&-\widetilde{P}(X_1,Z_3)\,\widetilde{g}(TZ_1,Z_2)+\widetilde{P}(X_1,Z_2)\,\widetilde{g}(TZ_1,Z_3)\\
&\qquad\qquad +2\widetilde{P}(X_1,Z_1)\,\widetilde{g}(TZ_2,Z_3),
\end{aligned}$$

$$\begin{aligned}
4\widetilde{R}(Z_1,Z_2,Z_3,Z_4) = c\big\{&\widetilde{g}(Z_1,Z_4)\,\widetilde{g}(Z_2,Z_3)-\widetilde{g}(Z_1,Z_3)\,\widetilde{g}(Z_2,X_Z)+\\
&+\widetilde{g}(TZ_1,Z_4)\,\widetilde{g}(TZ_2,Z_3)-\widetilde{g}(TZ_1,Z_3)\,\widetilde{g}(TZ_2,Z_4)-\\
&\qquad\qquad -2\widetilde{g}(TZ_1,Z_2)\,\widetilde{g}(TZ_3,X_4)\big\}+\\
&+3\big\{P(Z_1,Z_4)\,\widetilde{g}(Z_2,Z_3)-P(Z_1,Z_3)\,\widetilde{g}(Z_2,Z_4)-\\
&\qquad\qquad +P(Z_2,Z_3)\widetilde{g}(Z_1,Z_4)\big\}-\\
&-\widetilde{P}(Z_1,Z_4)\,\widetilde{g}(TZ_2,Z_3)+\widetilde{P}(Z_1,Z_3)\,\widetilde{g}(TZ_2,Z_4)-
\end{aligned}$$

$$
\begin{aligned}
&- \widetilde{P}(Z_2, Z_3)\,\widetilde{g}(TZ_1, Z_4) + \widetilde{P}(Z_2, Z_4)\,\widetilde{g}(TZ_1, Z_3) + \\
&+ 2\big\{\widetilde{P}(Z_1, Z_2)\,\widetilde{g}(TZ_3, Z_4) + \widetilde{P}(Z_3, Z_4)\,\widetilde{g}(TZ_1, Z_2)\big\}
\end{aligned}
$$

for any $X_1, X_2, X_3, X_4 \in \mathcal{D}$ and $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$.

Thus we have from (4.8) and (4.9) that

$$
\begin{aligned}
&4\{\widetilde{g}(\sigma(X_1, X_4), \sigma(X_2, X_3)) - \widetilde{g}(\sigma(X_1, X_3), \sigma(X_2, X_4))\} = \\
&\qquad - 4e^{f_1^2} R^{\mathcal{D}}(X_1, X_2, X_3, X_4) \\
&\qquad + \{c + 4f_1^2 \|\nabla^{\mathcal{D}^\theta} \log f_1\|^2\}\{\widetilde{g}(X_1, X_4)\widetilde{g}(X_2, X_3) \\
&\qquad - \widetilde{g}(X_1, X_3)\widetilde{g}(X_2, X_4)\} + c\{\widetilde{g}(TX_1, X_4)\widetilde{g}(TX_2, X_3) \\
&\qquad - \widetilde{g}(TX_1, X_3)\widetilde{g}(TX_2, X_4) - 2\widetilde{g}(TX_1, X_2)\widetilde{g}(TX_3, X_4)\} \\
&\qquad + 3\{P(X_1, X_4)\widetilde{g}(X_2, X_3) - P(X_1, X_3)\widetilde{g}(X_2, X_4) \\
&\qquad + P(X_2, X_3)\widetilde{g}(X_1, X_4) - P(X_2, X_4)\widetilde{g}(X_1, X_3)\} \\
&\qquad - \widetilde{P}(X_1, X_4)\widetilde{g}(TX_2, X_3) + \widetilde{P}(X_1, X_3)\widetilde{g}(TX_2, X_4) \\
&\qquad - \widetilde{P}(X_2, X_3)\widetilde{g}(TX_1, X_4) + \widetilde{P}(X_2, X_4)\widetilde{g}(TX_1, X_3) \\
&\qquad + 2\{\widetilde{P}(X_1, X_2)\widetilde{g}(TX_3, X_4) + \widetilde{P}(X_3, X_4)\widetilde{g}(TX_1, X_2)\},
\end{aligned}
$$

$$
\begin{aligned}
&4\{\widetilde{g}(\sigma(X_1, Z_1), \sigma(X_2, X_3)) - \widetilde{g}(\sigma(X_1, X_3), \sigma(X_2, Z_1))\} = \\
&\qquad 3\{P(X_1, Z_1)\widetilde{g}(X_2, X_3) - P(X_2, Z_1)\widetilde{g}(X_1, X_3)\} \qquad (4.10) \\
&\qquad - \widetilde{P}(X_1, Z_1)\widetilde{g}(TX_2, X_3) + \widetilde{P}(X_2, Z_1)\widetilde{g}(TX_1, X_3) \\
&\qquad + 2\widetilde{P}(X_3, Z_1)\widetilde{g}(TX_1, X_2),
\end{aligned}
$$

$$
\begin{aligned}
&2\{\widetilde{g}(\sigma(X_1, Z_2), \sigma(X_2, Z_1)) - \widetilde{g}(\sigma(X_1, Z_1), \sigma(X_2, Z_2))\} = \\
&\qquad - c\widetilde{g}(TX_1, X_2)\widetilde{g}(TZ_1, Z_2) + \widetilde{P}(X_1, X_2)\widetilde{g}(TZ_1, Z_2) \\
&\qquad + \widetilde{P}(Z_1, Z_2)\widetilde{g}(TX_1, X_2),
\end{aligned}
$$

$$
\begin{aligned}
&4\{\widetilde{g}(\sigma(X_1, X_2), \sigma(Z_1, Z_2)) - \widetilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, X_2))\} = \\
&\qquad 4f_1^2 e^{f_1^2}\{(2 + f_1^2)(Z_1 \log f_1)(Z_2 \log f_1) \\
&\qquad + \nabla^{\mathcal{D}^\theta}{}_{Z_1} \nabla^{\mathcal{D}^\theta}{}_{Z_2} \log f_1\} g_{\mathcal{D}}(X_1, X_2) \\
&\qquad + c\{\widetilde{g}(X_1, X_2)\widetilde{g}(Z_1, Z_2) + \widetilde{g}(TX_1, X_2)\widetilde{g}(TZ_1, Z_2)\} \\
&\qquad + 3\{P(X_1, X_2)\widetilde{g}(Z_1, Z_2) + P(Z_1, Z_2)\widetilde{g}(X_1, X_2)\} \\
&\qquad - \widetilde{P}(X_1, X_2)\widetilde{g}(TZ_1, Z_2) - \widetilde{P}(Z_1, Z_2)\widetilde{g}(TX_1, X_2),
\end{aligned}
$$

$$
\begin{aligned}
&4\{\widetilde{g}(\sigma(X_1, Z_3), \sigma(Z_1, Z_2)) - \widetilde{g}(\sigma(X_1, Z_2), \sigma(Z_1, Z_3))\} = \\
&\qquad 3\{P(X_1, Z_3)\widetilde{g}(Z_1, Z_2) - P(X_1, Z_2)\widetilde{g}(Z_1, Z_3)\}
\end{aligned}
$$

$$
\begin{aligned}
&- \widetilde{P}(X_1, Z_3)\widetilde{g}(TZ_1, Z_2) + \widetilde{P}(X_1, Z_2)\widetilde{g}(TZ_1, Z_3)\\
&+ 2\widetilde{P}(X_1, Z_1)\widetilde{g}(TZ_2, Z_3),
\end{aligned}
$$

$$
\begin{aligned}
4\{&\widetilde{g}(\sigma(Z_1, Z_4), \sigma(Z_2, Z_3)) - \widetilde{g}(\sigma(Z_1, Z_3), \sigma(Z_2, Z_4))\} =\\
&- 4R^{\mathcal{D}^\theta}(Z_1, Z_2, Z_3, Z_4)\\
&+ c\{\widetilde{g}(Z_1, Z_4)\widetilde{g}(Z_2, Z_3) - \widetilde{g}(Z_1, Z_3)\widetilde{g}(Z_2, Z_4)\\
&+ \widetilde{g}(TZ_1, Z_4)\widetilde{g}(TZ_2, Z_3) - \widetilde{g}(TZ_1, Z_3)\widetilde{g}(TZ_2, Z_4)\\
&- 2\widetilde{g}(TZ_1, Z_2)\widetilde{g}(TZ_3, X_4)\} + 3\{P(Z_1, Z_4)\widetilde{g}(Z_2, Z_3)\\
&- P(Z_1, Z_3)\widetilde{g}(Z_2, Z_4) + P(Z_2, Z_3)\widetilde{g}(Z_1, Z_4)\\
&P(Z_2, Z_4)\widetilde{g}(Z_1, Z_3\} - \widetilde{P}(Z_1, Z_4)\widetilde{g}(TZ_2, Z_3)\\
&+ \widetilde{P}(Z_1, Z_3)\widetilde{g}(TZ_2, Z_4) - \widetilde{P}(Z_2, Z_3)\widetilde{g}(TZ_1, Z_4)\\
&+ \widetilde{P}(Z_2, Z_4)\widetilde{g}(TZ_1, Z_3) + 2\{\widetilde{P}(Z_1, Z_2)\widetilde{g}(TZ_3, Z_4)\\
&+ \widetilde{P}(Z_3, Z_4)\widetilde{g}(TZ_1, Z_2)\},
\end{aligned}
$$

for any $X_1, X_2, X_3, X_4 \in \mathcal{D}$ and $Z_1, Z_2, Z_3, Z_4 \in \mathcal{D}^\theta$. Let

$$
\begin{aligned}
&e_1, \ldots, e_p,\ {e_1}^*, \ldots, {e_p}^*,\\
&e_{2p+1}, \ldots e_{2p+q},\ {e_{2p+1}}^*, \ldots, {e_{2p+q}}^*,\\
&e_{n+q+1}, \ldots, e_{n+q+s},\ {e_{n+q+1}}^*, \ldots, {e_{n+q+s}}^*
\end{aligned}
$$

be a generalized adapted local frame of $\widetilde{M}$, [4].

Using this frame, the Gauss equation (4.10) is written as

$$
\begin{aligned}
4\{&\widetilde{g}(\sigma_{kh}, \sigma_{ji}) - \widetilde{g}(\sigma_{ki}, \sigma_{jh})\} =\\
&- 4e^{f_1^2}{R^{\mathcal{D}}}_{kjih} + f_1^2\|\nabla^{\mathcal{D}^\theta} \log f_1\|^2(\delta_{kh}\delta_{ji} - \delta_{ki}\delta_{jh})\\
&- 3(P_{kh}\delta_{ji} - P_{ki}\delta_{jh} + P_{ji}\delta_{kh} - P_{jh}\delta_{ki})\\
&+ P(Je_k, e_h)\widetilde{g}(Je_j, e_i) - P(Je_k, e_i)\widetilde{g}(Je_j, e_h)\\
&+ P(Je_j, e_i)\widetilde{g}(Je_k, e_h) - P(Je_j, e_h)\widetilde{g}(Je_k, e_i)\\
&- 2\{P(Je_k, e_j)\widetilde{g}(Je_i, e_h) + P(Je_i, e_h)\widetilde{g}(Je_k, e_j)\},
\end{aligned}
$$

$$
\begin{aligned}
4\{&\widetilde{g}(\sigma_{ja}, \sigma_{ih}) - \widetilde{g}(\sigma_{jh}, \sigma_{ia})\} = 3(P_{ja}\delta_{ih} - P_{ia}\delta_{jh})\\
&- P(Je_j, e_a)\widetilde{g}(Je_i, e_h) + P(Je_j, e_a)\widetilde{g}(Je_j, e_h)\\
&+ 2P(Je_h, e_a)\widetilde{g}(Je_j, e_i),
\end{aligned}
$$

$$
\begin{aligned}
2\{&\widetilde{g}(\sigma_{ia}, \sigma_{hb}) - \widetilde{g}(\sigma_{ib}, \sigma_{ha})\} = -c\widetilde{g}(Je_i, e_h)\widetilde{g}(Te_b, e_a)\\
&+ P(Je_b.e_a)\widetilde{g}(Je_i, e_h) + P(Je_i, e_h)\widetilde{g}(Te_b, e_a),
\end{aligned}
$$

$$
\begin{aligned}
4\{\widetilde{g}(\sigma_{ih},\sigma_{ba}) - \widetilde{g}(\sigma_{ia},\sigma_{bh})\} &= 4f_1^2\{(2+f_1^2)(e_b \log f_1)(e_a \log f_1) \\
&\quad + \nabla^{\mathcal{D}^\theta}{}_{e_b}\nabla^{\mathcal{D}^\theta}{}_{e_a} \log f_1\}\delta_{ih} + c\{\delta_{ih}\delta_{ba} + \widetilde{g}(Je_i,e_h)\widetilde{g}(Te_b,e_a)\} \\
&\quad 3(P_{ih}\delta_{ba} + P_{ba}\delta_{ih}) - P(Je_i,e_h)\widetilde{g}(Te_b,e_a) \\
&\quad - P(Je_b,e_a)\widetilde{g}(Je_i,e_h),
\end{aligned} \tag{4.11}
$$

$$
\begin{aligned}
4\{\widetilde{g}(\sigma_{ha},\sigma_{cb}) - \widetilde{g}(\sigma_{hb},\sigma_{ca})\} &= 3(P_{ha}\delta_{cb} - P_{hb}\delta_{ca}) \\
&\quad - P(Je_h,e_a)\widetilde{g}(Te_c,e_b) + P(Je_h,e_b)\widetilde{g}(Te_c,e_a) \\
&\quad + 2P(Je_h,e_c)\widetilde{g}(Te_b,e_a),
\end{aligned}
$$

$$
\begin{aligned}
4\{\widetilde{g}(\sigma_{da},\sigma_{cb}) - \widetilde{g}(\sigma_{db},\sigma_{ch})\} &= -4R^{\mathcal{D}^\theta}{}_{dcba} + c\{\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca} \\
&\quad + \widetilde{g}(Te_d,e_a)\widetilde{g}(Te_c,e_b) - \widetilde{g}(Te_d,e_b)\widetilde{g}(Te_c,e_a) \\
&\quad - 2\widetilde{g}(Te_d,e_c)\widetilde{g}(Te_b,e_a)\} \\
&\quad + 3(P_{da}\delta_{cb} - P_{db}\delta_{ca} + P_{cb}\delta_{da} - P_{ca}\delta_{db}) \\
&\quad - P(Je_d,e_a)\widetilde{g}(Te_c,e_b) + P(Je_d,e_b)\widetilde{g}(Te_c,e_a) \\
&\quad - P(Je_c,e_b)\widetilde{g}(Te_d,e_a) + P(Je_c,e_a)\widetilde{g}(Te_d,e_b) \\
&\quad + 2\{P(Je_d,e_c)\widetilde{g}(Te_b,e_a) + P(Je_b,e_a)\widetilde{g}(Te_d,e_c)\},
\end{aligned}
$$

for any

$$k, j, i, h \in \{1, 2, \ldots, 2p\}$$

and

$$d, c, b, a \in \{2p+1, 2p+2, \ldots, 2p+q\},$$

where we put $\sigma(e_\mu, e_\lambda) = \sigma_{\mu\lambda}$, etc.

The mean curvature vector H and the mean curvature $\|H\|$ are respectively given by

$$H = \frac{1}{n}\sum_{\mu=1}^{n} \sigma_{\mu\mu}, \qquad \|H\|^2 = \frac{1}{n^2}\sum_{\mu,\lambda=1}^{n} \widetilde{g}(\sigma_{\mu\mu},\sigma_{\lambda\lambda}) \tag{4.12}$$

and the length $\|\sigma\|$ of the second fundamental form σ is given by

$$\|\sigma\|^2 = \sum_{\mu,\lambda=1}^{n} \widetilde{g}(\sigma_{\mu\lambda},\sigma_{\mu\lambda}) = \sum_{\mu,\lambda=1}^{n}\sum_{r=n+1}^{m} \{\widetilde{g}(\sigma_{\mu\lambda},e_r)\}^2 \tag{4.13}$$

for any local orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ of $T\widetilde{M}$.

By virtue of the Gauss equations, we have

$$\sum_{\mu,\lambda=1}^{n} (R_{\mu\lambda\mu\lambda} - \widetilde{R}_{\mu\lambda\mu\lambda}) = \|\sigma\|^2 - n\|H\|^2. \tag{4.14}$$

On the other hand, we have from (4.10) that

$$\begin{aligned} 4\sum_{\mu,\lambda=1}^{n}\widetilde{R}_{\mu\lambda\mu\lambda} &= -(n^2+2n-3q)c-6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} \\ &- 6\sum_{b=2p+1}^{2p+q}P_{bb}-3c\sum_{b,a=2p+1}^{2p+q}T_{ba}T_{ba}+6\sum_{b,a=2p+1}^{2p+q}P(Je_b,e_a)T_{ba}, \end{aligned} \tag{4.15}$$

where $T_{ba}=\widetilde{g}(T_b{}^c e_c, e_a)$ for any $c,b,a\in\{2p+1,2p+2,\ldots,2p+q=n\}$. We know T_{ba} is skew-symmetric.

Moreover, we have from (1.9) and (4.7) that

$$\begin{aligned} 4\sum_{\mu,\lambda=1}^{n}R_{\mu\lambda\mu\lambda} &= -(e^{f_1^2}\tau^{\mathcal{D}}+\tau^{\mathcal{D}^\theta})+8pf_1^2\{(2p-1)f_1^2\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2+ \\ &+2(2+f_1^2)\sum_{a=2p+1}^{2p+q}(e_a\log f_1)^2+2\sum_{a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}{}_{e_a}\nabla^{\mathcal{D}^\theta}{}_{e_a}\log f_1\}, \end{aligned} \tag{4.16}$$

where $\tau^{\mathcal{D}}$ (resp. $\tau^{\mathcal{D}^\theta}$) denotes the scalar curvature with respect to $g_{\mathcal{D}}$ (resp. $g_{\mathcal{D}^\theta}$).

Substituting (4.15) and (4.16) into (4.14), we obtain

$$\begin{aligned} 4\|\sigma\|^2 &= 4n\|H\|^2+8pf\{(2p-1)f_1^2\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2+ \\ &+2(2+f_1^2)\sum_{a=2p+1}^{2p+q}(e_a\log f_1)^2\}+(n^2+2n-3q)c+ \\ &+3c\sum_{b,a=2p+1}^{2p+q}(T_{ba})^2-4(e^{f_1^2}\tau^{\mathcal{D}}+\tau^{\mathcal{D}^\theta})+ \\ &+16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}_{e_a}\nabla^{\mathcal{D}^\theta}_{e_a}\log f_1+6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu}+ \\ &+6\sum_{a=2p+1}^{2p+q}P_{aa}-6\sum_{b,a=2p+1}^{2p+q}P(Je_b,e_a)T_{ba}. \end{aligned} \tag{4.17}$$

Thus we have

Theorem 4.1. *In a first type warped product semi-slant submanifold in an l.c.K.-space form, the mean curvature satisfies the inequality*

$$
\begin{aligned}
4n\|H\|^2 &+ 8pf_1^2\{(2p-1)f_1^2\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2 \\
&+ 2(2+f_1^2)\sum_{a=2p+1}^{2p+q}(e_a\log f_1)^2\} + (n^2+2n-3q)c \\
&+ 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f_1^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) \\
&+ 16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}_{e_a}\nabla^{\mathcal{D}^\theta}_{e_a}\log f_1 + 6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} \\
&+ 6\sum_{a=2p+1}^{2p+q}P_{aa} - 6\sum_{b,a=2p+1}^{2p+q}P(Je_b,e_a)T_{ba} \geq 0.
\end{aligned}
\tag{4.18}
$$

Corollary 4.2. *Under the same condition with Theorem 4.1, the equality case of* (4.18) *is that the submanifold is locally totally geodesic and the warping function f_1 satisfies*

$$
\begin{aligned}
8pf_1^2\{&(2p-1)f_1^2\|\nabla^{\mathcal{D}^\theta}\log f_1\|^2 + \\
&+ 2(2+f_1^2)\sum_{a=2p+1}^{2p+q}(e_a\log f_1)^2\} + (n^2+2n-3q)c + \\
&+ 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f_1^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\
&+ 16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}{}_{e_a}\nabla^{\mathcal{D}^\theta}{}_{e_a}\log f_1 + 6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} + \\
&+ 6\sum_{a=2p+1}^{2p+q}P_{aa} - 6\sum_{b,a=2p+1}^{2p+q}P(Je_b,e_a)T_{ba} = 0.
\end{aligned}
$$

and

$$
\begin{aligned}
(n^2+2n-3q)c &+ 3c\sum_{b,a=2p+1}^{2p+q}\{T_{ba}\}^2 - 4(e^{f_1^2}\tau^{\mathcal{D}} + \tau^{\mathcal{D}^\theta}) + \\
&16pf_1^2\sum_{b,a=2p+1}^{2p+q}\nabla^{\mathcal{D}^\theta}{}_{e_a}\nabla^{\mathcal{D}^\theta}{}_{e_a}\log f_1 + 6(n-2)\sum_{\mu=1}^{n}P_{\mu\mu} +
\end{aligned}
$$

$$+ 6 \sum_{a=2p+1}^{2p+q} P_{aa} - 6 \sum_{b,a=2p+1}^{2p+q} P(Je_b, e_a) T_{ba} \leq 0.$$

5. Acknowledgements

The author is thankful to the referee for his/her valuable comments which have improved the manuscript.

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Received: July 3, 2017, accepted: September 2, 2017.

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