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On weakly two-convex sets in three-dimensional Euclidean space

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Abstract. The present work considers the properties of generalized convex sets in real three-dimensional Euclidean space $\mathbb{R}^3$, known as weakly 2-convex. An open set in $\mathbb{R}^3$ is called weakly 2-convex if through every boundary point of the set, there passes a plane that does not intersect the given set. A closed set in $\mathbb{R}^3$ is called weakly 2-convex if it is approximated from the outside by a family of open weakly 2-convex sets. A point in the complement of a set in $\mathbb{R}^3$ (with respect to the entire space) is a 2-nonconvexity point of the set if every plane passing through the point intersects the set. It is proved that the non-empty interior of a closed weakly 2-convex set is weakly 2-convex. It is also shown that for any convex polyhedron there exists an open weakly 2-convex set such that its 2-nonconvexity-point set coincides with the polyhedron interior. However, there exists an open weakly 2-convex set such that its open bounded convex 2-nonconvexity-point set differs from the interior of a convex polyhedron.

Анотація. У роботі розглядаються властивості узагальнено опуклих множин у дійсному тривимірному евклідовому просторі $\mathbb{R}^3$, відомих як слабко 2-опуклі. Відкрита множина в $\mathbb{R}^3$ називається слабко 2-опуклою, якщо через кожну точку межі множини проходить площина, яка не перетинає задану множину. Замкнена множина в $\mathbb{R}^3$ називається слабко 2-опуклою, якщо вона апроксимується ззовні сім'єю відкритих слабко 2-опуклих множин. Точка доповнення множини в $\mathbb{R}^3$ до всього простору називається точкою 2-неопуклості цієї множини, якщо кожна площина, яка проходить через цю точку, перетинає множину. Доведено, що непорожня внутрішність замкненої слабко 2-опуклої множини слабко 2-опукла. Показано, що для всякого опуклого многогранника існує відкрита слабко 2-опукла множина така, що її множина точок 2-неопуклості співпадає з внутрішністю многогранника, однак існує відкрита слабко 2-опукла множина, у якої відкрита обмежена опукла множина точок 2-неопуклості відмінна від внутрішності опуклого многогранника.

Keywords: Convex set, convex polyhedron, weakly 2-convex set, 2-nonconvexity-point set, real Euclidean space
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1. Introduction

Weakly $(n-1)$-convex sets in the real space $\mathbb{R}^n$, $n \geqslant 2$, with the Euclidean norm, can be seen as a generalization of convex sets (*any set is convex if the entire segment joining two arbitrary points of the set is contained in the set*, see [2]). The notion was coined by Yurii Zelinskii [8].

First, recall that any $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$, $n \geqslant 2$, is called a *hyperplane*. A hyperplane in $\mathbb{R}^3$ is also known as a *plane*, and it is a *straight line* in the plane $\mathbb{R}^2$.

We will use the following standard notation. For a subset $A \subset \mathbb{R}^n$, let $\overline{A}$ be its closure, $\operatorname{Int} A$ be its interior, and $\partial A = \overline{A} \setminus \operatorname{Int} A$ be its boundary.

Definition 1.1 ([8])**.** An open subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, is called *weakly* $(n-1)$-*convex* if for any point $x \in \partial E$ there exists a hyperplane L such that $x \in L$ and $L \cap E = \varnothing$.

Any open convex set is *weakly* $(n-1)$-*convex*, since through every boundary point of any convex set, there passes a supporting hyperplane for the set. The converse statement is not always true, which will be shown further.

Definition 1.2. A set A is said to *be approximated from the outside* by a family of open sets A_k, $k = 1, 2, \ldots$, if $\overline{A}_{k+1}$ is contained in A_k, and $A = \bigcap\limits_k A_k$.

It can be proved that any set approximated from the outside by a family of open sets is closed.

Definition 1.3 ([8])**.** A closed subset $E \subset \mathbb{R}^n$ is called *weakly* $(n-1)$-*convex* if it can be approximated from the outside by a family of open weakly $(n-1)$-convex sets.

Among the closed weakly $(n-1)$-convex sets there are also those with empty interior:

$$A = \overline{A} = \overline{A} \setminus \operatorname{Int} A = \partial A.$$

The class of open and closed weakly $(n-1)$-convex sets in $\mathbb{R}^n$ is denoted by $\mathbf{WC^n_{n-1}}$.

Definition 1.4 ([5])**.** For a subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, a point $x \in \mathbb{R}^n \setminus E$ is called an $(n-1)$-*nonconvexity point* if every hyperplane passing through x intersects E. The $(n-1)$-*nonconvexity-point set* corresponding to E is denoted by $E^{\triangle}_{n-1}$.

The class of open and closed weakly $(n-1)$-convex sets in $\mathbb{R}^n$ with non-empty $(n-1)$-nonconvexity-point sets is denoted by $\mathbf{WC^n_{n-1}} \setminus \mathbf{C^n_{n-1}}$. The sets belonging to that class have the following properties.

Lemma 1.5 ([1, 3, 4]). *An open or closed set belonging to the class* $\mathbf{WC^n_{n-1}}\backslash\mathbf{C^n_{n-1}}$ *consists of not less than three connected components.*

In particular, from Lemma 1.5, it follows that any open set belonging to the class $\mathbf{WC^n_{n-1}}\backslash\mathbf{C^n_{n-1}}$ is disconnected and, therefore, not convex. Interestingly, the class $\mathbf{WC^n_{n-1}}\backslash\mathbf{C^n_{n-1}}$ does not contain open weakly $(n-1)$-convex sets with smooth boundary either; see [1, Proposition 2.3.7].

By a *convex polyhedron* we mean the convex hull of a finite collection of points in $\mathbb{R}^n$ with non-empty interior with respect to $\mathbb{R}^n$, which is closed and bounded obviously, see [2]. A convex polyhedron in $\mathbb{R}^2$ is also known as a *convex polygon*.

The following results are obtained in the plane.

Lemma 1.6 ([7, 6]). *Let* $E \subset \mathbb{R}^2$ *be a closed subset such that* $\operatorname{Int} E \neq \varnothing$*. If* E *is weakly* 1*-convex, then* $\operatorname{Int} E$ *is weakly* 1*-convex.*

Lemma 1.7 ([5]). *Let*

$$E := (D\backslash \operatorname{Int} P)\backslash \bigcup_{j=1}^{p} \gamma^j \subset \mathbb{R}^2, \tag{1.1}$$

where P *is a convex polygon;* γ^j*,* $j = \overline{1,p}$*, are the straight lines containing all sides of* P*;* $D \subset \mathbb{R}^2$ *is an open convex subset containing* P*. Then*

$$E \in \mathbf{WC^2_1}\backslash\mathbf{C^2_1} \quad \textit{and} \quad E^{\triangle}_1 = \operatorname{Int} P.$$

Lemma 1.8 ([5]). *Let a subset* $E \subset \mathbb{R}^2$ *be open and bounded, consist of a finite collection of components, and belong to* $\mathbf{WC^2_1}\backslash\mathbf{C^2_1}$*. Then any connected component of* $E^{\triangle}_1$ *is the interior of a convex polygon.*

In this study, we focus on establishing general topological properties of closed weakly 2-convex sets, and convexity properties of 2-nonconvexity-point sets corresponding to open weakly 2-convex sets in $\mathbb{R}^3$.

Theorem 2.1 generalizes Lemma 1.6 to subsets of $\mathbb{R}^3$. It proves that the nonempty interior of a closed weakly 2-convex subset of $\mathbb{R}^3$ is also weakly 2-convex.

Theorem 2.2 is a spatial analogue of Lemma 1.7. In particular, it proposes a method for constructing a broad subclass of $\mathbf{WC^3_2}\backslash\mathbf{C^3_2}$.

The similarity of the above properties of weakly $(n-1)$-convex sets in the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ immediately suggests the possible validity of the spatial analogue of Lemma 1.8 as well. However, the hypothesis that *each connected component of the* 2*-nonconvexity-point set corresponding to an open bounded weakly* 2*-convex subset of* $\mathbb{R}^3$ *consisting of a finite number of components is the interior of a convex polyhedron* is refuted by Theorem 2.5, in the proof of which, an example of an open bounded weakly 2-convex subset of $\mathbb{R}^3$ consisting of 17 components and having an open bounded convex 2-nonconvexity-point set distinct from the interior of a convex polyhedron is constructed.

Probably the most natural next question to study would be generalizing the results of this paper to the sets belonging to $\mathbf{WC^n_{n-1}}\backslash\mathbf{C^n_{n-1}}$, $\forall n > 3$.

2. Main results

Given two points $x, y \in \mathbb{R}^n$, we will denote by xy the open line segment between those points, by $\|x-y\|$ its length, and by $\langle x, y\rangle$ the scalar product of the radius vectors corresponding to those points. Let also

$$\begin{aligned} S^2 &:= \{z \in \mathbb{R}^3 : \|z\| = 1\}; \\ U(y,\varepsilon) &:= \{x \in \mathbb{R}^3 : \|x-y\| < \varepsilon\}, && y \in \mathbb{R}^3, \\ L_a(y) &:= \{x \in \mathbb{R}^3 : \langle a, x-y\rangle = 0\}, && \varepsilon > 0, \\ \eta_a(y) &:= \{ta + y : t \in [0,+\infty)\}, && a \in S^2. \\ \gamma_a(y) &:= \{ta + y : t \in (-\infty,+\infty)\}, \end{aligned}$$

Theorem 2.1. *Let $E \subset \mathbb{R}^3$ be a closed subset such that* $\operatorname{Int} E \neq \varnothing$. *If E is weakly 2-convex, then* $\operatorname{Int} E$ *is weakly 2-convex.*

Proof. We will prove the theorem by contradiction. Suppose that $\operatorname{Int} E$ is not weakly 2-convex. Then there exists a point $y \in \partial E \cap (\operatorname{Int} E)^{\triangle}_2$.

First, show that for y and any plane $L_a(y)$, $a \in S^2$, there exist points $x_a \in L_a(y) \cap \operatorname{Int} E$ and a constant $d > 0$ such that $U(x_a, d) \subset \operatorname{Int} E$, moreover, d depends only on y and does not depend on $a \in S^2$ and on $x_a \in L_a(y)$.

Without loss of generality, suppose that $y = O$, where $O \in \mathbb{R}^3$ is the origin.

Consider the homeomorphism

$$\phi\colon \mathbb{R}^3\backslash\{O\} \to S^2 \times (0,+\infty), \qquad z \mapsto \left(\frac{z}{\|z\|}, \|z\|\right).$$

Let also $\sigma\colon S^2 \times (0,+\infty) \to S^2$ be the central projection on the sphere, i.e.,

$$\sigma(z) := \frac{z}{\|z\|}.$$

Then σ is open.

Let z_a be an arbitrary fixed point of $L_a(y) \cap \operatorname{Int} E$, $a \in S^2$. Since $\operatorname{Int} E$ is open, there exist open balls $U_a := U(z_a, \varepsilon_a)$, $a \in S^2$, such that $\overline{U_a} \subset \operatorname{Int} E$. Then the images $\sigma(U_a)$, $a \in S^2$, are open subsets of S^2.

Let $V_a := \left\{b \in S^2 : L_b(y) \cap U_a \neq \varnothing\right\}$, $a \in S^2$. It is not difficult to verify that V_a, $a \in S^2$, are open too. Indeed, let L be a plane passing through yz_a.

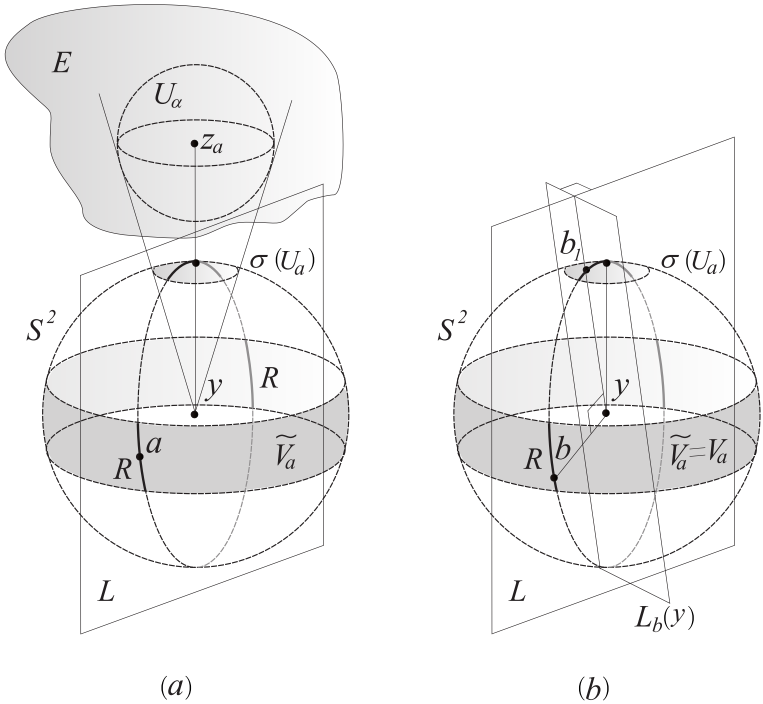


FIGURE 2.1.

Let $a_1 \in L \cap \sigma(U_a)$ and $a_2, a_3 \in L \cap S^2$ be perpendicular to a_1. Then there is a one-to-one correspondence between points a_1 and the pairs $\{a_2, a_3\}$ in every plane L. Define the set of all points a_2, a_3 with respect to L by R. The set R is open with respect to $L \cap S^2$. Rotating R around yz_a, we obtain the open set $\widetilde{V}_a$ on S^2; see Figure 2.1(a). Any plane $L_b(y)$, $b \in \widetilde{V}_a$, intersects U_a, since $b \in L \cap \widetilde{V}_a$, $L \perp L_b(y)$, and $b \mapsto b_1 = L_b(y) \cap L \cap \sigma(U_a)$; see Figure 2.1 (b). On the other hand, if $L_b(y) \cap U_a \neq \varnothing$, $b \in S^2$, then

$$b_1 = (L_b(y) \cap \sigma(U_a)) \cap L, \quad \text{where} \quad L \perp L_b(y),$$

is perpendicular to b. Therefore, $b \in \widetilde{V}_a$. Thus, $\widetilde{V}_a = V_a, \quad a \in S^2$. Moreover, $\bigcup_{a \in S^2} V_a$ is a cover of the unit sphere S^2, since for any $a \in S^2$ there exists an open subset $V_a \subset S^2$ containing a. By the Heine-Borel theorem, there exists a subcover $\bigcup\limits_{j \in M} V_{a_j}$ of S^2 such that $a_j \in S^2$, $j \in M$ and M is finite.

Let E_i, $i \in N \subseteq \mathbb{N}$, be the components of E with non-empty interior. Then for any $j \in M$ there exists $i(j) \in N$ such that $\overline{U_{a_j}} \subset \operatorname{Int} E_{i(j)}$.

Consider the distance functions

$$d_i(x) := \inf_{x^0 \in \partial(\operatorname{Int} E_i)} \|x - x^0\|, \quad x \in \operatorname{Int} E_i, \quad i \in N.$$

They are continuous in the domains $\operatorname{Int} E_i$, $i \in N$. Then their restrictions to the compacts $\overline{U_{a_j}}$ attain their minimum values $d_j > 0$ on those compacts, i.e.,

$$d_j := \min_{x \in \overline{U_{a_j}}} d_{i(j)}(x), \quad j \in M.$$

Since M is finite, there exists

$$d := \min_{j \in M} d_j > 0.$$

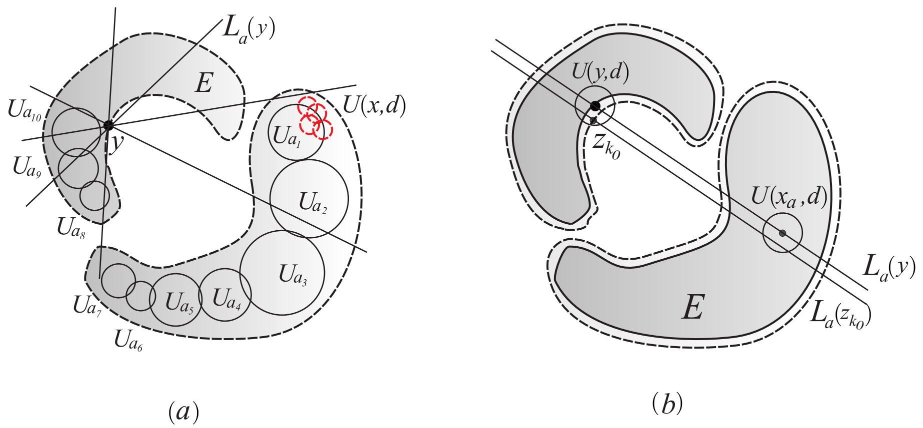


Figure 2.2.

Then $U(x, d) \subset \operatorname{Int} E$ for any point $x \in \overline{U_{a_j}}$, $j \in M$; see Figure 2.2 (a). Moreover, for any $a \in S^2$ there exists $j \in M$ such that $L_a(y) \cap \overline{U_{a_j}} \neq \varnothing$ by the construction.

Now, consider the neighborhood $U(y, d)$ of the point y; see Figure 2.2 (b). Since E is weakly 2-convex, there is a family of open weakly 2-convex sets G_k, $k = 1, 2, \ldots$, approximating E from the outside. Then there exists an index $k_0 \in \mathbb{N}$ such that $\partial G_{k_0} \cap U(y, d) \neq \varnothing$. Otherwise, $U(y, d) \subset G_k$ for all $k \in \mathbb{N}$. In this case, considering Definition 1.2, $U(y, d) \subset \bigcap_k G_k = E$, which contradicts the fact that $y \in \partial E$.

Choose a point $z_{k_0} \in \partial G_{k_0} \cap U(y, d)$ and draw an arbitrary plane $L_a(z_{k_0})$, $a \in S^2$, through z_{k_0}. Consider the plane $L_a(y)$ parallel to $L_a(z_{k_0})$. Since $U(x_a, d) \subset \operatorname{Int} E \subset E$ for the point x_a corresponding to $L_a(y)$, it follows that $L_a(z_{k_0}) \cap U(x_a, d) \neq \varnothing$ and, therefore, $L_a(z_{k_0}) \cap E \neq \varnothing$. Since $G_{k_0} \supset E$, we also have $L_a(z_{k_0}) \cap G_{k_0} \neq \varnothing$.

The plane $L_a(z_{k_0})$ is arbitrary, so $z_{k_0} \in \partial G_{k_0}$ is a 2-nonconvexity point of G_{k_0}, which gives a contradiction with the fact that G_{k_0} is weakly 2-convex. □

Theorem 2.2. *Let $P \subset \mathbb{R}^3$ be the union of convex polyhedron interiors, and $L \subset \mathbb{R}^3$ be the union of a finite collection of planes, satisfying the following conditions:*

(1) *$L \supset \partial P$;*
(2) *the set $L \cup P$ does not contain planes intersecting P.*

Let $D \subset \mathbb{R}^3$ be an open convex set containing $\overline{P}$. Then

$$E(D, P, L) := (D \backslash P) \backslash L \in \mathbf{WC^3_2} \backslash \mathbf{C^3_2}, \quad E(D, P, L)^{\triangle}_2 = P.$$

Proof. First, prove that $E := E(D, P, L)$ is weakly 2-convex. Since the collection of planes generating L is finite, then

$$\partial L = L.$$

Considering also the general properties of set boundaries and condition (1), we obtain that

$$\partial E \subset \partial D \cup \partial P \cup \partial L = \partial D \cup L.$$

Since D is convex and open, through every point of ∂D, there passes a plane not intersecting D and, therefore, not intersecting $E \subset D$. And, through every point of L, there passes a plane not intersecting E obviously. Thus, E is open and $E \in \mathbf{WC^3_2}$.

Now prove that $E^{\triangle}_2 = P$, which automatically shows that $E \in \mathbf{WC^3_2} \backslash \mathbf{C^3_2}$.

From condition (1) it follows that $L \bigcup P = L \bigcup \overline{P}$. Then, by condition (2)

$$L_a(x) \not\subset \overline{P} \quad \forall x \in P, \quad \forall a \in S^2.$$

Moreover, $D \supset \overline{P}$ and D is open. Then

$$L_a(x) \cap \left(D \backslash \overline{P}\right) \neq \varnothing, \quad x \in P, \quad a \in S^2.$$

Furthermore, since $L_a(x)$ is not contained in L by (1), and the number of planes generating L is finite, it follows that

$$L_a(x) \cap \left(\left(D \backslash \overline{P}\right) \backslash L\right) = L_a(x) \cap E \neq \varnothing, \quad x \in P, \quad a \in S^2.$$

Thus, $P \subset E^{\triangle}_2$.

Notice that $E^{\triangle}_2 \subset \mathbb{R}^3 \backslash E = P \cup \partial E \cup (\mathbb{R}^3 \backslash D)$. Since $E \in \mathbf{WC^3_2}$ and D is convex, then $\partial E \cap E^{\triangle}_2 = \varnothing$ and $(\mathbb{R}^3 \backslash D) \cap E^{\triangle}_2 = \varnothing$. Thus, $E^{\triangle}_2 \subset P$. □

Corollary 2.3. *For any convex polyhedron P,*

$$E(D, \operatorname{Int} P, L) \in \mathbf{WC^3_2} \backslash \mathbf{C^3_2}, \qquad E(D, \operatorname{Int} P, L)^{\triangle}_2 = \operatorname{Int} P.$$

Despite its simplicity, Theorem 2.2 is needed to show that, firstly, for any set P such that there is a collection of planes satisfying conditions (1) and (2), there exists a set $E \in \mathbf{WC^3_2}\backslash\mathbf{C^3_2}$ such that $E^{\triangle}_2 = P$. Secondly, it demonstrates that there is no one-to-one correspondence between the open sets belonging to $\mathbf{WC^3_2}\backslash\mathbf{C^3_2}$ and their 2-nonconvexity-point sets. Indeed, since $\mathbb{R}^3$ is convex, it implies that

$$E(\mathbb{R}^3, P, L) \in \mathbf{WC^3_2}\backslash\mathbf{C^3_2} \quad \text{and} \quad E(\mathbb{R}^3, P, L)^{\triangle}_2 = P$$

for any set P and any collection of planes satisfying conditions (1) and (2). Moreover, if the set P is bounded, then there exist infinitely many sets $D \supset \overline{P}$, and, therefore, many sets $E(D, P, L) \in \mathbf{WC^3_2}\backslash\mathbf{C^3_2}$ such that

$$E(D, P, L)^{\triangle}_2 = P.$$

Lemma 2.4. *Suppose that E is defined by* (1.1)*. Suppose that $H_1 \subset \mathbb{R}^2$ is the half-plane bounded by the straight line γ^1 and such that $\mathbb{R}^2\backslash H_1 \supset P$. Then*

$$A := E\backslash H_1 \in \mathbf{WC^2_1}\backslash\mathbf{C^2_1}, \qquad\qquad A^{\triangle}_1 = \operatorname{Int} P.$$

Proof. The set A is weakly 1-convex. Indeed, $A \subset E$, $\partial A \subset \partial E$, and $E \in \mathbf{WC^2_1}$, therefore, the straight line passing through a point $y \in \partial A$ and not intersecting E does not intersect A. We now prove that $A^{\triangle}_1 = \operatorname{Int} P$.

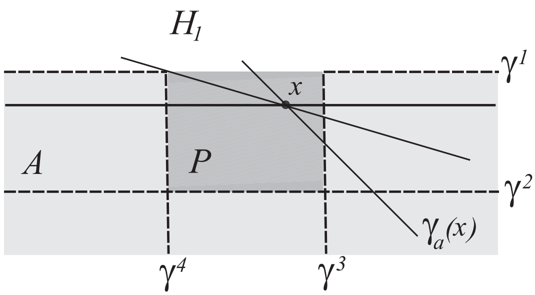


FIGURE 2.3.

Let $x \in \operatorname{Int} P$; see Figure 2.3. Since P is closed and bounded, D is open, and $P \subset D$, it follows that

$$\eta_a(x) \cap (D\backslash P) \neq \varnothing \quad \forall a \in S^1.$$

Moreover, the collection of straight lines γ^j, $j = \overline{1,p}$, containing ∂P is finite, and $\gamma^j \cap \operatorname{Int} P = \varnothing$. In this case, $\eta_a(x) \cap E \neq \varnothing$, $a \in S^1$.

Let $\eta'_a(x) := \eta_a(x)$ such that $\eta_a(x) \subset \mathbb{R}^2\backslash H_1$. Then $\eta'_a(x) \cap (E\backslash H_1) \neq \varnothing$. The union of all rays $\eta'_a(x)$ generates the closed half-plane that is contained in $\mathbb{R}^2 \backslash H_1$. Therefore,

$$\gamma_a(x) \cap A = \gamma_a(x) \cap (E \backslash H_1) \neq \varnothing, \quad a \in S^1, \quad x \in \operatorname{Int} P.$$

Thus, $A_1^{\triangle} \supset \operatorname{Int} P$.

Notice that $A_1^{\triangle} \subset \mathbb{R}^2 \backslash A = H_1 \cup (\mathbb{R}^2 \backslash D) \cup \bigcup_j \gamma^j \cup \operatorname{Int} P$.

The half-plane H_1 is the union of the straight lines parallel to γ^1, then $A_1^{\triangle} \cap H_1 = \varnothing$. Since D is convex, it implies that $A_1^{\triangle} \cap (\mathbb{R}^2 \backslash D) = \varnothing$. It is also obvious that $A_1^{\triangle} \cap \bigcup_j \gamma^j = \varnothing$. Thus, $A_1^{\triangle} \subset \operatorname{Int} P$. □

Theorem 2.5. *There exists a bounded open set $E \in \mathbf{WC}_2^3 \backslash \mathbf{C}_2^3$ with a finite collection of components, and such that $E_2^{\triangle}$ is open, bounded, convex, and distinct from the interior of a convex polyhedron.*

Proof. We will prove the theorem by constructing an example of a set satisfying its conditions.

Consider the following sets:

$$\begin{aligned} G^0 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 < -(x_2)^2 + 4\}, \\ G^1 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -3 < x_1 < 0\}, \\ G^2 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -1 < x_2 < 1\}, \\ G^3 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3\}, \end{aligned} \qquad G = \bigcap_{k=0}^{3} G^k.$$

The set G is open and convex as the intersection of a finite collection of open convex sets. It is also bounded and distinct from the interior of a convex polyhedron; see Figure 2.4 (a).

Consider the following supporting planes for G:

$$\begin{aligned} L^1 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = -3, 0\}, \\ L^2 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = -1, 1\}, \\ L^3 &:= \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = 0\}, \end{aligned} \qquad L := \bigcup_{k=1}^{3} L^k.$$

Let also

$$H := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : -4 < x_1 < 1, -1 < x_3\};$$

Now, consider the open set

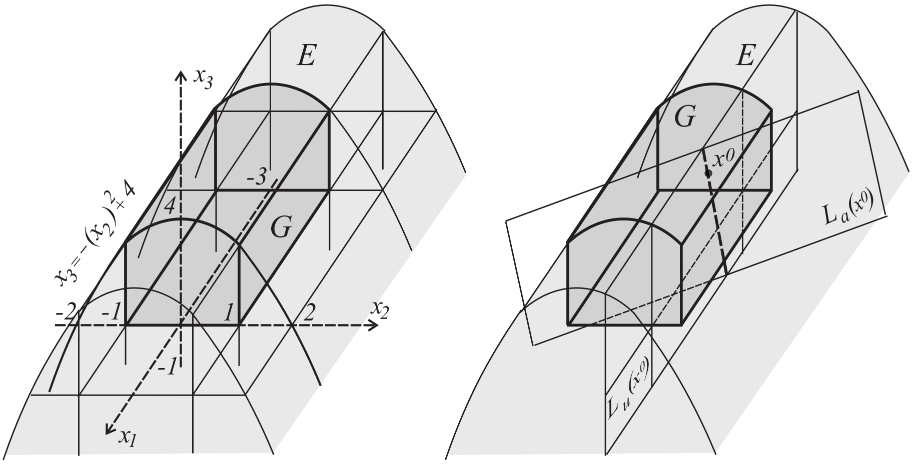
$$E := (G^0 \cap H) \backslash (G \cup L).$$

Since $\partial G \subset (\partial G^0 \cup L)$ and $\partial L = L$, it implies that

$$\partial E \subset \partial G^0 \cup \partial H \cup \partial G \cup \partial L \subset \partial G^0 \cup L \cup \partial H.$$

Moreover, through every point $y = (y_1, y_2, y_3) \in \partial G^0$, there passes the plane

$$L(y) := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = -2y_2 x_2 - (y_2)^2 + 4\}$$



(a) (b)

FIGURE 2.4.

not intersecting G^0, and, therefore, not intersecting $E \subset G^0$, since $L(y)$ is generated by the parallel translation of the tangent straight line to the parabola determining G^0 along the axis Ox_1. Furthermore, $L \cup \partial H$ is contained in the union of the planes not intersecting E. Thus, $E \in \mathbf{WC_2^3}$.

Show that $E_2^{\triangle} = G$. Let $x^0 = (x_1^0, x_2^0, x_3^0) \in G$; see Figure 2.4 (b). Draw the plane $L_u(x^0)$, $u = (0, 1, 0)$, parallel to the coordinate plane x_1Ox_3, i.e.,

$$L_u(x^0) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 = x_2^0\}.$$

Any straight line lying in $L_u(x^0)$ and passing through x^0 intersects E. Indeed, $\big(L_u(x^0) \cap E\big) \in \mathbf{WC_1^2} \backslash \mathbf{C_1^2}$ with respect to $L_u(x^0)$ by Lemma 2.4 as $A = \big(L_u(x^0) \cap E\big)$. Moreover,

$$L_u(x^0) \cap G = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{array}{r} -3 < x_1 < 0,\ x_2 = x_2^0 \\ 0 < x_3 < -(x_2^0)^2 + 4 \end{array} \right\}$$
$$= \big(L_u(x^0) \cap E\big)_1^{\triangle}.$$

Now consider an arbitrary plane $L_a(x^0)$, $a \in S^2$. It either coincides with $L_u(x^0)$, or intersects $L_u(x^0)$ by a straight line passing through x^0.

Therefore,

$$L_a(x^0) \cap E \neq \varnothing, \quad a \in S^2, \quad x^0 \in G.$$

Thus, $G \subset E_2^{\triangle}$.

Notice that $E_2^{\triangle} \subset \mathbb{R}^3 \backslash E = \big(\mathbb{R}^3 \backslash (G^0 \cap H)\big) \cup G \cup L$.

Since $\mathbb{R}^3 \backslash (G^0 \cap H)$ is the complement of the convex set to $\mathbb{R}^3$, it follows that

$E_2^{\triangle} \cap \big(\mathbb{R}^3 \backslash (G^0 \cap H)\big) = \varnothing$. Since L is the union of planes not intersecting E, it implies that $E_2^{\triangle} \cap L = \varnothing$. Then $E_2^{\triangle} \subset G$. □

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