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On distortion of the transfinite diameter under regular homeomorphisms

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Abstract. In this paper, a lower bound of exponential type for the distortion of the transfinite diameter of the image of a disk under regular homeomorphisms belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property (in the sense of Luzin) is obtained. Moreover, the problem of minimizing the distortion functional of the transfinite diameter of the image of a disk in a certain class of such mappings is solved.

Анотація. У даній роботі отримано нижню оцінку експоненціального типу для спотворення трансфінітного діаметра образу круга при регулярних гомеоморфізмах, що належать класу Соболєва $W^{1,1}_{\mathrm{loc}}$ та володіють N-властивістю (за Лузіним). Крім того, розв'язано задачу про мінімізацію функціонала спотворення трансфінітного діаметра образу круга на деякому класі таких відображень.

1. Introduction

We recall some definitions. Let G be a domain in the complex plane $\mathbb{C}$. A mapping $f\colon G \to \mathbb{C}$ is said to be *regular at a point* $z_0 \in G$ if f has a total differential at z_0 and its Jacobian $J_f = |f_z|^2 - |f_{\bar z}|^2 \neq 0$, see [6, I. 1.6]. Here $f_z = \frac{1}{2}(f_x - if_y)$ and $f_{\bar z} = \frac{1}{2}(f_x + if_y)$ denote the formal derivatives of f with respect to z and $\overline{z}$, respectively, whereas f_x and f_y stand for the partial derivatives of f with respect to the variables x and y. A homeomorphism f belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ is called *regular* if $J_f > 0$ almost everywhere.

Hereafter, $|E|$ denotes the Lebesgue measure of a set $E \subset \mathbb{C}$. A mapping $f\colon G \to \mathbb{C}$ is said to *possess the N-property* (in the sense of Luzin) if $|E| = 0$ implies $|f(E)| = 0$.

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For $z_0 \in \mathbb{C}$, we introduce the following notations:

$$\gamma(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}, \qquad \gamma_r = \gamma(0, r),$$
$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}, \qquad B_r = B(0, r), \quad \mathbb{B} = B(0, 1),$$
$$\overline{B(z_0, r)} = \{z \in \mathbb{C} : |z - z_0| \leqslant r\}, \qquad \overline{B}_r = \overline{B(0, r)}.$$

Let $f\colon G \to \mathbb{C}$ be a regular homeomorphism belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$, $z_0 \in G$, and let $p > 1$. The p-*angular dilatation* of f at a point $z = z_0 + re^{i\theta} \in G \backslash \{z_0\}$ with respect to z_0 is defined by

$$D_{p,f}(z, z_0) = \frac{|f_\theta(z_0 + re^{i\theta})|^p}{r^p J_f(z_0 + re^{i\theta})},$$

where f_θ denotes the partial derivative of f with respect to θ. Note that the p-angular dilatation is used in the study of the asymptotic properties of regular homeomorphic solutions to nonlinear Beltrami equations, see [2, 3, 7, 8, 9, 12, 13, 16], and of homeomorphisms of finite distortion, see [17].

For $D_{p,f}(z, z_0)$ and $t \in (0, d_0)$, where $d_0 = \operatorname{dist}(z_0, \partial G)$, we define the circular average by

$$d_{p,f}(z_0, t) = \Big(\frac{1}{2\pi t} \int\limits_{\gamma(z_0,t)} D_{p,f}^{\frac{1}{p-1}}(z, z_0)|dz|\Big)^{p-1}.$$

In what follows, we assume that

$$D_{p,f}(z, z_0) = D_{p,f}(z), \qquad d_{p,f}(z_0, t) = d_{p,f}(t), \qquad z_0 = 0.$$

The following statement provides a differential inequality for the function $S_f(z_0, r) = |f(B(z_0, r))|$ under the assumption that f is a regular Sobolev homeomorphism, see [16, Lemma 2.2], cf. [3, Lemma 2.1].

Proposition 1.1. *Let G be a domain in the complex plane $\mathbb{C}$, $z_0 \in G$, $p > 1$, and let $f\colon G \to \mathbb{C}$ be a regular homeomorphism belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property. Then*

$$S'_f(z_0, r) \geqslant 2\pi^{\frac{2-p}{2}} r^{1-p} d_{p,f}^{-1}(z_0, r)\, S_f^{\frac{p}{2}}(z_0, r) \tag{1.1}$$

for a.e. (almost every) $r \in (0, d_0)$, where $d_0 = \operatorname{dist}(z_0, \partial G)$.

Let E be a bounded closed set in the complex plane $\mathbb{C}$. The quantity

$$d_n(E) = \max_{z_i, z_j \in E} \Big\{ \prod_{1 \leqslant i \leqslant j \leqslant n} |z_i - z_j|^{\frac{2}{n(n-1)}} \Big\}$$

is called the n-*th diameter* of the set E. Note that $d_n(E)$ is computed among all n-tuples of points from the set E. In particular, $d_2(E) = \operatorname{diam} E$ is the Euclidean diameter of the set E.

We note that the sequence $\{d_n(E)\}$ is non-increasing, that is,

$$d_{n+1}(E) \leqslant d_n(E), \quad n = 2, 3, \ldots$$

Therefore, the limit

$$d(E) = \lim_{n\to\infty} d_n(E)$$

is called the *transfinite diameter* of the set E. If E is a finite set, then $d(E) = 0$. It is known that the quantity $d(E)$ coincides with the Chebyshev constant of the set E and with its logarithmic capacity, see, e.g. [4, pp. 285–305], [5, pp. 209–217].

We recall some well-known properties of the transfinite diameter.

1. If $E_1 \subset E_2$, then $d(E_1) \leqslant d(E_2)$.
2. The transfinite diameter of a closed disk $\overline{B}_r$ is equal to its radius r, and the transfinite diameter of a segment $[a, b]$ is equal to $(b - a)/4$.
3. According to Pólya's theorem, see [10, p. 10], for any compact set $E \subset \mathbb{C}$, the following estimate holds:

$$|E| \leqslant \pi d^2(E). \tag{1.2}$$

In this paper, using the method of isoperimetry, we obtain a lower bound of exponential type for the distortion of the transfinite diameter of the image of a disk under regular homeomorphisms belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property. We note that distortion estimates of power type for the transfinite diameter, for other classes of mappings, were obtained by means of the non-conformal modulus method in [14].

2. Distortion of the transfinite diameter of the image of a disk

Below we present a result on the distortion of the transfinite diameter of the image of a disk under regular homeomorphisms in the Sobolev class $W^{1,1}_{\mathrm{loc}}$ whose circular average $d_{p,f}(z_0, r)$ satisfies a power-exponential growth condition.

Theorem 2.1. *Let G be a bounded domain in the complex plane $\mathbb{C}$, $z_0 \in G$, $1 < p < 2$, and let $f\colon G \to \mathbb{C}$ be a regular homeomorphism belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property. If, for some numbers $K = K(z_0) > 0$, $\alpha > 0$, and $\beta > 0$, the following condition*

$$d_{p,f}(z_0, t) \leqslant K t^{\beta+2-p} e^{\alpha/t^{\beta}} \tag{2.1}$$

holds for a.e. $t \in (0, d_0)$, where $d_0 = \operatorname{dist}(z_0, \partial G)$, then

$$d\Big(f\big(\overline{B(z_0, r)}\big)\Big) \geqslant \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\Big(-\frac{\alpha}{(2-p)r^{\beta}}\Big) \tag{2.2}$$

for every $r \in (0, d_0)$.

Proof. The differential inequality (1.1) in Proposition 1.1, together with the condition (2.1), implies that the following relation

$$\frac{S'_f(z_0,t)}{S_f^{\frac{p}{2}}(z_0,t)} \geqslant 2\pi^{\frac{2-p}{2}} t^{1-p} d_{p,f}^{-1}(z_0,t) \geqslant 2\pi^{\frac{2-p}{2}} K^{-1} t^{-\beta-1} e^{-\alpha/t^{\beta}}$$

holds for a.e. $t \in (0, d_0)$, where $d_0 = \operatorname{dist}(z_0, \partial G)$. Integrating this inequality over the segment $[\varepsilon, r]$ with $0 < \varepsilon < r < d_0$ leads to

$$\int_{\varepsilon}^{r} \frac{S'_f(z_0,t)dt}{S_f^{\frac{p}{2}}(z_0,t)} \geqslant \frac{2\pi^{\frac{2-p}{2}}}{K\alpha\beta} \left(e^{-\alpha/r^{\beta}} - e^{-\alpha/\varepsilon^{\beta}} \right). \tag{2.3}$$

Taking into account that the function $\Phi_{p,f}(z_0,t) = \frac{2}{2-p} S_f^{\frac{2-p}{2}}(z_0,t)$ is non-decreasing in t, we further derive

$$\begin{aligned} \int_{\varepsilon}^{r} \frac{S'_f(z_0,t)dt}{S_f^{\frac{p}{2}}(z_0,t)} &= \int_{\varepsilon}^{r} \Phi'_{p,f}(z_0,t)dt \\ &\leqslant \Phi_{p,f}(z_0,r) - \Phi_{p,f}(z_0,\varepsilon) \\ &= \frac{2}{2-p} \left(S_f^{\frac{2-p}{2}}(z_0,r) - S_f^{\frac{2-p}{2}}(z_0,\varepsilon) \right), \end{aligned} \tag{2.4}$$

see [11, Theorem IV.7.4].

A combination of the inequalities (2.3) and (2.4) yields

$$\begin{aligned} S_f^{\frac{2-p}{2}}(z_0,r) &\geqslant S_f^{\frac{2-p}{2}}(z_0,r) - S_f^{\frac{2-p}{2}}(z_0,\varepsilon) \\ &\geqslant \frac{\pi^{\frac{2-p}{2}}(2-p)}{K\alpha\beta} \left(e^{-\alpha/r^{\beta}} - e^{-\alpha/\varepsilon^{\beta}} \right). \end{aligned}$$

Letting $\varepsilon \to 0$ and simplifying the resulting expression, we arrive at

$$\left| f\left(\overline{B(z_0,r)}\right) \right| \geqslant S_f(z_0,r) \geqslant \pi \left(\frac{2-p}{K\alpha\beta} \right)^{\frac{2}{2-p}} \exp\Big(-\frac{2\alpha}{(2-p)r^{\beta}} \Big).$$

An application of Pólya's inequality (1.2) then gives

$$\begin{aligned} \pi d^2 \left(f\left(\overline{B(z_0,r)}\right) \right) &\geqslant \left| f\left(\overline{B(z_0,r)}\right) \right| \\ &\geqslant \pi \left(\frac{2-p}{K\alpha\beta} \right)^{\frac{2}{2-p}} \exp\Big(-\frac{2\alpha}{(2-p)r^{\beta}} \Big). \end{aligned}$$

Consequently, the estimate (2.2) follows. □

The following result establishes a lower bound for the distortion of the transfinite diameter of the image of a disk under a power-exponential growth condition on the p-angular dilatation $D_{p,f}(z, z_0)$.

Corollary 2.2. *Let G be a bounded domain in the complex plane $\mathbb{C}$, $z_0 \in G$, $1 < p < 2$, and let $f\colon G \to \mathbb{C}$ be a regular homeomorphism belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property. If, for some numbers $K = K(z_0) > 0$, $\alpha > 0$, and $\beta > 0$, the following condition*

$$D_{p,f}(z, z_0) \leqslant K|z - z_0|^{\beta+2-p}\, e^{\alpha/|z-z_0|^\beta}$$

holds for a.e. $z \in B(z_0, d_0)$, where $d_0 = \operatorname{dist}(z_0, \partial G)$, then

$$d\left(f\left(\overline{B(z_0, r)}\right)\right) \geqslant \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^\beta}\right)$$

for every $r \in (0, d_0)$.

Setting $G = \mathbb{B}$ and $z_0 = 0$ in Theorem 2.1, we obtain the following statement.

Corollary 2.3. *Let $f\colon \mathbb{B} \to \mathbb{C}$ be a regular homeomorphism belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property and let $1 < p < 2$. If, for some numbers $K > 0$, $\alpha > 0$, and $\beta > 0$, the following condition*

$$d_{p,f}(t) \leqslant K t^{\beta+2-p} e^{\alpha/t^\beta}$$

holds for a.e. $t \in (0, 1)$, then

$$d\left(f\left(\overline{B}_r\right)\right) \geqslant \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^\beta}\right)$$

for every $r \in (0, 1)$.

Corollary 2.4. *Let $f\colon \mathbb{B} \to \mathbb{C}$ be a regular homeomorphism belonging to the Sobolev class $W^{1,1}_{\mathrm{loc}}$ and possessing the N-property and let $1 < p < 2$. If, for some numbers $K > 0$, $\alpha > 0$, and $\beta > 0$, the following condition*

$$D_{p,f}(z) \leqslant K|z|^{\beta+2-p} e^{\alpha/|z|^\beta}$$

holds for a.e. $z \in \mathbb{B}$, then

$$d\left(f\left(\overline{B}_r\right)\right) \geqslant \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^\beta}\right)$$

for every $r \in (0, 1)$.

3. Minimization of the transfinite diameter

In this section, we solve an extremal problem concerning the minimization of the transfinite diameter of the image of a disk within the class of regular homeomorphisms under the condition that the circular average $d_{p,f}(t)$ satisfies a power-exponential growth condition.

Fix numbers $1 < p < 2$, $K > 0$, $\alpha > 0$, and $\beta > 0$, and denote by $\mathcal{H} = \mathcal{H}(p, K, \alpha, \beta)$ the class of all regular homeomorphisms $f\colon \mathbb{B} \to \mathbb{C}$ that belong to the Sobolev class $W^{1,1}_{\mathrm{loc}}$, possess the N-property, and satisfy the condition

$$d_{p,f}(t) \leqslant K t^{\beta+2-p} e^{\alpha/t^{\beta}} \tag{3.1}$$

for a.e. $t \in (0,1)$.

On the class $\mathcal{H}$, consider the functional of the distortion of the transfinite diameter of the image of a disk

$$\mathbf{d}_r(f) = d\left(f\left(\overline{B}_r\right)\right).$$

The following theorem provides the solution to the minimization problem for the functional $\mathbf{d}_r(f)$ over the class $\mathcal{H}$.

Theorem 3.1. *For every $r \in [0,1)$, the following equality*

$$\min_{f\in\mathcal{H}} \mathbf{d}_r(f) = \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^{\beta}}\right)$$

holds. Moreover, the minimum of this functional is attained by

$$\tilde{f}(z) = \begin{cases} \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)|z|^{\beta}}\right)\frac{z}{|z|}, & z \neq 0, \\ 0, & z = 0. \end{cases} \tag{3.2}$$

Proof. From Corollary 2.3 together with the condition (3.1), we deduce the following estimate

$$d\left(f\left(\overline{B}_r\right)\right) \geqslant \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^{\beta}}\right), \quad r \in [0,1).$$

Consequently, the inequality

$$\mathbf{d}_r(f) \geqslant \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^{\beta}}\right)$$

holds for any homeomorphism $f \in \mathcal{H}$.

Now consider the radial stretching $\tilde{f} \in \mathcal{H}$ defined by (3.2). Writing (3.2) in polar coordinates (r,θ), $z = re^{i\theta}$, we obtain

$$\tilde{f}\left(re^{i\theta}\right) = \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\left(-\frac{\alpha}{(2-p)r^{\beta}}\right)e^{i\theta}, \qquad \tilde{f}(0) = 0.$$

We verify that $\tilde{f}$ is a regular homeomorphism possessing the N-property. Indeed, $\tilde{f}$ is a diffeomorphism in $\mathbb{B}\backslash\{0\}$ and $\tilde{f} \in W^{1,1}_{\mathrm{loc}}\left(\mathbb{B}\backslash\{0\}\right)$.

Let $r_0 = \min\left\{\left(\frac{\alpha\beta}{2-p}\right)^{\frac{1}{\beta}}, 1\right\}$. Denote by λ_1 and λ_2 the minimal and maximal stretchings of $\tilde{f}$ at a point $z = re^{i\theta}$, where $0 < r < r_0$. Setting

$$R(r) = \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big),$$

we derive

$$\lambda_1 = \frac{R(r)}{r} = \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} r^{-1} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big)$$

and

$$\lambda_2 = \frac{dR}{dr} = K^{-\frac{1}{2-p}} \left(\frac{2-p}{\alpha\beta}\right)^{\frac{p-1}{2-p}} r^{-\beta-1} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big),$$

see [15, Proposition 1.4]. Next, we compute

$$\frac{\lambda_2}{\lambda_1} = \frac{\alpha\beta}{2-p} r^{-\beta} > 1, \quad r \in (0, r_0).$$

Here, $r_0 = 1$ if $\alpha\beta \geqslant 2-p$, and $r_0 = \left(\frac{\alpha\beta}{2-p}\right)^{\frac{1}{\beta}}$ if $\alpha\beta < 2-p$. Hence, $\lambda_2 > \lambda_1$ for every $z = re^{i\theta}$ with $0 < r < r_0$. Therefore,

$$\int\limits_{B_{r_0}} \lambda_2 dxdy = c\int\limits_0^{r_0} r^{-\beta} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big) dr < \infty,$$

where $c = 2\pi K^{-\frac{1}{2-p}}\left(\frac{2-p}{\alpha\beta}\right)^{\frac{p-1}{2-p}}$. Thus, $\tilde{f} \in W^{1,1}_{\mathrm{loc}}(\mathbb{B})$.

The homeomorphism $\tilde{f}$ maps the closed disk $\overline{B}_r$ onto the closed disk $\overline{B}_{r_*}$ of radius

$$r_*(r) = \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big).$$

It is well known that the transfinite diameter of a closed disk coincides with its radius. Hence,

$$d\left(\tilde{f}\left(\overline{B}_r\right)\right) = d\left(\overline{B}_{r_*}\right) = r_*(r).$$

Therefore,

$$\mathbf{d}_r(\tilde{f}) = d\left(\tilde{f}\left(\overline{B}_r\right)\right) = r_*(r) = \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big).$$

Computing the partial derivatives of $\tilde{f}$ with respect to r and θ

$$\tilde{f}_r = K^{-\frac{1}{2-p}} \left(\frac{2-p}{\alpha\beta}\right)^{\frac{p-1}{2-p}} r^{-\beta-1} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big) e^{i\theta},$$

$$\tilde{f}_\theta = i \left(\frac{2-p}{K\alpha\beta}\right)^{\frac{1}{2-p}} \exp\Big(-\frac{\alpha}{(2-p)r^\beta}\Big) e^{i\theta},$$

we obtain the Jacobian

$$\begin{aligned} J_{\tilde{f}}(re^{i\theta}) &= \frac{1}{r} \operatorname{Im}\Big(\overline{\tilde{f}_r(re^{i\theta})}\tilde{f}_\theta(re^{i\theta})\Big) \\ &= K^{-\frac{2}{2-p}} \left(\frac{2-p}{\alpha\beta}\right)^{\frac{p}{2-p}} r^{-\beta-2} \exp\Big(-\frac{2\alpha}{(2-p)r^\beta}\Big) > 0, \end{aligned}$$

see, e.g. [1, formula (21.52)].

The corresponding p-angular dilatation is

$$\begin{aligned} D_{p,\tilde{f}}(z) = D_{p,\tilde{f}}(re^{i\theta}) &= \frac{|\tilde{f}_\theta(re^{i\theta})|^p}{r^p J_{\tilde{f}}(re^{i\theta})} \\ &= Kr^{\beta+2-p} e^{\alpha/r^\beta} = K|z|^{\beta+2-p} e^{\alpha/|z|^\beta}. \end{aligned}$$

Hence, the circular average equals

$$d_{p,\tilde{f}}(t) = \left(\frac{1}{2\pi t} \int\limits_{\gamma_t} D_{p,\tilde{f}}^{\frac{1}{p-1}}(z)|dz|\right)^{p-1} = Kt^{\beta+2-p} e^{\alpha/t^\beta}$$

for every $t \in [0,1)$. Therefore, $\tilde{f} \in \mathcal{H}$. □

References

[1] K. Astala, T. Iwaniec, G. Martin. *Elliptic partial differential equations and quasiconformal mappings in the plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2009.

[2] A. Golberg, R. Salimov. Nonlinear Beltrami equation. *Complex Var. Elliptic Equ.*, 65(1):6–21, 2020. doi:10.1080/17476933.2019.1631292.

[3] A. Golberg, R. Salimov, M. Stefanchuk. Asymptotic dilation of regular homeomorphisms. *Complex Anal. Oper. Theory*, 13(6):2813–2827, 2019. doi:10.1007/s11785-018-0833-2.

[4] G. M. Goluzin. *Geometric theory of functions of a complex variable*, volume 26 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1969.

[5] N. S. Landkof. *Foundations of modern potential theory*, volume 180 of *Grundlehren der mathematischen Wissenschaften*. Heidelberg: Springer Berlin, 1972. URL: `https://link.springer.com/book/9783642651854`.

[6] O. Lehto, K. I. Virtanen. *Quasiconformal mappings in the plane*, volume 126 of *Grundlehren der mathematischen Wissenschaften*. Heidelberg: Springer Berlin, 1973. URL: `https://link.springer.com/book/9783642655159`.

[7] I. Petkov, R. Salimov, M. Stefanchuk. Asymptotic behavior of solutions of the nonlinear Beltrami equation with the Jacobian. *J. Math. Sci. (N.Y.)*, 280(6):971–983, 2024. doi:10.1007/s10958-024-07280-0.

[8] I. Petkov, R. Salimov, M. Stefanchuk. Nonlinear Beltrami equation: lower estimates of Schwarz lemma's type. *Canad. Math. Bull.*, 67(3):533–543, 2024. doi:10.4153/s0008439523000942.

[9] I. Petkov, R. Salimov, M. Stefanchuk. Sharp lower bounds for area distortion via growth control of angular dilatations. *Complex Anal. Oper. Theory*, 19(6):162, 2025, 21 pages. doi:10.1007/s11785-025-01787-3.

[10] G. Pólya, G. Szegö. *Isoperimetric inequalities in mathematical physics*, volume 27 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1951. URL: `https://www.jstor.org/stable/j.ctt1b9rzzn`.

[11] S. Saks. *Theory of the integral*, volume 7 of *Monografie Matematyczne*. Warszawa-Lwów: Instytut Matematyczny Polskiej Akademii Nauk, 1937.

[12] R. Salimov, M. Stefanchuk. On the local properties of solutions of the nonlinear Beltrami equation. *J. Math. Sci. (N.Y.)*, 248(2):203–216, 2020. (Translation of Ukr. Mat. Visn. 17(1):77–95, 2020). doi:10.1007/s10958-020-04870-6.

[13] R. Salimov, M. Stefanchuk. Logarithmic asymptotics of the nonlinear Cauchy-Riemann-Beltrami equation. *Ukrainian Math. J.*, 73(3):463–478, 2021. (Translation of Ukr. Mat. Zh. 73(3):395–407, 2021). doi:10.1007/s11253-021-01936-9.

[14] R. Salimov, L. Vyhivska, B. Klishchuk. On distortions of the transfinite diameter of disk image. *Ukrainian Math. J.*, 75(2):235–243, 2023. (Translation of Ukr. Mat. Zh. 75(2):207–214, 2023). doi:10.1007/s11253-023-02196-5.

[15] E. Sevost'yanov. *Mappings with direct and inverse Poletsky inequalities*, volume 78 of *Developments in Mathematics*. Springer, Cham, 2023. doi:10.1007/978-3-031-45418-9.

[16] M. Stefanchuk. On local properties of regular homeomorphisms. *J. Math. Sci. (N.Y.)*, 292(4):508–528, 2025. (Translation of Ukr. Mat. Visn. 22(2):264–289, 2025). doi:10.1007/s10958-025-07935-6.

[17] M. Stefanchuk. On the power growth order of homeomorphisms of certain classes. *Canad. Math. Bull.*, 2026, 22 pages. doi:10.4153/S0008439525101574.

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