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Recovering boundary conditions for star graphs

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Abstract. We consider the quantum graph spectral problem describing free motion of a particle in a star shaped waveguide with three edges of the same length under the standard conditions at the interior vertex and the Robin conditions at the pendant vertices. The corresponding inverse problem consist of recovering the coefficients in the Robin conditions using three eigenvalues. It is shown that knowing three specially chosen eigenvalues is sufficient to find the coefficients.

1. Introduction

By quantum graph theory, we mean the spectral problems generated by differential equations of quantum mechanics on metric graphs. Quantum graphs arise naturally as simplified models in mathematical physics and engineering when one considers propagation of waves of various types through a quasi-one-dimensional system, which resembles a thin neighborhood of a metric graph. It is customary to consider metric graphs with a finite number of edges of finite lengths. The usual matching conditions at the interior vertices are the continuity conditions and Kirchhoff‘s condition. The analog of Kirchhoff‘s condition at a pendant vertex is the Neumann condition. These are so-called standard conditions. However, the Dirichlet conditions and the Robin conditions are also physically motivated .

It should be mentioned that equations analogues to the quantum graph theory, appear in classical mechanics of small transverse vibrations of a net (graph) of strings.

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In this work, we restrict our attention to the case of the Sturm-Liouville equations (or the Schrödinger equations) on all edges with zero potentials. In this case, the corresponding operator is sometimes called Hamiltonian.

There are three types of inverse problems in the theory of quantum graphs.

In the first type of inverse problems, the shape of a graph and the matching conditions at the vertices are known together with the spectrum. We need to find the potentials of the Sturm-Liouville equations on the edges. For work on this problem see, e. g., [9], [1].

In problems of the second type, conditions at the vertices and the spectrum are known. We need to find the shape of the graph. For example, [13], [5], [7], [8], [10] we dealing with this question. Research in this direction was initiated in the famous article by M. Kac кCan one hear the shape of a drum?њ [6].

In the third type of problem, the shape of the graph and the spectrum are known and we need to find the coefficients in conditions at the vertices, see e. g.[3], [11], [12], [2]. Our paper is about the third type of the inverse problems.

The number of the coefficients in the matching conditions is usually finite. Therefore, there is a conjecture that (at least in some cases) the knowing a finite number of eigenvalues is sufficient to recover the coefficients in vertex conditions.

In the present paper, we prove this conjecture for the case of a star graph with three edges. It should be mentioned that in [14] the case of arbitrary trees was considered. It was proved that $2^p - 1$ eigenvalues uniquely determine Robin conditions at p vertices. In our case $2^p - 1 = 7$. However, we show in our paper that due to symmetry of our graph the number of necessary eigenvalues is 3. Also, we show how to choose these eigenvalues.

2. Main results

Let us consider the boundary value problem on an equilateral metric star graph that has three edges of length 1 each (Fig.2.1):

$$-y_j'' = \lambda^2 \, y_j, \quad x \in [0;1], \quad j = 1,2,3, \tag{2.1}$$

$$y_1(1) = y_2(1) = y_3(1), \tag{2.2}$$

$$y_1'(1) + y_2'(1) + y_3'(1) = 0, \tag{2.3}$$

$$y_1'(0) - b_1 y_1(0) = 0, \tag{2.4}$$

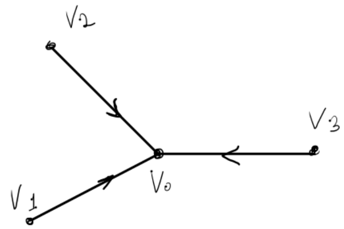

Figure 2.1.

$$y_2'(0) - b_2 y_2(0) = 0, \tag{2.5}$$

$$y_3'(0) - b_3 y_3(0) = 0, \tag{2.6}$$

where b_1, b_2, b_3 are real numbers.

This problem describes free motion of a quantum particle in a thin waveguide of star-graph form.

The boundary condition $y_j'(0) + b_j y_j(0) = 0$, where $b_j \in \mathbb{R}$ is called the Robin condition. The conditions (2.2) and (2.3) are called continuity conditions and Kirchhoff's condition, respectively. The physical meaning of these vertex conditions is explained, e.g., in [12].

We express the general solution of equation (2.1) in the following form:

$$y_j = A_j \frac{\sin \lambda x}{\lambda} + B_j \cos \lambda x, \tag{2.7}$$

where A_j, B_j real numbers. Substituting (2.7) into (2.4)-(2.6) we arrive at $A_j = b_j B_j$. Thus,

$$y_j = b_j B_j \frac{\sin \lambda x}{\lambda} + B_j \cos \lambda x. \tag{2.8}$$

Substituting (2.8) into (2.2)-(2.3), we obtain the following system of equations for the unknowns B_1, B_2 B_3:

$$\begin{cases} B_1(b_1 \frac{\sin\lambda}{\lambda} + \cos\lambda) = B_2\left(b_2 \frac{\sin\lambda}{\lambda} + \cos\lambda\right), \\ B_2(b_2 \frac{\sin\lambda}{\lambda} + \cos\lambda) = B_3(b_3 \frac{\sin\lambda}{\lambda} + cos\lambda), \\ B_1\left(b_1 \cos\lambda - \lambda \sin\lambda\right) + B_2\left(b_2 \cos\lambda - \lambda \sin\lambda\right) + B_3\left(b_3 \cos\lambda - \lambda \sin\lambda\right) = 0. \end{cases} \tag{2.9}$$

If the determinant of this system is not equal to zero, then the solution is unique, namely $B_1 = B_2 = B_3 = 0$. In order to find nontrivial solutions, we consider the case, where the determinant equals zero:

$$\det\begin{pmatrix} b_1\frac{\sin\lambda}{\lambda}+\cos\lambda & -(b_2\frac{\sin\lambda}{\lambda}+\cos\lambda) & 0 \\ 0 & b_2\frac{\sin\lambda}{\lambda}+\cos\lambda & -(b_3\frac{\sin\lambda}{\lambda}+\cos\lambda) \\ b_1\cos\lambda-\lambda\sin\lambda & b_2\cos\lambda-\lambda\sin\lambda & b_3\cos\lambda-\lambda\sin\lambda \end{pmatrix}=0. \tag{2.10}$$

We call the determinant in (2.10) the characteristic function. This way we have the following characteristic equation:

$$-3\lambda sin\lambda\cos^2\lambda+(b_1+b_2+b_3)\left(\cos^3\lambda-2cos\lambda\sin^2\lambda\right)+$$
$$+3\left(b_1b_2+b_2b_3+b_1b_3\right)\frac{\left(\cos^2\lambda sin\lambda-\sin^3\lambda\right)}{\lambda}+$$
$$+3b_1b_2b_3\frac{\sin^2\lambda cos\lambda}{\lambda^2}=0. \tag{2.11}$$

It was shown in [12], that there is a sequence of zeros of the function of λ in the left hand-side of (2.11) which has the form

$$\lambda_k \underset{k\to+\infty}{=} \pi k+\frac{x_1}{k}+O\left(\frac{1}{k^3}\right), \tag{2.12}$$

where $x_1=\frac{b_1+b_2+b_3}{3\pi}$.

In [12] we showed that knowing the asymptotics of the eigenvalues of problem (2.1)-(2.6) or, which is the same, of the zeros of the left hand side of (2.11) one can find $D_1:=b_1+b_2+b_3$, $D_2:=b_1b_2+b_1b_3+b_2b_3$ and $D_3:=b_1b_2b_3$, and, therefore, eventually find b_1,b_2 and b_3. However, solving the inverse problem in [12] we used infinitely many eigenvalues to find the three coefficients. In the present paper, we show how to find b_1,b_2 and b_3 using only three eigenvalues.

Lemma 2.1. The number $\lambda_k=\pi k$ for some $k\in\mathbb{Z}$ belongs to the set of eigenvalues of problem (2.1)-(2.6) if and only if $b_1+b_2+b_3=0$.

Proof. Necessity. Let us substitute $\lambda_k=\pi k$ into (2.11). Thus, we have

$$(-1)^k(b_1+b_2+b_3)=0$$

or

$$b_1+b_2+b_3=0.$$

Sufficiency. Plugging $b_1+b_2+b_3=0$ into (2.11) we arrive at

$$\sin\lambda\left(3\lambda\cos^2\lambda-3D_2\frac{\cos^2\lambda-\sin^2\lambda}{\lambda}-3D_3\frac{\sin\lambda\cos\lambda}{\lambda^2}\right)=0. \tag{2.13}$$

So, $\lambda_k = \pi k$ satisfies this equation. $\square$

Lemma 2.2. The number $\lambda_k = \frac{\pi}{2} + \pi k$ for some $k \in \mathbb{Z}$ belongs to the set of eigenvalues of problem (2.1)-(2.6) if and only if $b_1b_2 + b_2b_3 + b_1b_3 = 0$.

Proof. Necessity. Let us substitute $\lambda_k = \frac{\pi}{2} + \pi k$ into (2.11). Thus, we have

$$(-1)^k(b_1b_2 + b_2b_3 + b_1b_3) = 0$$

or

$$b_1b_2 + b_2b_3 + b_1b_3 = 0.$$

Sufficiency. Plugging $b_1b_2 + b_2b_3 + b_1b_3 = 0$ into (2.11) we arrive at

$$\cos\lambda\left(-3\lambda\sin\lambda\cos\lambda + D_1\left(\cos^2\lambda - 2\sin^2\lambda\right) + 3D_3\frac{\sin^2\lambda}{\lambda^2}\right) = 0 \quad (2.14)$$

So, $\lambda_k = \frac{\pi}{2} + \pi k$ satisfies this equation. $\square$

Remark 2.3. 1. It follows from Lemma 2.1 that if there exists $k \in \mathbb{Z}$ such that πk is an eigenvalue of problem (2.1)-(2.6) then $b_1 + b_2 + b_3 = 0$.

2. It follows from Lemma 2.2 that if there exists $r \in \mathbb{Z}$ such that $\frac{\pi}{2} + \pi r$ is an eigenvalue of problem (2.1)-(2.6) then $b_1b_2 + b_2b_3 + b_1b_3 = 0$.

Theorem 2.4. One can choose a triple of pairwise distinct eigenvalues of problem (2.1)-(2.6) such that they uniquely determine the coefficients b_1, b_2 and b_3 in the Robin conditions (2.4)-(2.6).

Proof 1. Let us consider the first case: assume that πk is an eigenvalue for some $k \in \mathbb{Z}$ and $\frac{\pi}{2} + \pi r$ is an eigenvalue for some r. Then, by Remark 2.3 $D_1 = D_2 = 0$, i. e. $b_1 + b_2 + b_3 = b_1b_2 + b_2b_3 + b_1b_3 = 0$ which implies $b_1 = b_2 = b_3 = 0$. In this case we need only two distinct eigenvalues to recover the coefficients b_1, b_2 and b_3.

Proof 2. Assume that $\lambda_k = \pi k$ is an eigenvalue for some value of k and then for all $k \in \mathbb{Z}$, but there is no eigenvalue of the form $\frac{\pi}{2} + \pi r$. Then $\cos\lambda_i \neq 0$ and $\sin\lambda_i \neq 0$ for the eigenvalues, which do not belong to the branch $\{\pi k\}$. By Lemma 2.1 in our case $D_1 = 0$. Therefore, we choose two distinct eigenvalues λ_i and λ_j ($\lambda_i \neq \pi k$ and $\lambda_j \neq \pi k$), which satisfy the following equations

$$D_2\frac{\cos^2\lambda_i - \sin^2\lambda_i}{\lambda_i} + D_3\frac{\sin\lambda_i\cos\lambda_i}{\lambda_i^2} = \lambda_i\cos^2\lambda_i, \quad (2.15)$$

$$D_2\frac{\cos^2\lambda_j - \sin^2\lambda_j}{\lambda_j} + D_3\frac{\sin\lambda_j\cos\lambda_j}{\lambda_j^2} = \lambda_j\cos^2\lambda_j, \quad (2.16)$$

where $\lambda_i \neq \lambda_j$. Let us choose λ_i arbitrarily and consider λ_j tending to infinity. Then using equation (2.16) from [12] we have

$$\lambda_j \underset{j\to+\infty}{=} \pi j + \frac{\pi}{2} + o(1), \tag{2.17}$$

and, therefore, $\cos^2\lambda_j \underset{j\to+\infty}{=} o(1)$, $\sin^2\lambda_j \underset{j\to+\infty}{=} 1+o(1)$;

$$\frac{\cos 2\lambda_j}{\lambda_j} \underset{j\to+\infty}{=} -\frac{1}{\pi j} + o\left(\frac{1}{j}\right), \quad \frac{\sin 2\lambda_j}{2\lambda_j^2} \underset{j\to+\infty}{=} \frac{1}{\pi^2 j^2} + o\left(\frac{1}{j^2}\right);$$

$$\frac{\cos 2\lambda_j}{\lambda_j}\cdot\frac{\sin 2\lambda_i}{2\lambda_i^2} \underset{j\to+\infty}{=} \frac{\sin 2\lambda_i}{2\lambda_i^2}\left(-\frac{1}{\pi j} + o\left(\frac{1}{j}\right)\right),$$

$$\frac{\cos 2\lambda_i}{\lambda_i}\cdot\frac{\sin 2\lambda_j}{2\lambda_j^2} \underset{j\to+\infty}{=} \frac{\cos 2\lambda_i}{\lambda_i}\left(\frac{1}{\pi^2 j^2} + o\left(\frac{1}{j^2}\right)\right),$$

then the determinant of system (2.15)-(2.16)

$$\Delta = \frac{\cos 2\lambda_i}{\lambda_i}\cdot\frac{\sin 2\lambda_j}{2\lambda_j^2} - \frac{\cos 2\lambda_j}{\lambda_j}\cdot\frac{\sin 2\lambda_i}{2\lambda_i^2} \underset{j\to+\infty}{=} \frac{\sin 2\lambda_i}{2\lambda_i^2}\cdot\frac{1}{\pi j} + o\left(\frac{1}{j}\right) \tag{2.18}$$

We see that we may fix any λ_i and choose j large enough to obtain $\Delta \neq 0$ and system (2.15)-(2.16) has a unique solution. Thus, three eigenvalues $\lambda_k = \pi k$, λ_i and λ_j uniquely determine D_1, D_2 and D_3 and, therefore, b_1, b_2 and b_3 as solutions of the cubic equation

$$z^3 + D_2 z - D_3 = 0.$$

Proof 3. Let us consider the case where $D_1 \neq 0$. In this case, we choose three distinct eigenvalues λ_i, λ_j, λ_s. That means these λ_i, λ_j, λ_s satisfy equation (2.11). So we have

$$D_1\left(\cos^3\lambda_i - 2cos\lambda_i\sin^2\lambda_i\right) + 3D_2\frac{sin\lambda_i cos2\lambda_i}{\lambda_i} + 3D_3\frac{\sin^2\lambda_i cos\lambda_i}{\lambda_i^2} =$$
$$= \lambda_i sin\lambda_i\cos^2\lambda_i \tag{2.19}$$

$$D_1\left(\cos^3\lambda_j - 2cos\lambda_j\sin^2\lambda_j\right) + 3D_2\frac{sin\lambda_j cos2\lambda_j}{\lambda_j} + 3D_3\frac{\sin^2\lambda_j cos\lambda_j}{\lambda_j^2} =$$
$$3\lambda_j sin\lambda_j\cos^2\lambda_j, \tag{2.20}$$

$$D_1\left(\cos^3\lambda_s - 2cos\lambda_s\sin^2\lambda_s\right) + 3D_2\frac{sin\lambda_s cos2\lambda_s}{\lambda_s} + 3D_3\frac{\sin^2\lambda_s cos\lambda_s}{\lambda_s^2} =$$
$$= 3\lambda_s sin\lambda_s\cos^2\lambda_s. \tag{2.21}$$

It is quite possible that the determinant of this system is not equal to zero for any distinct λ_i, λ_j, λ_s. However, we cannot prove it because of the tremendous calculations needed. Thus, we will prove that the eigenvalues λ_i, λ_j, λ_s can be chosen such that the determinant of this system is not zero. For convenience, we denote the terms in (2.19)-(2.21) in the following way

$$\Lambda_m^1 = \cos^3\lambda_m - 2cos\lambda_m \sin^2\lambda_m,$$

$$\Lambda_m^2 = 3\frac{sin\lambda_m cos2\lambda_m}{\lambda_m},$$

$$\Lambda_m^3 = 3\frac{\sin^2\lambda_m cos\lambda_m}{\lambda_m^2},$$

$$R_m = 3\lambda_m sin\lambda_m \cos^2\lambda_m,$$

where $m = i, j, s$. So, the determinant has the following form

$$\Delta_{ijs} = \det\begin{pmatrix} \Lambda_i^1 & \Lambda_i^2 & \Lambda_i^3 \\ \Lambda_j^1 & \Lambda_j^2 & \Lambda_j^3 \\ \Lambda_s^1 & \Lambda_s^2 & \Lambda_s^3 \end{pmatrix}. \tag{2.22}$$

Let us show that λ_i, λ_j, λ_s can be chosen such that Δ_{ijs} is not equal to zero. We choose λ_i, λ_j, λ_s from the subsequence $\{\lambda_k\}$ described by (2.12) and we set $\lambda_i = \lambda_k$, $\lambda_j = \lambda_{k^2}$, $\lambda_s = \lambda_{k^3}$. Then, with $x_1 = \frac{b_1+b_2+b_3}{3\pi}$,

$$\lambda_i \underset{k\to+\infty}{=} 2\pi k + \frac{x_1}{k} + O\left(\frac{1}{k^3}\right), \tag{2.23}$$

$$\lambda_j \underset{k\to+\infty}{=} 2\pi k^2 + \frac{x_1}{k^2} + O\left(\frac{1}{k^6}\right), \tag{2.24}$$

$$\lambda_s \underset{k\to+\infty}{=} 2\pi k^3 + \frac{x_1}{k^3} + O\left(\frac{1}{k^9}\right). \tag{2.25}$$

We rewrite (2.22) in the form

$$\Delta_{ijs} = -\Lambda_s^1\Lambda_i^3\Lambda_j^2 + \Lambda_j^1\Lambda_i^3\Lambda_s^2 + \Lambda_s^1\Lambda_i^2\Lambda_j^3 - \Lambda_j^1\Lambda_i^2\Lambda_s^3 - \Lambda_i^1\Lambda_j^3\Lambda_s^2 + \Lambda_i^1\Lambda_j^2\Lambda_s^3. \tag{2.26}$$

It is clear that $\Lambda^1_m \underset{m\to+\infty}{=} 1+o(1)$. Using λ_i, λ_j, λ_s with the asymptotics (2.23)-(2.25) we arrive at the following expressions:

$$-\Lambda^1_s\Lambda^3_i\Lambda^2_j \underset{k\to+\infty}{=} -\frac{9x_1^3}{8\pi^3k^8}+o\left(\frac{1}{k^8}\right),$$

$$\Lambda^1_j\Lambda^3_i\Lambda^2_s \underset{k\to+\infty}{=} \frac{9x_1^3}{8\pi^3k^{10}}+o\left(\frac{1}{k^{10}}\right),$$

$$\Lambda^1_s\Lambda^2_i\Lambda^3_j \underset{k\to+\infty}{=} \frac{9x_1^3}{8\pi^3k^{10}}+o\left(\frac{1}{k^{10}}\right),$$

$$-\Lambda^1_j\Lambda^2_i\Lambda^3_s \underset{k\to+\infty}{=} -\frac{9x_1^3}{8\pi^3k^{14}}+o\left(\frac{1}{k^{14}}\right),$$

$$-\Lambda^1_i\Lambda^3_j\Lambda^2_s \underset{k\to+\infty}{=} -\frac{9x_1^3}{8\pi^3k^{14}}+o\left(\frac{1}{k^{14}}\right),$$

$$\Lambda^1_i\Lambda^2_j\Lambda^3_s \underset{k\to+\infty}{=} \frac{9x_1^3}{8\pi^3k^{16}}+o\left(\frac{1}{k^{16}}\right).$$

Since the terms contain k in different degrees, and the term $-\Lambda^1_s\Lambda^3_i\Lambda^2_j$ is dominating, we have

$$\Delta_{ijs} \underset{k\to+\infty}{=} -\frac{9x_1^3}{8\pi^3k^8}+o\left(\frac{1}{k^8}\right).$$

That means (since $x_1\neq 0$ due to $D_1\neq 0$) that $\Delta_{ijs}\neq 0$ for sufficiently large k and three eigenvalues λ_i, λ_j, λ_s chosen such as described above. Thus, system (2.19)-(2.21) has a unique solution.

Knowing D_1, D_2, and D_3 we find b_1, b_2, b_3 as solutions of the cubic equation

$$z^3-D_1z^2+D_2z-D_3=0.$$

3. Conclutions

In our previous publication [12] we showed that the asymptotics of the eigenvalues of problem (2.1)-(2.6) uniquely determine the coefficients in the Robin conditions. Thus, we used infinitely many eigenvalues. Also, we showed that since the number of vertices of our tree which is 4 is less than the critical value (which, according to [4], is 8) we concluded that the two first terms of the asymptotics uniquely determine the shape of our tree.

In the present paper, we assume that the shape of the tree is given, but we use only 3 eigenvalues to find the coefficients in the Robin conditions.

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