Proceedings of the International Geometry Center
Volume 19 · Number 2 · (2026) · Paper 8 · 20 pages
https://doi.org/10.15673/pigc.3328

On semigroups which admit only discrete left-continuous Hausdorff topology

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Dedicated to the memory of my teacher
Igor Guran (23.02.1955–08.04.2025)

Abstract. We give a sufficient condition under which every left-continuous (right-continuous) Hausdorff topology on a semigroup S is discrete. We construct a submonoid $\mathscr{C}_+(a,b)$ (resp., $\mathscr{C}_-(a,b)$) of the bicyclic monoid which contains a family $\{S_\alpha\colon \alpha\in\mathfrak{c}\}$ of continuum many subsemigroups with the following properties: (i) every left-continuous (resp., right-continuous) Hausdorff topology on S_α is discrete; (ii) every semigroup S_α admits a non-discrete right-continuous (resp., left-continuous) Hausdorff topology which is not left-continuous (resp., right-continuous); (iii) every semigroup S_α isomorphically embeds into a Hausdorff compact topological semigroup. Also we construct a submonoid $\mathscr{C}_{\mathbb{Z}}^+$ (resp., $\mathscr{C}_{\mathbb{Z}}^-$) of the extended bicyclic semigroup which contains a family $\{S_\alpha\colon \alpha\in\mathfrak{c}\}$ of continuum many subsemigroups with the above described properties.

Анотація. Знайдено достатні умови за виконання яких кожна неперервна зліва (неперервна справа) гаусдорфова топологія на напівгрупі S дискретна. Побудовано підмоноїд $\mathscr{C}_+(a,b)$ (відп., $\mathscr{C}_-(a,b)$) біциклічного моноїда, який містить сім'ю $\{S_\alpha\colon \alpha\in\mathfrak{c}\}$ потужності континуум піднапівгруп з такими властивостями: (i) кожна неперервна зліва (відп., неперервна справа) гаусдорфова топологія на напівгрупі S_α дискретна; (ii) на кожній напівгрупі S_α існує недискретна неперервна справа (відп., неперервна зліва) гаусдорфова топологія, яка не є неперервною зліва (відп., неперервною справа); (iii) кожна напівгрупа S_α ізоморфно занурюється в гаусдорфову компактну топологічну напівгрупу. Також побудовано підмоноїд $\mathscr{C}_{\mathbb{Z}}^+$ (відп., $\mathscr{C}_{\mathbb{Z}}^-$) розширеної біциклічної напівгрупи, який містить сім'ю $\{S_\alpha\colon \alpha\in\mathfrak{c}\}$ потужності континуум піднапівгруп з вище описаними властивостями.

Keywords: Semitopological semigroup, topological semigroup, left topological semigroup, right topological semigroup, discrete topology, bicyclic monoid, compact, extended bicyclic semigroup, embedding

MSC2020: 22A15, 54C08, 54D10, 54D30, 54E52, 54H10

ISSN 2072-9812 (print)
ISSN 2409-8906 (online)

1. Introduction, motivation and main definitions

In this paper we shall follow the terminology of [7, 9, 12, 29].

By ω we denote the set of all non-negative integers and by $\mathbb{Z}$ the set of all integers. Throughout these notes we always assume that all topological spaces involved are Hausdorff – unless explicitly stated otherwise.

Definition 1.1. Let X, Y and Z be topological spaces. A map

$$f\colon X\times Y\to Z, \qquad (x,y)\mapsto f(x,y),$$

is called

(i) *right* (*left*) *continuous* if it is continuous in the right (left) variable; i.e., for every fixed $x_0\in X$ ($y_0\in Y$) the map $Y\to Z$, $y\mapsto f(x_0,y)$, ($X\to Z$, $x\mapsto f(x,y_0)$) is continuous;

(ii) *separately continuous* if it is both left and right continuous;

(iii) *jointly continuous* if it is continuous as a map between the product space $X\times Y$ and the space Z.

Definition 1.2 ([7, 29]). Let S be a non-void topological space which is provided with an associative multiplication (a semigroup operation)

$$\mu\colon S\times S\to S, \qquad (x,y)\mapsto \mu(x,y)=xy$$

Then the pair (S,μ) is called

(i) a *right topological semigroup* if the map μ is right continuous, i.e., all interior left shifts $\lambda_s\colon S\to S$, $x\mapsto sx$, are continuous maps, $s\in S$;

(ii) a *left topological semigroup* if the map μ is left continuous, i.e., all interior right shifts $\rho_s\colon S\to S$, $x\mapsto xs$, are continuous maps, $s\in S$;

(iii) a *semitopological semigroup* if the map μ is separately continuous;

(iv) a *topological semigroup* if the map μ is jointly continuous.

We usually omit the reference to μ and write simply S instead of (S,μ). It goes without saying that every topological semigroup is also semitopological and every semitopological semigroup is both a right and left topological semigroup.

A topology τ on a semigroup S is called:

- a *semigroup* topology if (S,τ) is a topological semigroup;
- a *shift-continuous* topology if (S,τ) is a semitopological semigroup;
- a *left-continuous* topology if (S,τ) is a left topological semigroup;
- a *right-continuous* topology if (S,τ) is a right topological semigroup.

If S is a semigroup then by $E(S)$ we denote the set of all idempotents of S.

The *bicyclic monoid* $\mathscr{C}(a,b)$ is the semigroup with the identity 1 generated by two elements a and b subjected only to the condition $ab=1$. The semigroup operation on $\mathscr{C}(a,b)$ is determined as follows:

$$b^k a^l \cdot b^m a^n = \begin{cases} b^{k-l+m}a^n, & \text{if } l < m; \\ b^k a^n, & \text{if } l = m; \\ b^k a^{l-m+n}, & \text{if } l > m. \end{cases}$$

It is well known that the bicyclic monoid $\mathscr{C}(a,b)$ is a bisimple (and hence simple) combinatorial E-unitary inverse semigroup and every non-trivial congruence on $\mathscr{C}(a,b)$ is a group congruence, see [9].

On $\mathbf{B}_\omega = \omega \times \omega$ we define the semigroup operation “ $\cdot$ ” in the following way

$$(k,l)\cdot(m,n) = \begin{cases} (k-l+m,n), & \text{if } l < m; \\ (k,n), & \text{if } l = m; \\ (k,l-m+n), & \text{if } l > m, \end{cases} \tag{1.1}$$

$k,l,m,n \in \omega$. It is obvious that the bicyclic monoid $\mathscr{C}(a,b)$ is isomorphic to the semigroup $\mathbf{B}_\omega$ by the mapping $b^i a^j \mapsto (i,j)$. The set $\mathscr{C}_\mathbb{Z} = \mathbb{Z}\times\mathbb{Z}$ with the semigroup operation (1.1) is called the *extended bicyclic semigroup*, see [32].

A semigroup S is called *inverse* if for any $s \in S$ there exists a unique $t \in S$ such that $sts = s$ and $tst = t$. In this case the element t is called *inverse of* s and it is denoted by s^{-1}. Every inverse semigroup S admits the *natural partial order*:

$$s \preccurlyeq t \quad \text{if and only if there exists an idempotent } e \in S \text{ such that } s = et.$$

It is well known that the bicyclic monoid and the extended bicyclic semigroup are inverse semigroups. Also the natural partial order $\preccurlyeq$ on $\mathscr{C}_\mathbb{Z}$ and $\mathbf{B}_\omega$ is determined in the following way

$$(k,l) \preccurlyeq (m,n) \iff (k-l = m-n \text{ and } m \leqslant k) \iff (k-l = m-n \text{ and } n \leqslant l).$$

For semigroups S and T a map $\alpha\colon S \to T$ is said to be an *anti-homomorphism* if $(s\cdot t)\alpha = (t)\alpha\cdot(s)\alpha$. A bijective anti-homomorphism of semigroups is called an *anti-isomorphism*.

It is well known that topological algebra studies how the topological properties of its objects influence their algebraic properties, and vice versa. There are two main problems in topological algebra: the problem of non-discrete topologization and the problem of embedding into objects with some topological-algebraic properties.

In mathematical literature the question about non-discrete (Hausdorff) topologization was posed by Markov in [27]. Pontryagin gave well known conditions on a base at the unity of a group for its non-discrete topologization (see Theorem 4.5 of [21]). Various authors have refined Markov's question:

> Can a given infinite group G endowed with a non-discrete group topology be embedded into a compact topological group?

Again, for an arbitrary Abelian group G the answer is affirmative, but there is a non-Abelian topological group that cannot be embedded into any compact topological group (see Section 9 of [10]).

Also, Ol'shanskiy [28] constructed an infinite countable group G such that every Hausdorff group topology on G is discrete. In [23], the authors constructed the first examples of infinite hereditarily non-topologizable groups. In [33], Ye. Zelenyuk proved that on every countable group there exists a non-discrete regular topology with continuous shifts and inversion. Also, every countable locally finite group admits a non-discrete Hausdorff topology (see [5]). Taimanov presented in [30], a commutative semigroup $\mathfrak{T}$ which admits only the discrete Hausdorff semigroup topology. Also in [31] he gave sufficient conditions on a commutative semigroup to have a non-discrete semigroup topology. In [15], it is proved that each shift-continuous T_1-topology on the Taimanov semigroup $\mathfrak{T}$ is discrete. The bicyclic monoid admits only the discrete shift-continuous Hausdorff topology (see [11, 6]). In [8], non-discrete shift-continuous (semigroup, semigroup inverse) T_1-topologies on the bicyclic monoid $\mathscr{C}(a,b)$ are constructed. In [8] sufficient conditions are presented, under which a shift-continuous T_1-topology on $\mathscr{C}(a,b)$ is discrete. Also every shift-continuous Hausdorff topology on the extended bicyclic semigroup and any interassociate of the bicyclic monoid is discrete (see [13, 18]).

It is known by [1, 22, 24] that stable and Γ-compact topological semigroups do not contain the bicyclic semigroup. The problem of embedding the bicyclic monoid into compact-like topological semigroups was studied in [2, 3, 4, 20].

In [26], the anti-isomorphic subsemigroups of the bicyclic monoid

$$\mathscr{C}_{+}(a,b) = \left\{b^i a^j \in \mathscr{C}(a,b) \colon i \leqslant j\right\},$$
$$\mathscr{C}_{-}(a,b) = \left\{b^i a^j \in \mathscr{C}(a,b) \colon i \geqslant j\right\}$$

are studied. In [17] it was proven that every Hausdorff left-continuous (right-continuous) topology on the monoid $\mathscr{C}_{+}(a,b)$ ($\mathscr{C}_{-}(a,b)$) is discrete and there exists a compact Hausdorff topological monoid S which contains $\mathscr{C}_{+}(a,b)$ ($\mathscr{C}_{-}(a,b)$) as a submonoid. Also, in [17] it was constructed a non-discrete right-continuous (left-continuous) topology τ_p^{+} (τ_p^{-}) on the semigroup $\mathscr{C}_{+}(a,b)$ ($\mathscr{C}_{-}(a,b)$) which is not left-continuous (right-continuous).

Similarly as in [26] we define the following two subsemigroups

$$\mathscr{C}_{\mathbb{Z}}^{+} = \{(i,j) \in \mathscr{C}_{\mathbb{Z}} \colon i \leqslant j\} \quad \text{and} \quad \mathscr{C}_{\mathbb{Z}}^{-} = \{(i,j) \in \mathscr{C}_{\mathbb{Z}} \colon i \geqslant j\}$$

of $\mathscr{C}_{\mathbb{Z}}$. The semigroup operation of $\mathscr{C}_{\mathbb{Z}}$ implies that $\mathscr{C}_{\mathbb{Z}}^{+}$ and $\mathscr{C}_{\mathbb{Z}}^{-}$ are anti-isomorphic subsemigroups of $\mathscr{C}_{\mathbb{Z}}$.

In the paper, we give a sufficient condition under which every left-continuous (right-continuous) Hausdorff topology on a semigroup S is discrete. We construct a submonoid $\mathscr{C}_{+}(a,b)$ (resp., $\mathscr{C}_{-}(a,b)$) of the bicyclic monoid which contains a family $\{S_\alpha\colon \alpha \in \mathfrak{c}\}$ of continuum many subsemigroups with the following properties:

(i) every left-continuous (resp., right-continuous) Hausdorff topology on S_α is discrete;
(ii) every semigroup S_α admits a non-discrete right-continuous (resp., left-continuous) Hausdorff topology which is not left-continuous (resp., right-continuous);
(iii) every semigroup S_α isomorphically embeds into a Hausdorff compact topological semigroup.

Also we construct a submonoid $\mathscr{C}_{\mathbb{Z}}^{+}$ (resp., $\mathscr{C}_{\mathbb{Z}}^{-}$) of the extended bicyclic semigroup which contains a family $\{S_\alpha\colon \alpha \in \mathfrak{c}\}$ of continuum many subsemigroups with the above described properties.

2. On topologizations of subsubsemigroups of the semigroups $\mathscr{C}_{+}(a,b)$ and $\mathscr{C}_{-}(a,b)$

The following example shows that the bicyclic monoid $\mathscr{C}(a,b)$ admits a Hausdorff right-continuous (left-continuous) topology τ_p^r (τ_p^l) such that left (right) translations are not continuous in $(\mathscr{C}(a,b),\tau_p^r)$ $((\mathscr{C}(a,b),\tau_p^l))$.

Example 2.1. Fix an arbitrary prime positive integer p. For any $b^ia^j \in \mathscr{C}(a,b)$ and any $n\in\omega$ we denote

$$U_n^r(b^ia^j)=\left\{b^ia^{j+p^n\cdot k}\colon k\in\omega\right\}.$$

Then the family $\mathscr{B}_p^r=\left\{\mathscr{B}_p^r(b^ia^j)\colon i,j\in\omega\right\}$, where

$$\mathscr{B}_p^r(b^ia^j)=\left\{U_n^r(b^ia^j)\colon n\in\omega\right\}$$

for $b^ia^j\in\mathscr{C}(a,b)$, satisfies the properties (BP1)–(BP4) of [12], and hence it generates a Hausdorff topology τ_p^r on the bicyclic monoid $\mathscr{C}(a,b)$. It is obvious that τ_p^r is a non-discrete topology on $\mathscr{C}(a,b)$, and moreover all points of $(\mathscr{C}(a,b),\tau_p^r)$ are non-isolated.

We claim that τ_p^r is a right-continuous topology on $\mathscr{C}(a,b)$. Indeed, for any $b^{i_1}a^{j_1}, b^{i_2}a^{j_2} \in \mathscr{C}(a,b)$ and any $n \in \omega$ we have that

$$
\begin{aligned}
b^{i_1}a^{j_1} \cdot U_n^r(b^{i_2}a^{j_2}) &= \left\{ b^{i_1}a^{j_1} \cdot b^{i_2}a^{j_2+p^n\cdot k} \colon k \in \omega \right\} \\
&= \begin{cases}
\{b^{i_1-j_1+i_2}a^{j_2+p^n\cdot k} : k \in \omega\}, & \text{if} \quad j_1 < i_2; \\
\{b^{i_1}a^{j_2+p^n\cdot k} : k \in \omega\}, & \text{if} \quad j_1 = i_2; \\
\{b^{i_1}a^{j_1-i_2+j_2+p^n\cdot k} : k \in \omega\}, & \text{if} \quad j_1 > i_2
\end{cases} \\
&= \begin{cases}
U_n^r(b^{i_1-j_1+i_2}a^{j_2}), & \text{if} \quad j_1 < i_2; \\
U_n^r(b^{i_1}a^{j_2}), & \text{if} \quad j_1 = i_2; \\
U_n^r(b^{i_1}a^{j_1-i_2+j_2}), & \text{if} \quad j_1 > i_2,
\end{cases}
\end{aligned}
$$

and hence the semigroup operation is right-continuous in $(\mathscr{C}(a,b), \tau_p^r)$.

Fix arbitrary $b^{i_1}a^{j_1}, b^{i_2}a^{j_2} \in \mathscr{C}(a,b)$ such that $j_1 < i_2$. Then we have that $b^{i_1}a^{j_1} \cdot b^{i_2}a^{j_2} = b^{i_1-j_1+i_2}a^{j_2}$. For any $n \in \omega$ there exists k_0 such that the neighbourhood $U_n^r(b^{i_1}a^{j_1})$ contains the elements $b^{i_1}a^{j_1+p^n\cdot k}$ with the property that $j_1 + p^n \cdot k > i_2$ for all $k > k_0$. This implies that

$$
b^{i_1}a^{j_1+p^n\cdot k} \cdot b^{i_2}a^{j_2} = b^{i_1}a^{j_1+p^n\cdot k-i_2+j_2}
$$

and hence we get that

$$
U_n^r(b^{i_1}a^{j_1}) \cdot b^{i_2}a^{j_2} \nsubseteq U_m^r(b^{i_1-j_1+i_2}a^{j_2})
$$

for any $n, m \in \omega$. Thus, the semigroup operation is not left-continuous in $(\mathscr{C}(a,b), \tau_p^r)$.

The anti-isomorphism of the bicyclic monoid

$$
\begin{aligned}
\mathfrak{AI} \colon \mathscr{C}(a,b) &\to \mathscr{C}(a,b) \\
b^i a^j &\mapsto b^j a^i
\end{aligned}
\qquad \forall i, j \in \omega
$$

generated a left-continuous topology on $\mathscr{C}(a,b)$ in the following way. For any $b^i a^j \in \mathscr{C}(a,b)$ and any $n \in \omega$ we denote

$$
U_n^l(b^i a^j) = (U_n^r(b^j a^i))\mathfrak{AI}.
$$

Then the family

$$
B_p^l = \left\{ \mathscr{B}_p^r(b^i a^j) \colon b^i a^j \in \mathscr{C}(a,b), n \in \omega \right\},
$$

where $B_p^l(b^i a^j) = \left\{ U_n^r(b^i a^j) \colon n \in \omega \right\}$, satisfies the properties (BP1)–(BP4) of [12], and hence it generates a non-discrete Hausdorff topology τ_p^l on the bicyclic monoid $\mathscr{C}(a,b)$. Since $\mathfrak{AI} \colon \mathscr{C}(a,b) \to \mathscr{C}(a,b)$ is the anti-isomorphism, the semigroup operation is left-continuous in $(\mathscr{C}(a,b), \tau_p^l)$ but it is not right-continuous.

The following theorem gives a sufficient condition on a semigroup S under which every Hausdorff left-continuous (right-continuous) topology on S is discrete.

Theorem 2.2. *Let S be an infinite semigroup. If for any $s \in S$ there exists an idempotent $e_s \in S$ such that $s \in S \setminus Se_s$ ($s \in S \setminus e_sS$) and the set $S \setminus Se_s$ ($S \setminus e_sS$) is finite, then every Hausdorff left-continuous (right-continuous) topology on S is discrete.*

Proof. Fix an arbitrary $s \in S$. It is obvious that for the idempotent e_s of the semigroup S the right translation $\rho_{e_s} \colon S \to S$, $s \mapsto se_s$ is a continuous retraction of the topological space S. Hausdorffness of S and Exercise 1.5.C of [12] imply that Se_s is a closed subspace of S. Since the set $S \setminus Se_s$ is finite and open, the point s is isolated in S.

The proof of the dual statement is similar. □

Theorem 2.2 implies the following corollary.

Corollary 2.3. *Let S be a subsemigroup of $\mathscr{C}_+(a,b)$ ($\mathscr{C}_-(a,b)$) such that the set $E(S)$ is infinite. Then every Hausdorff left-continuous (right-continuous) topology on S is discrete.*

Proof. Fix an arbitrary $b^i a^{i+k} \in S$, $i,k \in \omega$. Since the set $E(S)$ is infinite, there exists a positive integer p such that $p > k$ and $b^{i+p}a^{i+p} \in E(S)$. The semigroup operation of $\mathscr{C}_+(a,b)$ (and hence of S) implies that for any $b^s a^t \in S$ we have that

$$b^s a^t \cdot b^{i+p}a^{i+p} = \begin{cases} b^{s-t+i+p}a^{i+p}, & \text{if} \quad t < i+p; \\ b^s a^{i+p}, & \text{if} \quad t = i+p; \\ b^s a^t, & \text{if} \quad t > i+p. \end{cases}$$

By the above equalities we get that $b^i a^{i+k} \notin S \cdot b^{i+p}a^{i+p}$, and moreover the set $\mathscr{C}_+(a,b) \setminus (\mathscr{C}_+(a,b) \cdot b^{i+p}a^{i+p})$ is finite. This implies that $S \setminus S \cdot b^{i+p}a^{i+p}$ is finite, as well. Next we apply Theorem 2.2.

In the case of semigroup $\mathscr{C}_-(a,b)$ the proof is similar. □

The following example and later statements show that the semigroups $\mathscr{C}_+(a,b)$ and $\mathscr{C}_-(a,b)$ contain continuum many subsemigroups which satisfy the assumptions of Theorem 2.2.

Example 2.4. For an arbitrary non-negative integer k we define

$$\mathscr{C}_{+k}(a,b) = \left\{ b^i a^{i+s} \in \mathscr{C}_+(a,b) \colon s \geqslant k, s \in \omega \right\}.$$

The semigroup operation of $\mathscr{C}_+(a,b)$ implies that $\mathscr{C}_{+k}(a,b)$ is a subsemigroup of $\mathscr{C}_+(a,b)$. Fix an arbitrary infinite subset X of ω. Later we shall assume that $X = \{x_i \colon i \in \omega\}$ where $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω. Put

$$\mathscr{C}^X_{+k}(a,b) = \mathscr{C}_{+k}(a,b) \cup \{b^{x_i}a^{x_i} \in \mathscr{C}_+(a,b) \colon i \in \omega\}\,.$$

Since

$$b^j a^{j+s} \cdot b^{x_i}a^{x_i} = \begin{cases} b^{x_i-s}a^{x_i}, & \text{if} \quad j+s < x_i; \\ b^j a^{j+s}, & \text{if} \quad j+s \geqslant x_i \end{cases}$$

and

$$b^{x_i}a^{x_i} \cdot b^j a^{j+s} = \begin{cases} b^j a^{j+s}, & \text{if} \quad x_i < j; \\ b^{x_i}a^{x_i+s}, & \text{if} \quad x_i \geqslant j \end{cases}$$

for all $i,j,s \in \omega$, the inequality $s \geqslant k$ implies that $\mathscr{C}^X_{+k}(a,b)$ is a subsemigroup of $\mathscr{C}_+(a,b)$. It is obvious that the semigroup $\mathscr{C}^X_{+k}(a,b)$ contains infinitely many idempotents. Hence the conditions of Corollary 2.3 hold for the semigroup $\mathscr{C}^X_{+k}(a,b)$.

Lemma 2.5. *For an arbitrary infinite subset X of ω the semilattice $(X,\max)$ is isomorphic to the semilattice $(\omega,\max)$.*

Proof. Since $(\omega,\max)$ is a linearly ordered semilattice, $(X,\max)$ is a semilattice, too.

Without loss of generality we may assume that $X = \{x_i \colon i \in \omega\}$ where $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω. We define the map $\iota_X \colon \omega \to X$ by the formula $(i)\iota_X = x_i$, $i \in \omega$. Since the sets ω and X are well ordered by the usual linear order $\leqslant$, the map $\iota_X \colon \omega \to X$ is well defined. It is obvious that $\iota_X \colon (\omega,\max) \to (X,\max)$ is a semilattice isomorphism. Indeed, since $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω, the inequality $i \leqslant j$ implies $x_i = (i)\iota_X \leqslant (j)\iota_X = x_j$. It is easy to see that the map $\iota_X \colon \omega \to X$ is bijective and its inverse is steadily increasing. $\square$

Lemma 2.6. *Let $\{x_i\}_{i\in\omega}$ and $\{y_i\}_{i\in\omega}$ be steadily increasing sequences in ω. Then the sets $X = \{x_i \colon i \in \omega\}$ and $Y = \{y_i \colon i \in \omega\}$ with the induced semilattice operation from $(\omega,\max)$ are isomorphic semilattices, and moreover the isomorphism $\iota \colon (X,\max) \to (Y,\max)$ is defined by the formula $(x_i)\iota = y_i$, $i \in \omega$.*

Proof. By the proof of Lemma 2.5 the semilattice isomorphism

$$\iota_X \colon (\omega,\max) \to (X,\max)$$

is defined by the formula $(i)\iota_X = x_i$, $i \in \omega$. These arguments imply that the isomorphism $\iota \colon (X,\max) \to (Y,\max)$ is determined in the following way $(x_i)\iota = ((x_i)\iota_X^{-1})\iota_Y = y_i$, $i \in \omega$. Since the isomorphisms ι_X and ι_Y are well defined, so is ι. $\square$

Lemma 2.7. *Let $\{x_i\}_{i\in\omega}$ and $\{y_i\}_{i\in\omega}$ be steadily increasing sequences in ω and let $X=\{x_i\colon i\in\omega\}$ and $Y=\{y_i\colon i\in\omega\}$. Then the subsemigroups $\mathscr{C}_{+1}^{X}(a,b)$ and $\mathscr{C}_{+1}^{Y}(a,b)$ of the monoid $\mathscr{C}_{+}(a,b)$ are isomorphic if and only if $x_i=y_i$ for all $i\in\omega$.*

Proof. The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that the semigroups $\mathscr{C}_{+1}^{X}(a,b)$ and $\mathscr{C}_{+1}^{Y}(a,b)$ are isomorphic. First we observe that the semigroup operation of $\mathscr{C}_{+}(a,b)$ implies that for any $i\in\omega$ the set of solution of the equality $b^ia^i\cdot z=b^ia^{i+1}$ in $\mathscr{C}_{+}(a,b)$ is the following

$$\mathscr{S}_{b^ia^{i+1}}^{b^ia^i}=\left\{a,\ldots,b^ia^{i+1}\right\},$$

i.e., the cardinality of the set $\mathscr{S}_{b^ia^{i+1}}^{b^ia^i}$ is equal to $i+1$. This implies that the same holds for solutions of the equality $b^{x_i}a^{x_i}\cdot z=b^{x_i}a^{x_i+1}$ in the semigroup $\mathscr{C}_{+1}^{X}(a,b)$ and for solutions of the equality $b^{y_i}a^{y_i}\cdot z=b^{y_i}a^{y_i+1}$ in the semigroup $\mathscr{C}_{+1}^{X}(a,b)$.

If $\mathfrak{I}\colon \mathscr{C}_{+1}^{X}(a,b)\to\mathscr{C}_{+1}^{Y}(a,b)$ is an isomorphism, then by Lemma 2.6, we get that $(b^{x_i}a^{x_i})\mathfrak{I}=b^{y_i}a^{y_i}$ for any $i\in\omega$. This and the above arguments imply that the set of solutions of the equality $b^{x_i}a^{x_i}\cdot z=b^{x_i}a^{x_i+1}$ in $\mathscr{C}_{+1}^{X}(a,b)$ and the set of solutions of the equality $b^{y_i}a^{y_i}\cdot z=b^{y_i}a^{y_i+1}$ in $\mathscr{C}_{+1}^{Y}(a,b)$ have the same cardinality $x_i+1=y_i+1$, and hence they coincide. This implies that $x_i=y_i$ for all $i\in\omega$. □

It is well known that the set ω contains continuum many distinct infinite subsets. This and Lemma 2.7 imply that there exist continuum many non-isomorphic subsemigroups in the monoid $\mathscr{C}_{+}(a,b)$ of the forms $\mathscr{C}_{+k}^{X}(a,b)$, where $k\in\omega\setminus\{0\}$ and X is an infinite subset of ω. Then using Theorem 2.2 we obtain the following theorem.

Theorem 2.8. *The monoid $\mathscr{C}_{+}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{+k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, such that every left-continuous Hausdorff topology on $\mathscr{C}_{+k}^{X}(a,b)$ is discrete.*

Proof. Let $X=\{x_i\colon i\in\omega\}$ be any infinite subset of ω such that $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω. For any element $b^ja^{j+s}\in\mathscr{C}_{+k}^{X}(a,b)$ there exists $x_i\in X$ such that $x_i\geqslant j+s+1$. Then we have that

$$b^ja^{j+s}\notin\mathscr{C}_{+k}^{X}(a,b)\cdot b^{x_i}a^{x_i}\subset\mathscr{C}_{+}(a,b)\cdot b^{x_i}a^{x_i}$$

and the set

$$\mathscr{C}_{+k}^{X}(a,b)\setminus\left(\mathscr{C}_{+k}^{X}(a,b)\cdot b^{x_i}a^{x_i}\right)\subset\mathscr{C}_{+}(a,b)\setminus\left(\mathscr{C}_{+}(a,b)\cdot b^{x_i}a^{x_i}\right)$$

is infinite. Next we apply Theorem 2.2. □

By the dual way as in Example 2.4 for an arbitrary infinite subset X of ω and any positive integer k we construct the subsemigroup $\mathscr{C}_{-k}^{X}(a,b)$ of the monoid $\mathscr{C}_{-}(a,b)$. We observe that the monoids $\mathscr{C}_{+}(a,b)$ and $\mathscr{C}_{-}(a,b)$ are anti-isomorphic by the mapping

$$\mathfrak{AI}\colon b^i a^j \mapsto b^j a^i, \quad i,j\in\omega, i\leqslant j$$

For more details see [17, 26]. Simple verifications show that the restriction of $\mathfrak{AI}$ onto $\mathscr{C}_{+k}^{X}(a,b)$ is an anti-isomorphism from $\mathscr{C}_{+k}^{X}(a,b)$ to $\mathscr{C}_{-k}^{X}(a,b)$ for an arbitrary infinite subset X of ω and any positive integer k. By the above arguments and Theorem 2.8 we get the following result.

Theorem 2.9. *The monoid $\mathscr{C}_{-}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{-k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, such that every right-continuous Hausdorff topology on $\mathscr{C}_{-k}^{X}(a,b)$ is discrete.*

We observe that for any positive integer k and any infinite subset X of ω we have that for every $b^i a^j\in\mathscr{C}_{+k}^{X}(a,b)$ there exists a basic neighbourhood $U_n^r(b^i a^j)$ of $b^i a^j$ in $(\mathscr{C}(a,b),\tau_p^l)$ such that $U_n^r(b^i a^j)\subset\mathscr{C}_{+k}^{X}(a,b)$, and moreover, $U_m^r(b^i a^j)\subset\mathscr{C}_{+k}^{X}(a,b)$ for any positive integer $m>n$. This implies that the topology of the space $(\mathscr{C}(a,b),\tau_p^l)$ induces on $\mathscr{C}_{+k}^{X}(a,b)$ a non-discrete topology. Later we denote this topology on $\mathscr{C}_{+k}^{X}(a,b)$ by τ_p^l. Then the arguments presented in Example 2.1 imply the following proposition.

Proposition 2.10. *The monoid $\mathscr{C}_{+}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{+k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, and the semigroup operation is left-continuous and is not right-continuous in $(\mathscr{C}_{+k}^{X}(a,b),\tau_p^l)$.*

Also, by the dual way we get the following proposition.

Proposition 2.11. *The monoid $\mathscr{C}_{-}(a,b)$ contains continuum many non-isomorphic subsemigroups of the forms $\mathscr{C}_{-k}^{X}(a,b)$, where k is a positive integer and X is an infinite subset of ω, and the semigroup operation is right-continuous and is not left-continuous in $(\mathscr{C}_{-k}^{X}(a,b),\tau_p^r)$.*

3. On topologizations of subsubsemigroups of the semigroups $\mathscr{C}_{\mathbb{Z}}^{+}$ and $\mathscr{C}_{\mathbb{Z}}^{-}$

For arbitrary $(k,l)\in\mathscr{C}_{\mathbb{Z}}$ we denote

$$\begin{aligned}
{\uparrow_{\preccurlyeq}}(k,l) &:= \{(i,j)\in\mathscr{C}_{\mathbb{Z}}\colon (k,l)\preccurlyeq(i,j)\}\\
{\downarrow_{\preccurlyeq}}(k,l) &:= \{(i,j)\in\mathscr{C}_{\mathbb{Z}}\colon (i,j)\preccurlyeq(k,l)\},
\end{aligned}$$

where $\preccurlyeq$ is the natural partial order on $\mathscr{C}_{\mathbb{Z}}$.

We observe that the semilattice $E(\mathscr{C}_{\mathbb{Z}})$ of idempotents of $\mathscr{C}_{\mathbb{Z}}$ is isomorphic to the semilattice $(\mathbb{Z}, \max)$ by the mappings $(i,i) \mapsto i$, $i \in \mathbb{Z}$.

Lemma 3.1. *Every Hausdorff shift-continuous topology τ on $(\mathbb{Z}, \max)$ is discrete.*

Proof. Fix an arbitrary integer n and suppose that n is a non-isolated point of $(\mathbb{Z}, \max, \tau)$. Since the semilattice operation max on $(\mathbb{Z}, \tau)$ is separately continuous, the set $\uparrow_{\geqslant}(n-1) = \{i \in \mathbb{Z} \colon n-1 \geqslant i\}$ is closed in the space $(\mathbb{Z}, \tau)$ as the full preimage of the point $(n-1)$ under the continuous shift $k \mapsto \max\{k, n-1\}$. This implies that the point n has an open neighbourhood $U(n)$ in $(\mathbb{Z}, \tau)$ such that $U(n) \subseteq \downarrow_{\geqslant} n = \{i \in \mathbb{Z} \colon i \geqslant n\}$.

Again, since the shift $k \mapsto \max\{k, n+1\}$ is a continuous retraction of the Hausdorff space $(\mathbb{Z}, \tau)$, by Exercise 1.5.C of [12] we get that the set $\downarrow_{\geqslant}(n+1) = \{i \in \mathbb{Z} \colon i \geqslant n+1\}$ is a closed subset of $(\mathbb{Z}, \tau)$. Hence n is an isolated point of $(\mathbb{Z}, \tau)$, which implies the statement of the lemma. $\square$

Remark 3.2.

(1) Since the semilattices $(\mathbb{Z}, \max)$ and $(\mathbb{Z}, \min)$ are isomorphic by the map $n \mapsto -n$, the statement Lemma 3.1 holds for the semilattice $(\mathbb{Z}, \min)$.
(2) The first part of the proof of Lemma 3.1 implies that every shift-continuous T_1-topology τ on $(\mathbb{N}, \min)$ is discrete.
(3) The second part of the proof of Lemma 3.1 implies that every Hausdorff shift-continuous topology τ on $(\mathbb{N}, \max)$ is discrete. It is obvious that the semilattice $(\mathbb{N}, \max)$ admits a non-discrete semigroup T_1-topology.

Lemma 3.3. *For any $k \in \mathbb{Z}$ the subsemigroup $\mathscr{C}_{\mathbb{Z}}^{+}[k] = \big\{(s,t) \in \mathscr{C}_{\mathbb{Z}}^{+} \colon s \geqslant k\big\}$ of $\mathscr{C}_{\mathbb{Z}}^{+}$ is isomorphic to the semigroup $\mathscr{C}_{+}(a,b)$ by the mapping*

$$\mathfrak{h}_k \colon \mathscr{C}_{+}(a,b) \to \mathscr{C}_{\mathbb{Z}}^{+}[k], \qquad b^i a^j \mapsto (i+k, j+k).$$

Proof. The semigroup operation of $\mathscr{C}_{\mathbb{Z}}$ implies that $\mathscr{C}_{\mathbb{Z}}^{+}[k]$ is a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{+}$. By definition of $\mathfrak{h}_k$, we have the equalities

$$\begin{aligned}
(b^{i_1}a^{j_1} \cdot b^{i_2}a^{j_2})\mathfrak{h}_k &= \begin{cases} (b^{i_1-j_1+i_2}a^{j_2})\mathfrak{h}_k, & \text{if} \quad j_1 < i_2; \\ (b^{i_1}a^{j_2})\mathfrak{h}_k, & \text{if} \quad j_1 = i_2; \\ (b^{i_1}a^{j_1-i_2+j_2})\mathfrak{h}_k, & \text{if} \quad j_1 > i_2; \end{cases} \\
&= \begin{cases} (i_1-j_1+i_2+k, j_2+k), & \text{if} \quad j_1 < i_2; \\ (i_1+k, j_2+k), & \text{if} \quad j_1 = i_2; \\ (i_1+k, j_1-i_2+j_2+k), & \text{if} \quad j_1 > i_2; \end{cases}
\end{aligned}$$

and

$$
\begin{aligned}
(b^{i_1}a^{j_1})\mathfrak{h}_k \cdot (b^{i_2}a^{j_2})\mathfrak{h}_k &= (i_1+k, j_1+k)\cdot(i_2+k, j_2+k) \\
&= \begin{cases} (i_1+k-(j_1+k)+i_2+k, j_2+k), & \text{if} \quad j_1+k<i_2+k; \\ (i_1+k, j_2+k), & \text{if} \quad j_1+k=i_2+k; \\ (i_1+k, j_1+k-(i_2+k)+j_2+k), & \text{if} \quad j_1+k>i_2+k; \end{cases} \\
&= \begin{cases} (i_1-j_1+i_2+k, j_2+k), & \text{if} \quad j_1<i_2; \\ (i_1+k, j_2+k), & \text{if} \quad j_1=i_2; \\ (i_1+k, j_1-i_2+j_2+k), & \text{if} \quad j_1>i_2; \end{cases}
\end{aligned}
$$

which imply the statement of the lemma. □

If $\preccurlyeq$ is the natural partial order on $\mathscr{C}_{\mathbb{Z}}$ and S is a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{+}$, then in this section we denote by $\preccurlyeq^{S}$ the restriction of the binary relation $\preccurlyeq$ onto S, i.e., $(i,j)\preccurlyeq^{S}(k,l)$ if and only if $(i,j)\preccurlyeq(k,l)$ in $\mathscr{C}_{\mathbb{Z}}$. Put

$$
\left.\begin{aligned}
{\uparrow}_{\preccurlyeq^S}(m,n) &= \left\{(p,q)\in S\colon (m,n)\preccurlyeq^{S}(p,q)\right\} \\
{\downarrow}_{\preccurlyeq^S}(m,n) &= \left\{(p,q)\in S\colon (p,q)\preccurlyeq^{S}(m,n)\right\},
\end{aligned}\right. \qquad \forall (m,n)\in S
$$

Theorem 3.4. *Let S be a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{+}$ ($\mathscr{C}_{\mathbb{Z}}^{-}$) such that the band $E(S)$ of S is isomorphic to the semilattice $(\mathbb{Z},\max)$. Then every Hausdorff left-continuous (right-continuous) topology τ on S is discrete.*

Proof. If $S=E(S)$ then the statement of the theorem follows from Lemma 3.1. Hence we assume that $S\neq E(S)$.

Fix an arbitrary $(i,j)\in S$. Since the band $E(S)$ is isomorphic to the semilattice $(\mathbb{Z},\max)$, the semigroup operation of $\mathscr{C}_{\mathbb{Z}}^{+}$ implies that there exists the maximum integer k such that $(k,k)\in E(S)$ and $k\leqslant i$. By Lemma 3.3, the subsemigroup $\mathscr{C}_{\mathbb{Z}}^{+}[k]$ of $\mathscr{C}_{\mathbb{Z}}^{+}$ is isomorphic to the semigroup $\mathscr{C}_{+}(a,b)$, and hence by Corollary 2.3, $S_k=S\cap\mathscr{C}_{\mathbb{Z}}^{+}[k]$ is a discrete subsemigroup of S.

Again, since $E(S)$ is isomorphic to $(\mathbb{Z},\max)$, the semigroup operation of $\mathscr{C}_{\mathbb{Z}}^{+}$ implies that there exists the minimum integer n such that $(n,n)\in E(S)$ and $j<n$. Then we have that $(i,j)\cdot(n,n)=(n-j+i,n)$. We observe that

$$
\begin{aligned}
{\uparrow}_{\preccurlyeq^S}(n-j+i,n) &= \{(s,t)\in S\colon (s,t)\cdot(n,n)=(n-j+i,n)\} \\
&= \{(n-j+i-p, n-p)\colon p\in\omega\}.
\end{aligned}
$$

By Lemma 3.3 and Corollary 2.3, the point $(n-j+i,n)$ is isolated in the space $S_n=S\cap\mathscr{C}_{\mathbb{Z}}^{+}[n]$. Then the equality

$$
(s,t)\cdot(n,n)=\begin{cases} (n-t+s,n), & \text{if} \quad t\leqslant n; \\ (s,t), & \text{if} \quad t>n \end{cases}
$$

and the left-continuity of the semigroup operation of (S,τ) imply that the set ${\uparrow}_{\preccurlyeq_S}(n-j+i,n)$ is open in (S,τ). It is obvious that $(i,j)\in{\uparrow}_{\preccurlyeq_S}(n-j+i,n)$. If the set ${\uparrow}_{\preccurlyeq_S}(n-j+i,n)$ is finite, then the proof is complete.

Suppose the set ${\uparrow}_{\preccurlyeq_S}(n-j+i,n)$ is infinite. Since $E(S)$ is isomorphic to $(\mathbb{Z},\max)$, the semigroup operation of $\mathscr{C}_{\mathbb{Z}}^{+}$ implies that there exists the maximum integer $s<j$ such that $(s,s)\in E(S)$ and $(i,j)\cdot(s,s)=(i,j)$. We observe that the semigroup operation of $\mathscr{C}_{\mathbb{Z}}^{+}$ implies that the set ${\uparrow}_{\preccurlyeq_S}(i,j)$ is infinite, and hence there exists $(x,y)\in{\uparrow}_{\preccurlyeq_S}(i,j)$ such that $(i,j)\neq(x,y)\cdot(s,s)$. Without loss of generality we may assume that $y<s$ and hence we have that $(x,y)\cdot(s,s)=(s-y+x,s)$. Then Hausdorffness and left-continuity of the semigroup operation of (S,τ) imply that the set ${\uparrow}_{\preccurlyeq_S}(s-y+x,s)$ is closed in (S,τ) as the full preimage of the point $(s-y+x,s)$ under the right shift

$$\rho_{(s,s)}\colon S\to S,\qquad (a,b)\mapsto(a,b)\cdot(s,s)$$

This implies the point (i,j) has a finite open neighbourhood in (S,τ), and hence is isolated.

The dual statement of the theorem follows from the fact that the semigroups $\mathscr{C}_{\mathbb{Z}}^{+}$ and $\mathscr{C}_{\mathbb{Z}}^{-}$ are anti-isomorphic by the mapping

$$\mathfrak{AI}\colon\mathscr{C}_{\mathbb{Z}}^{+}\to\mathscr{C}_{\mathbb{Z}}^{-},\qquad (i,j)\mapsto(j,i).\qquad\square$$

The following example shows that the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ admits a Hausdorff right-continuous (left-continuous) topology τ_p^r (τ_p^l) such that left (right) translations are not continuous in $(\mathscr{C}_{\mathbb{Z}},\tau_p^r)$ ($(\mathscr{C}_{\mathbb{Z}},\tau_p^l)$).

Example 3.5. Fix an arbitrary prime positive integer p. For any $(i,j)\in\mathscr{C}_{\mathbb{Z}}$ and any $n\in\omega$ we denote

$$U_n^r(i,j)=\{(i,j+p^n\cdot k)\colon k\in\omega\}.$$

Then the family $\mathscr{B}_p^r=\{\mathscr{B}_p^r(i,j)\colon i,j\in\mathbb{Z}\}$, where $\mathscr{B}_p^r(i,j)=\{U_n^r(i,j)\colon n\in\omega\}$ for $(i,j)\in\mathscr{C}_{\mathbb{Z}}$, satisfies the properties (BP1)–(BP4) of [12], and hence it generates a Hausdorff topology τ_p^r on the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$. It is obvious that τ_p^r is a non-discrete topology on $\mathscr{C}_{\mathbb{Z}}$, and moreover all points of $(\mathscr{C}_{\mathbb{Z}},\tau_p^r)$ are non-isolated.

The proofs of the statements that the semigroup operation on $(\mathscr{C}_{\mathbb{Z}},\tau_p^r)$ is right-continuous and it is not left-continuous are similar to the corresponding statements in Example 2.1. Also, by the anti-isomorphism $\mathfrak{AI}\colon\mathscr{C}_{\mathbb{Z}}\to\mathscr{C}_{\mathbb{Z}}$, $(i,j)\mapsto(j,i)$, we get a Hausdorff left-continuous topology τ_p^l such that right translations are not continuous in $(\mathscr{C}_{\mathbb{Z}},\tau_p^l)$.

Remark 3.6. (1) We observe that for any integer i_0 the subset

$$\widetilde{L}_{i_0}=\{(i_0,i_0+i)\colon i\in\mathbb{Z}\}$$

is not a subsemigroup of $\mathscr{C}_{\mathbb{Z}}$ because

$$(i_0, i_0 - 1) \cdot (i_0, i_0 - 1) = (i_0 + 1, i_0 - 1) \notin \widetilde{L}_{i_0}.$$

(2) It is obvious that the topology τ_p^r (τ_p^l) of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ induces on the semigroup $\mathscr{C}_{\mathbb{Z}}^+$ ($\mathscr{C}_{\mathbb{Z}}^-$) a Hausdorff non-discrete right-continuous (left-continuous) topology which is not left-continuous (right-continuous).

The following example shows that the statement of Theorem 3.4 does not hold in the case when the band $E(S)$ of a subsemigroup S of $\mathscr{C}_{\mathbb{Z}}^+$ is infinite.

Example 3.7. We define $S = S_0 \cup S_1$, where $S_0 = \{(-i, -i) \in \mathscr{C}_{\mathbb{Z}}^+ : i \in \mathbb{N}\}$ and $S_1 = \{(0, k) \in \mathscr{C}_{\mathbb{Z}}^+ : k \in \omega\}$. Simple verifications show that S is a commutative subsemigroup of $\mathscr{C}_{\mathbb{Z}}^+$. Moreover, S_1 is an ideal of S such that $(-i, -i) \cdot (0, k) = (0, k)$ for all $(-i, -i) \in S_0$ and $(0, k) \in S_1$.

We define a topology τ on S in the following way. All points of the form $(-i, -i)$, $i \in \mathbb{N}$, are isolated in (S, τ). Fix an arbitrary prime integer p. The family $B_\tau = \{U_n(0, k) : n \in \mathbb{N}\}$, where

$$U_n(0, k) = \{(0, k + p^n s) \in S_1 : s \in \omega\},$$

is the base of the topology τ at the point $(0, k) \in S_1$. Simple verifications show that τ is a Hausdorff topology on S_α.

Since

$$(-i, -i) \cdot (-j, -j) = (\max\{-i, -j\}, \max\{-i, -j\}),$$

$(-i, -i) \cdot U_n(0, k) = U_n(0, k)$, and $U_n(0, k) \cdot U_n(0, l) \subseteq U_n(0, k + l)$ for any $i, j, k \in \mathbb{N}$ and $k, l \in \omega$, we conclude that the semigroup operation on (S, τ) is continuous.

By Lemma 3.3 the semigroup $\mathscr{C}_{\mathbb{Z}}^+$ contains infinitely many isomorphic copies of the semigroup $\mathscr{C}_+(a, b)$ and hence by Theorem 2.8 $\mathscr{C}_{\mathbb{Z}}^+$ has continuum many subsemigroups $\{S_\alpha : \alpha \in \mathfrak{c}\}$ such that for any $\alpha \in \mathfrak{c}$ the following conditions hold:

(1) the band $E(S_\alpha)$ is isomorphic to $(\omega, \max)$;
(2) for any idempotent (i, i) of S_α the set $S_\alpha \setminus (S_\alpha \cdot (i, i))$ is finite;
(3) every left-continuous Hausdorff topology on S_α is discrete.

Example 3.8 and Theorem 3.10 imply that $\mathscr{C}_{\mathbb{Z}}^+$ has continuum many subsemigroups $\{S_\alpha : \alpha \in \mathfrak{c}\}$ such that for any $\alpha \in \mathfrak{c}$ the following conditions hold:

(i$^\circ$) the band $E(S_\alpha)$ is isomorphic to $(\mathbb{Z}, \max)$;
(ii$^\circ$) for any idempotent (i, i) of S_α the set $S_\alpha \setminus (S_\alpha \cdot (i, i))$ is infinite;
(iii$^\circ$) every left-continuous Hausdorff topology on S_α is discrete.

Example 3.8. Later we shall assume that $X = \{x_i \colon i \in \omega\}$ where $\{x_i\}_{i\in\omega}$ is a steadily increasing sequence in ω. For any positive integer k we define

$$\mathscr{C}_{\mathbb{Z}}^{+}(k,X) = \{(-i,-i)\colon i \in \omega\} \cup \{(x_i,x_i)\colon i \in \omega\} \cup \left\{(m,n) \in \mathscr{C}_{\mathbb{Z}}^{+}[0]\colon m \geqslant k\right\}.$$

It is not difficult to prove that $\mathscr{C}_{\mathbb{Z}}^{+}(k,X)$ is a subsemigroup of $\mathscr{C}_{\mathbb{Z}}^{+}$ for any positive integer k.

Lemma 3.9. *Let $\{x_i\}_{i\in\omega}$ and $\{y_i\}_{i\in\omega}$ be steadily increasing sequences in ω and let $X = \{x_i \colon i \in \omega\}$ and $Y = \{y_i \colon i \in \omega\}$. Then the subsemigroups $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)$ and $\mathscr{C}_{\mathbb{Z}}^{+}(1,Y)$ of $\mathscr{C}_{\mathbb{Z}}^{+}$ are isomorphic if and only if $x_i = y_i$ for all $i \in \omega$.*

Proof. The implication $(\Leftarrow)$ is trivial.

$(\Rightarrow)$ Suppose that $\mathfrak{I}\colon \mathscr{C}_{\mathbb{Z}}^{+}(1,X) \to \mathscr{C}_{\mathbb{Z}}^{+}(1,Y)$ is an isomorphism. Put

$$S_X = \mathscr{C}_{\mathbb{Z}}^{+}(1,X) \cap \mathscr{C}_{\mathbb{Z}}^{+}[0], \qquad S_Y = \mathscr{C}_{\mathbb{Z}}^{+}(1,Y) \cap \mathscr{C}_{\mathbb{Z}}^{+}[0].$$

Simple verifications show that S_X and S_Y are subsemigroups of $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)$ and $\mathscr{C}_{\mathbb{Z}}^{+}(1,Y)$, respectively. Moreover S_X is isomorphic to $\mathscr{C}_{+1}^{X}(a,b)$ and S_Y is isomorphic to $\mathscr{C}_{+1}^{Y}(a,b)$ by the mapping $(m,n) \mapsto b^m a^n$. The semigroup operation of $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)$ and $\mathscr{C}_{\mathbb{Z}}^{+}(1,Y)$ implies that the equation

$$z \cdot (i,i+p) = (i,i+p)$$

(in $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)$ and $\mathscr{C}_{\mathbb{Z}}^{+}(1,Y)$), where $i \in \omega$ and $p \in \mathbb{N}$, has finitely many solutions, we conclude that the restriction $\mathfrak{I}_{S_X} \colon S_X \to S_Y$ is an isomorphism, and hence by Lemma 2.7, we get that $x_i = y_i$ for all $i \in \omega$. Then the semilattices $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)\backslash S_X$ and $\mathscr{C}_{\mathbb{Z}}^{+}(1,Y)\backslash S_Y$ are isomorphic. It is obvious that $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)\backslash S_X$ and $\mathscr{C}_{\mathbb{Z}}^{+}(1,Y)\backslash S_Y$ are isomorphic to the semilattice $(\omega,\min)$ by the mapping $(-i,-i) \mapsto i$. Since the natural partial order on the semilattice $(\omega,\min)$ is complete, i.e., $(\omega,\min,\leqslant)$, we have that $(-i,-i)\mathfrak{I} = (-i,-i)$ for any $i \in \omega$. □

We observe that for any steadily increasing sequence $\{x_i\}_{i\in\omega}$ in ω the semigroup $\mathscr{C}_{\mathbb{Z}}^{+}(1,X)$, where $X = \{x_i \colon i \in \omega\}$, satisfies the conditions (i°), (ii°) and (iii°). Then Lemma 3.9 implies the following theorem.

Theorem 3.10. *The semigroup $\mathscr{C}_{\mathbb{Z}}^{+}$ contains continuum many non-isomorphic subsemigroups S_α such that every left-continuous Hausdorff topology on S_α is discrete and S_α satisfies the conditions* (i°), (ii°) *and* (iii°).

4. Embedding of the semigroup $\mathscr{C}_{\mathbb{Z}}^{+}$ into a compact topological semigroup

The following example of the closure of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ is presented in [13].

Example 4.1 ([13, Example 5])**.** We denote by $G_1(1)$ and G_0 the additive groups of integers. We denote by $(n)^1$ and $(m)^0$ the elements of the groups $G_1(1)$ and G_0, respectively, $n, m \in \mathbb{Z}$. This means that $(m)^1 \cdot (n)^1 = (m+n)^1$ and $(m)^0 \cdot (n)^0 = (m+n)^0$ for any $m, n \in \mathbb{Z}$. Put $S = G_1(1) \sqcup \mathscr{C}_{\mathbb{Z}} \sqcup G_0$ and extend the semigroup operation from $\mathscr{C}_{\mathbb{Z}}$, $G_1(1)$ and G_0 in the following way:

$$\begin{aligned}
(n)^1 \cdot (i,j) &= (-n+i, j) \in \mathscr{C}_{\mathbb{Z}};\\
(i,j) \cdot (n)^1 &= (i, j+n) \in \mathscr{C}_{\mathbb{Z}};\\
(n)^1 \cdot (m)^0 &= (n+m)^0 \in G_0;\\
(m)^0 \cdot (n)^1 &= (m+n)^0 \in G_0;\\
(m)^0 \cdot (i,j) &= (m+j-i)^0 \in G_0;\\
(i,j) \cdot (m)^0 &= (m+j-i)^0 \in G_0,
\end{aligned}$$

for any $(n)^1 \in G_1(1)$, $(i,j) \in \mathscr{C}_{\mathbb{Z}}$, and $(m)^0 \in G_0$.

The topology τ on S is determined in the following way:

(1) all elements (i,j) of the extended bicyclic semigroup $\mathscr{C}_{\mathbb{Z}}$ are isolated in (S, τ);

(2) the family $\mathscr{B}((n)^1) = \left\{U_p((n)^1) \colon p \in \mathbb{N}\right\}$, where

$$U_p((n)^1) = \left\{(n)^1\right\} \cup \left\{(-q, -q+n) \in \mathscr{C}_{\mathbb{Z}} \colon q \geqslant p\right\},$$

is a base of the topology τ at the point $(n)^1 \in G_1(1)$;

(3) the family $\mathscr{B}((m)^0) = \left\{V_p((m)^0) \colon p \in \mathbb{N}\right\}$, where

$$V_p((m)^0) = \begin{cases} \left\{(m)^0\right\} \cup \left\{(q, q+m) \in \mathscr{C}_{\mathbb{Z}} \colon q \geqslant p\right\}, & \text{if} \quad m \geqslant 0;\\ \left\{(m)^0\right\} \cup \left\{(q-m, q) \in \mathscr{C}_{\mathbb{Z}} \colon q \geqslant p\right\}, & \text{if} \quad m \leqslant 0, \end{cases}$$

is a base of the topology τ at the point $(m)^0 \in G_0$.

Then (S, τ) is a Hausdorff locally compact topological inverse semigroup, see [13].

Example 4.2. Let $G_1(1)$ and G_0 be the groups which are defined in Example 4.1. We denote

$$G_1^+ = \left\{(n)^1 \in G_1(1) \colon n \geqslant 0\right\} \quad \text{and} \quad G_0^+ = \left\{(m)^0 \in G_0 \colon m \geqslant 0\right\}.$$

The semigroup operation of S implies that the subset $S^+ = G_1^+ \sqcup \mathscr{C}_{\mathbb{Z}}^+ \sqcup G_0^+$ of S with the induced from S multiplication is a subsemigroup of S.

Let τ^+ be the topology on S^+ which is induced from (S, τ). Then (S^+, τ^+) is a Hausdorff topological semigroup. Since (S^+, τ^+) is a closed subspace of (S, τ), Theorem 3.3.8 of [12] implies that the space (S^+, τ^+) is locally compact.

We define $S^+_{\mathscr{O}}$ to be the semigroup S^+ with the adjoined zero $\mathscr{O}$, i.e., $\mathscr{O}\cdot s = s\cdot\mathscr{O} = \mathscr{O}\cdot\mathscr{O} = \mathscr{O}$ for any $s\in S^+$. We define a topology $\tau^+_{\mathscr{O}}$ on the semigroup $S^+_{\mathscr{O}}$ as follows. We extend the topology τ^+ onto the semigroup $S^+_{\mathscr{O}}$ in the following way: the family $\mathscr{B}^+(\mathscr{O}) = \{W_p(\mathscr{O})\colon p\in\mathbb{N}\}$, where

$$\begin{aligned} W_p(\mathscr{O}) = \{\mathscr{O}\} &\sqcup \left\{(n)^1\in G_1(1)\colon n\geqslant p\right\}\sqcup\\ &\sqcup \left\{(i,j)\in\mathscr{C}^+_{\mathbb{Z}}\colon j-i\geqslant p\right\}\sqcup\\ &\sqcup \left\{(m)^0\in G_0\colon m\geqslant p\right\}, \end{aligned}$$

is the base of the topology $\tau^+_{\mathscr{O}}$ at the point $\mathscr{O}$. It is obvious that $\tau^+_{\mathscr{O}}$ is a compact Hausdorff topology on $S^+_{\mathscr{O}}$. Since the semigroup $S^+_{\mathscr{O}}$ is countable, Theorems 3.1.21 and 4.2.8 from [12] imply that the space $(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$ is metrizable.

Proposition 4.3. *$(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$ is a topological semigroup.*

Proof. Since (S^+,τ^+) is a Hausdorff topological semigroup and (S^+,τ^+) is an open subspace of the topological space $(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$, it is enough to show that the semigroup operation is continuous on $(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$ in the following seven cases.

(1) In the case $(n)^1\cdot\mathscr{O}$, where $(n)^1\in G_1(1)$, we have that

$$U_{p_1}((n)^1)\cdot W_{p_2}(\mathscr{O})\subseteq W_{n+p_2}(\mathscr{O})$$

for any non-negative integer n and any positive integers p_1 and p_2.

(2) In the case $\mathscr{O}\cdot(n)^1$, where $(n)^1\in G_1(1)$, we obtain that

$$W_{p_1}(\mathscr{O})\cdot U_{p_2}((n)^1)\subseteq W_{p_1+n}(\mathscr{O})$$

for any non-negative integer n and any positive integers p_1 and p_2.

(3) In the case $(i,j)\cdot\mathscr{O}$, where $(i,j)\in\mathscr{C}^+_{\mathbb{Z}}$, we get that

$$(i,j)\cdot W_p(\mathscr{O})\subseteq W_{p+j-i}(\mathscr{O})$$

for any positive integer p.

(4) In the case $\mathscr{O}\cdot(i,j)$, where $(i,j)\in\mathscr{C}^+_{\mathbb{Z}}$, we have that

$$W_p(\mathscr{O})\cdot(i,j)\subseteq W_{p+j-i}(\mathscr{O})$$

for any positive integer p.

(5) In the case $(m)^0\cdot\mathscr{O}$, where $(m)^0\in G_0$, we obtain that

$$V_{p_1}((m)^0)\cdot W_{p_2}(\mathscr{O})\subseteq W_{m+p_2}(\mathscr{O})$$

for any non-negative integer m and any positive integers p_1 and p_2.

(6) In the case $\mathscr{O}\cdot(m)^0$, where $(m)^0\in G_0$, we get that

$$W_{p_1}(\mathscr{O})\cdot V_{p_2}((m)^0)\subseteq W_{p_1+m}(\mathscr{O})$$

for any non-negative integer m and any positive integers p_1 and p_2.

(7) In the case $\mathscr{O}\cdot\mathscr{O}$ we have that

$$W_{p_1}(\mathscr{O})\cdot W_{p_2}(\mathscr{O})\subseteq W_{p_1+p_2}(\mathscr{O})$$

for any positive integers p_1 and p_2.

The above arguments imply that the semigroup operation is continuous on $(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$. □

The definition of the topology $\tau^+_{\mathscr{O}}$ implies that $I_0 = G^+_0 \sqcup \{\mathscr{O}\}$ is a compact ideal of the Hausdorff compact topological semigroup $(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$. By the Lawson-Madison theorem (see [25] or [7, Theorem 1.57]) the quotient semigroup $S^+_{\mathscr{O}}/I_0$ with the quotient topology $\tau_{\mathbf{q}}$ is a Hausdorff compact topological semigroup. Since $\mathscr{C}^+_{\mathbb{Z}}$ is a dense subsemigroup of $(S^+_{\mathscr{O}},\tau^+_{\mathscr{O}})$ and $\mathscr{C}^+_{\mathbb{Z}}\cap I_0=\varnothing$ we get the following corollary.

Corollary 4.4. *The semigroup $\mathscr{C}^+_{\mathbb{Z}}$ is a dense subsemigroup of the Hausdorff compact topological semigroup $(S^+_{\mathscr{O}}/I_0,\tau_{\mathbf{q}})$.*

Since the closure of a subsemigroup S of a Hausdorff compact topological semigroup K is a compact topological semigroup, Proposition 4.3 implies the following two corollaries.

Corollary 4.5. *Every subsemigroup of semigroup $\mathscr{C}^+_{\mathbb{Z}}$ is a dense subsemigroup of a Hausdorff compact topological semigroup.*

Corollary 4.6. *The semigroup $\mathscr{C}_+(a,b)$ with adjoined zero admits a Hausdorff compact semigroup topology.*

Remark 4.7. (a) Similar statements to Proposition 4.3 and Corollaries 4.4 and 4.5 hold for the semigroup $\mathscr{C}^-_{\mathbb{Z}}$. Also, the statement of Corollary 4.6 holds for the semigroup $\mathscr{C}_-(a,b)$.

(b) In [14], it is proved that a Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero $\mathscr{C}^0$ is either compact or discrete. Similar statement holds for semitopological interassociates of the bicyclic monoid [18]. But the extended bicyclic semigroup with adjoined zero $\mathscr{C}^0_{\mathbb{Z}}$ admits continuum many shift-continuous locally compact topologies [19] and similar statements hold for the semigroups $\mathscr{C}_+(a,b)$ and $\mathscr{C}_-(a,b)$ [16].

Acknowledgements

The author acknowledges Alex Ravsky and the referee for their comments and suggestions.

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Received 27.01.2026. Accepted 15.04.2026. Published online 02.09.2026.

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