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Sets of numbers with unique representations in numeral systems with a natural base and a redundant alphabet

Mykola Pratsiovytyi, Oleh Vynnyshyn

Abstract. In this work, we study a numeral system with a natural base $s \geqslant 2$ and a redundant alphabet $A_r = \{0, 1, \ldots, r\}$, where $s \leqslant r \leqslant 2s-2$. We investigate the topological, metric, and fractal properties of the set of numbers in the interval $[0, \frac{r}{s-1}]$ that admit a unique representation $x = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} \equiv \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, $\alpha_n \in A_r$. A criterion for the uniqueness of a number's representation is established. We prove that the Hausdorff–Besicovitch dimension of the set of numbers with a unique representation is equal to $\frac{\ln(2s-r-1)}{\ln s}$. An analysis of the quantity of representations for numbers having purely periodic representations with a simple (single-digit) period is carried out. It is proved that the set of numbers that admit a continuum of distinct representations has full Lebesgue measure. Conditions for a number to belong to this set are given in terms of one of its representations.

Анотація. У роботі для системи числення з натуральною основою $s \geqslant 2$ і надлишковим алфавітом $A_r = \{0, 1, \ldots, r\}$, $s \leqslant r \leqslant 2s-2$, вивчаються тополого-метричні і фрактальні властивості множин чисел відрізка $[0; \frac{r}{s-1}]$, які мають єдине зображення:

$$x = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n} \equiv \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \quad \alpha_n \in A_r.$$

Обґрунтовано критерій єдиності зображення числа, доведено, що розмірність Гаусдорфа-Безиковича множини чисел з єдиним зображенням рівна $\frac{\ln(2s-r-1)}{\ln s}$. Наведено результати аналізу кількості зображень у чисел, які мають чисто періодичне зображення з простим періодом (періодом з однієї цифри). Доведено, що множина чисел, які мають континуальну множину різних зображень, є множиною повної

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міри Лебега. Наведено умови належності чисел цій множині за одним з їх зображень.

1. Introduction

Expansions of real numbers $x \in [0; \frac{m}{q-1}]$ of the form

$$x = \sum_{k=1}^{\infty} \frac{\alpha_k}{q^k} \equiv \Delta^q_{\alpha_1\alpha_2\ldots\alpha_k\ldots}$$

with an arbitrary (not necessarily integer) base $q \in (1; m+1)$ and a redundant alphabet $A_m = \{0, 1, \ldots, m\}$ have been studied since 1957. This line of research was initiated by Rényi [20] and Parry [12]. A new wave of interest in this topic was triggered by the work of Erdős and Joó [4], where it was shown that the set of bases in which 1 has a unique expansion has the cardinality of the continuum. Since then, many interesting results have been obtained, and dozens of papers have been published. The two main directions of research in this area are as follows.

1. For fixed q and A_m, one studies the sets of numbers having: a unique representation, a finite set of distinct representations, a countable set of representations, or a continuum of representations.
2. For a fixed $x > 0$, one studies the set of bases q for which x has a set of representations of various cardinalities.

For the sets mentioned above, researchers study their topological, metric, and fractal properties. We are interested in the first direction under the assumption that the base q is an integer. Let us note that numeral systems with a redundant alphabet have found applications in various areas of mathematics, including the theory of distributions of random variables (primarily singular ones), infinite Bernoulli convolutions [18, 14, 19], the theory of functions with a locally complex structure and fractal properties [17], fractal analysis [15], the geometry of numerical series [22, 8], ergodic theory, the theory of dynamical systems, and others. It is not the aim of the present paper to provide a comprehensive survey of known results, as there already exist relatively comprehensive surveys [21, 9, 13, 2].

Our interest in numeral systems with a natural base and a redundant alphabet arose from the study of convolutions of singular probability distributions and the distributions of random variables whose digits in a given numeral system are independent. It is known [7] that the convolution of two singular distributions may be singular, absolutely continuous, or a mixture of both. The problem of distinguishing these cases by explicit conditions, even within specific classes of random variables, is nontrivial, although examples in which this can be done are easy to construct. In [14], it was shown that the sum of two independent singularly distributed random variables, whose digits in the classical binary numeral system are independent random variables, has a singular distribution.

Our approach is based on two main ideas: the existence of mutually replaceable (alternative) pairs of consecutive digits in the representation of a number, and the use of the specific structure of overlaps of cylinder sets. A distinctive feature of this approach is that it yields explicit descriptions of the representations of numbers, together with direct, elementary number-theoretic proofs of the results. In what follows, we use the notation described below.

Let s and r be two natural numbers such that $2 \leqslant s \leqslant r \leqslant 2s-2$. Let $A_r = \{0, 1, \ldots, r\}$ be an alphabet (a set of digits), and let $L_r = A_r \times A_r \times \cdots$ be the set of sequences with elements from the alphabet. It is known [5, 18] that for any number $x \in [0; \frac{r}{s-1}]$, there exists a sequence $(a_k) \in L_r$ such that

$$x = \sum_{k=1}^{\infty} a_k s^{-k} = \Delta^{r_s}_{a_1 a_2 \ldots a_k \ldots}. \tag{1.1}$$

The symbolic (abbreviated) notation $x = \Delta^{r_s}_{a_1 a_2 \ldots a_n \ldots}$ of the number's expansion into the series (1.1) is called the r_s-*representation* of the number x, or the representation of x in the numeral system with base s and alphabet A_r. Hereafter, parentheses in the representation of a number denote a period. Obviously,

$$\begin{aligned} \Delta^{r_s}_{a_1 a_2 \ldots} &= \Delta^{r_s}_{a_1 a_2 \ldots a_m (0)} + \Delta^{r_s}_{0 \ldots 0 a_{m+1} a_{m+2} \ldots} \\ &= \Delta^{r_s}_{a_1 a_2 \ldots a_m (0)} + \frac{1}{s^m} \Delta^{r_s}_{a_{m+1} a_{m+2} \ldots}, \end{aligned} \tag{1.2}$$

and

$$\Delta^{r_s}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots} + \Delta^{r_s}_{\beta_1 \beta_2 \ldots \beta_n \ldots} = \Delta^{r_s}_{[\alpha_1+\beta_1][\alpha_2+\beta_2]\ldots[\alpha_n+\beta_n]\ldots},$$

with $[\alpha_n + \beta_n] \in A_r$. If

$$\frac{a}{s} + \frac{b}{s^2} = \frac{c}{s} + \frac{d}{s^2},$$

which is equivalent to

$$s(a-c) = d-b,$$

then the pairs (a, b) and (c, d) in the representation of a number are mutually replaceable (alternative). The redundancy of the alphabet A_r guarantees the existence of numbers that have an infinite set, or even a continuum, of different r_s-representations. Such systems of encoding real numbers [16] with an integer base $s \geqslant 2$ have been studied in various works, in particular in [5, 11]. Also, in the works [1, 10, 3], sets of numbers with a unique representation in numeral systems with a redundant alphabet and a real base were studied. However, we did not find an elementary exposition for integer bases in the literature. Therefore, we continue our previously initiated research, specifying certain results and providing more detailed arguments.

2. Numbers with purely periodic representations and a simple period

Lemma 2.1. *Every number x_0 that has a purely periodic r_s-representation with period (c), where $c \in \{0, r\} \cup \{r-s+2, \dots, s-2\}$, has a unique representation.*

Proof. If $c \in \{0, r\}$, then the statement is obvious. Consider the number $x_0 = \Delta^{r_s}_{(r-s+1+m)}$, where $m \in \{1, 2, \dots, 2s-r-3\}$. Assume that there exists another r_s-representation $\Delta^{r_s}_{\alpha_1\alpha_2\dots\alpha_n\dots}$ of the number x_0. Since the representations are different, there is a smallest index k such that $\alpha_k \neq c$. We proceed by considering the following cases:

(1) If $\alpha_k > c$, then, taking into account (1.2), we have

$$\Delta^{r_s}_{\alpha_k\alpha_{k+1}\alpha_{k+2}\dots} \geqslant \Delta^{r_s}_{\alpha_k(0)} \geqslant \Delta^{r_s}_{[c+1](0)} = \Delta^{r_s}_{c(s-1)} > \Delta^{r_s}_{c(c)},$$

since $c < s-1$. Thus we obtain $x_0 > x_0$, which is a contradiction.

(2) If $\alpha_k < c$, then

$$\Delta^{r_s}_{\alpha_k\alpha_{k+1}\alpha_{k+2}\dots} \leqslant \Delta^{r_s}_{\alpha_k(r)} \leqslant \Delta^{r_s}_{[c-1](r)} = \Delta^{r_s}_{c(r-(s-1))} < \Delta^{r_s}_{c(c)},$$

since $c > r-(s-1)$. Thus we obtain $x_0 > x_0$, which is a contradiction.

Lemma is proved. □

Lemma 2.2. *Every number x_0 that has a purely periodic r_s-representation with period (c), where $c \in \{1, 2, \dots, r-s, s, s+1, \dots, r-1\}$, has a continuum of distinct representations.*

Proof. If $c \in \{1, 2, \dots, r-s\}$, then for each pair of adjacent digits cc there exists a mutually replaceable pair $(c-1, c+s)$.

If $c \in \{s, s+1, \dots, r-1\}$, then for the pair cc the alternative one is $(c+1, c-s)$. Since there are infinitely many positions of adjacent digit pairs in the r_s-representation of the number x_0 that have alternatives, we conclude that there exists a continuum of distinct representations of x_0. □

Lemma 2.3. *Every number x_0 that has a purely periodic r_s-representation with period (c), where $c \in \{r-s+1, s-1\}$, has an infinite set of distinct r_s-representations.*

Proof. If $c = s-1$, then the equalities

$$\Delta^{r_s}_{(s-1)} = \Delta^{r_s}_{s(0)} = \Delta^{r_s}_{[s-1]s(0)} = \Delta^{r_s}_{[s-1][s-1]\dots[s-1]s(0)}$$

hold, meaning the given number has an infinite set of representations. If $c = r - s + 1$, then similarly,

$$\Delta^{r_s}_{(r-s+1)} = \Delta^{r_s}_{[r-s](r)} = \Delta^{r_s}_{[r-s+1][r-s](r)} = \Delta^{r_s}_{[r-s+1]\ldots[r-s+1][r-s](r)}.$$

Thus, the number x_0 has an infinite set of representations. □

3. Numbers with a unique representation

The following statement is obvious.

Lemma 3.1. *The number $x = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ has a unique r_s-representation if and only if the number $x' = \Delta^{r_s}_{[r-\alpha_1][r-\alpha_2]\ldots[r-\alpha_n]\ldots}$ has a unique representation.*

Proof. Indeed, if the number x has another representation $\Delta^{r_s}_{\beta_1\ldots\beta_n\ldots}$, where $\beta_k \neq \alpha_k$ for some k, then $\Delta^{r_s}_{[r-\beta_1][r-\beta_2]\ldots[r-\beta_n]\ldots}$ is another representation of the number x'. Conversely, the same logic holds in the opposite direction. This contradicts the uniqueness of the r_s-representation of the number. □

Theorem 3.2 (Criterion for the uniqueness of a number representation)**.** *For a number $x = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ in the interval $(0, \frac{r}{s-1})$ to have a unique r_s-representation, it is necessary and sufficient that, for every $k \in \mathbb{N}$, the following two conditions hold:*

(1) $\Delta^{r_s}_{\alpha_{k+1}\alpha_{k+2}\ldots\alpha_{k+n}\ldots} < 1$ *or* $\alpha_k = r$;
(2) $\Delta^{r_s}_{[r-\alpha_{k+1}][r-\alpha_{k+2}]\ldots[r-\alpha_{k+n}]\ldots} < 1$ *or* $\alpha_k = 0$.

Proof. **Necessity.** Let the number $x = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ have a unique r_s-representation. We prove that conditions (1) and (2) are met.

Since $0 \neq x \neq \frac{r}{s-1}$, there exists an index k such that $\alpha_k \neq r$. Consider the number $x_k = \Delta^{r_s}_{\alpha_{k+1}\ldots\alpha_{k+n}\ldots}$. Assume that $x_k \geqslant 1$.

If $x_k = 1$, then $x_k = \Delta^{r_s}_{(s-1)} = \Delta^{r_s}_{[s-1]s(0)}$, and hence

$$x = \Delta^{r_s}_{\alpha_1\ldots\alpha_k(s-1)} = \Delta^{r_s}_{\alpha_1\ldots\alpha_k[s-1]s(0)},$$

which contradicts the uniqueness of the r_s-representation of x.

Assume that $x_k > 1$. Then among the digits of the representation of x_k, there exists an $\alpha_{k+j} > s - 1$; we take the smallest such j. If $j = 1$, then

$$x = \Delta^{r_s}_{\alpha_1\ldots\alpha_{k-1}[\alpha_k+1][\alpha_{k+1}-s]\alpha_{k+2}\alpha_{k+3}\ldots}.$$

If $\alpha_{k+1} \leqslant s - 1$, then

$$x = \Delta^{r_s}_{\alpha_1\ldots\alpha_k\alpha_{k+1}\ldots[\alpha_{k+j-1}+1][\alpha_{k+j}-s]\alpha_{k+j+1}\ldots}.$$

In both cases, we obtain another r_s-representation of the number x, which contradicts its uniqueness. These contradictions prove condition (1).

Condition (2) is proved analogously. Moreover, it follows directly from condition (1) and Lemma 3.1.

Sufficiency. Assume that for the number $x = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$, both conditions (1) and (2) hold for every $k \in \mathbb{N}$, and that there exists another representation $x = \Delta^{r_s}_{\beta_1\beta_2\ldots\beta_n\ldots}$. Then there exists an index $k \in \mathbb{N}$ such that $\alpha_k \neq \beta_k$, while $\alpha_j = \beta_j$ for all $j < k$. Suppose that $\alpha_k > \beta_k$. Then $\alpha_k \neq 0$, and by condition (2) we have

$$\Delta^{r_s}_{(r)} - \Delta^{r_s}_{\alpha_{k+1}\alpha_{k+2}\ldots} = \Delta^{r_s}_{[r-\alpha_{k+1}][r-\alpha_{k+2}]\ldots[r-\alpha_{k+n}]\ldots} < 1.$$

Then $\Delta^{r_s}_{(r)} - 1 < \Delta^{r_s}_{\alpha_{k+1}\alpha_{k+2}\ldots\alpha_{k+n}\ldots}$, but

$$\Delta^{r_s}_{(r)} - 1 = \Delta^{r_s}_{(r)} - \Delta^{r_s}_{(s-1)} = \Delta^{r_s}_{(r-s+1)} < \Delta^{r_s}_{\alpha_{k+1}\alpha_{k+2}\ldots\alpha_{k+n}\ldots}.$$

Hence

$$\Delta^{r_s}_{\alpha_k(r-s+1)} = \frac{\alpha_k}{s} + \frac{1}{s}\Delta^{r_s}_{(r-s+1)} < \frac{\alpha_k}{s} + \frac{1}{s}\Delta^{r_s}_{\alpha_{k+1}\ldots\alpha_{k+n}\ldots} = \Delta^{r_s}_{\alpha_k\alpha_{k+1}\ldots\alpha_{k+n}\ldots}.$$

However,

$$\Delta^{r_s}_{\alpha_k(r-s+1)} = \Delta^{r_s}_{[\alpha_k-1](r)},$$

and therefore

$$\Delta^{r_s}_{[\alpha_k-1](r)} < \Delta^{r_s}_{\alpha_k\alpha_{k+1}\ldots\alpha_{k+n}\ldots}.$$

Consequently,

$$\Delta^{r_s}_{\beta_k\beta_{k+1}\beta_{k+2}\ldots} \leqslant \Delta^{r_s}_{\beta_k(r)} \leqslant \Delta^{r_s}_{[\alpha_k-1](r)} < \Delta^{r_s}_{\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots},$$

and hence

$$\Delta^{r_s}_{\beta_k\beta_{k+1}\beta_{k+2}\ldots} < \Delta^{r_s}_{\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots}.$$

Thus, in the case where $\alpha_k > \beta_k$, we obtain a contradiction to the assumption that there exists another representation of the number x.

Now suppose that $\alpha_k < \beta_k$. By condition (1) we have

$$\Delta^{r_s}_{\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots} < \Delta^{r_s}_{[\alpha_k+1](0)}$$

and $\alpha_k \neq r$. Moreover,

$$\Delta^{r_s}_{[\alpha_k+1](0)} \leqslant \Delta^{r_s}_{\beta_k\beta_{k+1}\beta_{k+2}\ldots},$$

and hence

$$\Delta^{r_s}_{\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots} < \Delta^{r_s}_{\beta_k\beta_{k+1}\beta_{k+2}\ldots}.$$

Thus, in this case as well, we obtain a contradiction to the assumption that there exists another representation of the number x. The sufficiency and hence the entire theorem are proved. □

Remark 3.3. In the case $r = s$, the described criterion coincides with [1, Theorem 4.2], where the set of numbers with a unique representation in numeral systems with one redundant digit is studied in detail.

Lemma 3.4. *Let α_k and α_{k+1} be consecutive digits in the representation of a number $x_0 = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_{k-1}\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots}$ such that*
(1) $\alpha_k > 0$ and $\alpha_{k+1} \leqslant r - s$, or
(2) $\alpha_k < r$ and $\alpha_{k+1} \geqslant s$.
Then the digit pair (α_k, α_{k+1}) can be replaced by the alternative pair
In case (1): $(\alpha_k, \alpha_{k+1}) \mapsto (\alpha_k - 1, \alpha_{k+1} + s) = (\alpha'_k, \alpha'_{k+1})$;
In case (2): $(\alpha_k, \alpha_{k+1}) \mapsto (\alpha_k + 1, \alpha_{k+1} - s) = (\alpha'_k, \alpha'_{k+1})$;
and the equality $x_0 = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_{k-1}\alpha'_k\alpha'_{k+1}\alpha_{k+2}\ldots}$ holds.

Proof. In case (1), we have the equality $\frac{a}{s^k} + \frac{b}{s^{k+1}} = \frac{a-1}{s^k} + \frac{b+s}{s^{k+1}}$. Therefore

$$\Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_{k-1}\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots} = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_{k-1}[\alpha_k-1][\alpha_{k+1}+s]\alpha_{k+2}\ldots}.$$

In case (2), we have the equality $\frac{a}{s^k} + \frac{b}{s^{k+1}} = \frac{a+1}{s^k} + \frac{b-s}{s^{k+1}}$. Therefore

$$\Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_{k-1}\alpha_k\alpha_{k+1}\alpha_{k+2}\ldots} = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_{k-1}[\alpha_k+1][\alpha_{k+1}-s]\alpha_{k+2}\ldots}.$$ □

Lemma 3.5. *Let $x_0 \in (0, \frac{r}{s-1})$ be a number with a unique r_s-representation $\Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$. If $\alpha_k = 0$ for some k, then $\alpha_i = 0$ for all $i < k$. The representation of x_0 has the form*

$$x_0 = \Delta^{r_s}_{0\ldots0\alpha_m\alpha_{m+1}\ldots\alpha_{m+n}\ldots},$$

where $0 < \alpha_m < s$ and $\alpha_{m+j} \in \{r - s + 1, r - s + 2, \ldots, s - 1\} \equiv N^{s-1}_{r-s+1}$, and the sequence (α_{m+n}) cannot have period $(r - s + 1)$ or $(s - 1)$.

Proof. Since $x_0 \neq 0$, its r_s-representation contains at least one nonzero digit. Assume that $\alpha_k = 0$ and that there exists an index $i < k$ such that $\alpha_i > 0$. Let $j = \max\{i : \alpha_i > 0, i < k\}$. The number x_0 has a unique r_s-representation, but the pair $(\alpha_j, \alpha_{j+1}) = (\alpha_j, 0)$ admits the replacement $(\alpha_j - 1, s)$, which is a contradiction. Thus, the number x_0 has the form

$$x_0 = \Delta^{r_s}_{0\ldots0\alpha_m\alpha_{m+1}\ldots}.$$

The digit α_m cannot exceed $s-1$, since otherwise the pair $(0, \alpha_m)$ would admit the alternative replacement $(1, \alpha_m - s)$. Thus, $0 < \alpha_m \leqslant s - 1$.

Suppose that there exists an $l > 0$ such that either $\alpha_{m+l} > s - 1$ or $\alpha_{m+l} < r - s + 1$. Let t be the smallest number with this property. In the case where $\alpha_{m+t} > s - 1$, the pair $(\alpha_{m+t-1}, \alpha_{m+t})$ can be replaced by $(\alpha_{m+t-1} + 1, \alpha_{m+t} - s)$. In the case where $\alpha_{m+t} < r - s + 1$, the pair $(\alpha_{m+t-1}, \alpha_{m+t})$ can be replaced by $(\alpha_{m+t-1} - 1, \alpha_{m+t} + s)$. This is a contradiction with the uniqueness of the representation.

Also, by Lemma 2.3, the number x_0 cannot have period $(r - s + 1)$ or $(s - 1)$. Taking into account the above observations, the statement is proven. $\square$

Lemma 3.6. *Let $x_0 \in (0, \frac{r}{s-1})$ be a number with a unique r_s-representation $\Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$. If $\alpha_k = r$ for some k, then $\alpha_i = r$ for all $i < k$. The representation of x_0 has the form*

$$x_0 = \Delta^{r_s}_{r\ldots r\alpha_m\alpha_{m+1}\ldots\alpha_{m+n}\ldots}, \quad \textit{where} \quad r - s < \alpha_m < r \ \textit{and}\ \alpha_{m+i} \in N^{s-1}_{r-s+1},$$

and the sequence (α_{m+n}) cannot have period $(s - 1)$ or $(r - s + 1)$.

Proof. Since $x_0 \neq \frac{r}{s-1}$, its r_s-representation contains a digit different from r. Assume that $\alpha_k = r$. Suppose that there exists an index $i < k$ such that $\alpha_i < r$, and let

$$j = \max\{i : \alpha_i < r, i < k\}.$$

Then the pair $(\alpha_j, \alpha_{j+1}) = (\alpha_j, r)$ admits an alternative replacement, i.e., the pair $(\alpha_j + 1, r - s)$, which contradicts the uniqueness of the r_s-representation of the number x_0. Therefore, the r_s-representation of the number x_0 has the form

$$x_0 = \Delta^{r_s}_{r\ldots r\alpha_m\alpha_{m+1}\ldots}.$$

If we assume that $\alpha_m \leqslant r - s$, then the pair (r, α_m) admits an alternative replacement, namely the pair $(r - 1, \alpha_m + s)$, which contradicts the uniqueness of the r_s-representation of the number x_0. Hence, $r - s < \alpha_m < r$.

Suppose that $\alpha_{m+l} < r - s + 1$ or $\alpha_{m+l} > s - 1$ for some $l > 0$, and let $m + t$ be the smallest index with this property. Then, for the digit pair $(\alpha_{m+t-1}, \alpha_{m+t})$, an alternative replacement can be specified, which again leads to a contradiction to the uniqueness of the r_s-representation of x_0. The same conclusion can be obtained based on Lemma 3.1, since the number $x = \Delta^{r_s}_{r\ldots r\alpha_m\alpha_{m+1}\ldots}$ has a unique representation if and only if the number

$$x_0' = \frac{r}{s-1} - x_0 = \Delta^{r_s}_{0\ldots 0[r-\alpha_m][r-\alpha_{m+1}][r-\alpha_{m+2}]\ldots}$$

has a unique r_s-representation. According to Lemma 3.5, the representation of x_0' must satisfy the conditions

$$0 < r - \alpha_m < s \quad \text{and} \quad r - s + 1 \leqslant r - \alpha_{m+1} \leqslant s - 1,$$

which are equivalent to the conditions of the present statement. Taking into account the above observation, the lemma is proven. $\square$

Theorem 3.7. *The set E of numbers $x \in (0, \frac{r}{s-1})$ that have a unique r_s-representation consists precisely of those numbers whose r_s-representations do* *not contain a period equal to $(s-1)$ or $(r-s+1)$ and have one of the following forms:*

$$\Delta^{r_s}_{c\beta_1\beta_2\ldots\beta_n\ldots},\quad \Delta^{r_s}_{00\ldots0a\beta_1\beta_2\ldots\beta_n\ldots},\quad \Delta^{r_s}_{rr\ldots rb\beta_1\beta_2\ldots\beta_n\ldots},$$

where $0 \leqslant a < s$, $r-s < b \leqslant r$, $0 \leqslant c \leqslant r$, and $\beta_i \in N^{s-1}_{r-s+1}$.

Proof. If the r_s-representation of a number with a unique representation contains the digit 0 or r, then, as established by the two preceding lemmas, the representation of the number has the form

$$\Delta^{r_s}_{00\ldots0a\beta_1\beta_2\ldots\beta_n\ldots} \quad \text{or} \quad \Delta^{r_s}_{rr\ldots rb\beta_1\beta_2\ldots\beta_n\ldots},$$

where $0 \leqslant a < s$, $r-s < b \leqslant r$, and $\beta_i \in N^{s-1}_{r-s+1}$. Consider the numbers that have a unique r_s-representation and do not use the digits 0 and r in their r_s-representations at all. Let $x_0 = \Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ be a number of this type. Suppose that for some $k > 1$ the digit is

$$\alpha_k \in \{1,\ldots,r-s\} \cup \{s, s+1,\ldots,r-1\},$$

then the pair (α_{k-1}, α_k) admits an alternative replacement. In the case $1 \leqslant \alpha_k \leqslant r-s$, this replacement is the pair $(\alpha_{k-1}-1, \alpha_k+s)$, and in the case $s \leqslant \alpha_k \leqslant r-1$, the alternative replacement is the pair $(\alpha_{k-1}+1, \alpha_k-s)$. In both cases, this contradicts the uniqueness of the r_s-representation of the number x_0. Hence,

$$\alpha_k \in N^{s-1}_{r-s+1} \quad \forall k \in \{2,3,\ldots\}.$$

The same conclusion can also be obtained by applying the criterion of uniqueness (Theorem 3.2). Note that the digit α_1 is not subject to these restrictions. □

Remark 3.8. We note that in [3, Lemma 4.1], a lexicographic criterion for the uniqueness of number representation was established for arbitrary real bases $q > 1$. Therefore, Theorem 3.2 is not new in itself. However, the proof presented above is direct and elementary, and does not rely on the theory of greedy or quasi-greedy expansions. The structural description of the set E of numbers having a unique r_s-representation is a particular case of a more general result established in [3]. At the same time, the proofs presented here are elementary and self-contained.

Theorem 3.9. *The set E of numbers $x \in (0, \frac{r}{s-1})$ that have a unique r_s-representation is symmetric about the point $\frac{r}{2(s-1)}$, has a Hausdorff–Besicovitch fractal dimension of $\frac{\ln(2s-r-1)}{\ln s}$, and is not closed for $r < 2s-2$.*

Proof. The symmetry of the set E is proved in Lemma 3.1. Let $r = 2s-2$, then by Theorem 3.7, all numbers $x \in (0; \frac{r}{s-1})$ with a unique representation have a tail $\beta_1\beta_2\ldots$, where $\beta_i \in N^{s-1}_{r-s+1} = \{s-1\}$, which means the representation is periodic with period $(s-1)$. However, by the same theorem, numbers from E cannot have period $(s-1)$ or $(r-s+1)$. Therefore $E = \varnothing$.

In the case $r < 2s-2$, each point of the form

$$\Delta^{r_s}_{\underbrace{0\ldots0}_{k}a\beta_1\beta_2\ldots\beta_n(c)},$$

where $c \in \{s-1, r-s+1\}$ and $k \in \mathbb{N}$, is a limit point of the set E, since in E there exists a point arbitrarily close to it of the form

$$\Delta^{r_s}_{\underbrace{0\ldots0}_{k}a\beta_1\beta_2\ldots\beta_n cc\ldots c([s-1][s-2])},$$

with a two-digit period. Moreover, each point of the form

$$\Delta^{r_s}_{\underbrace{0\ldots0}_{k}a\beta_1\beta_2\ldots\beta_n(c)}$$

does not belong to the set E (Theorem 3.7). Therefore, the set E is not closed.

We determine the Hausdorff–Besicovitch dimension of the set E. Let D denote the set of all numbers whose r_s-representation has period $(s-1)$ or $(r-s+1)$. As noted above, $E \cap D = \varnothing$. It is well known that every countable set has Lebesgue measure zero and Hausdorff–Besicovitch dimension zero. Therefore, such sets can be neglected in metric problems.

Let us introduce the following notation. For $n \in \mathbb{N} \cup \{0\}$, define

$$E^a_{0n} = \{x : \; x = \Delta^{r_s}_{\underbrace{0\ldots0}_{n}a\beta_1\beta_2\ldots\beta_k\ldots}, \beta_k \in N^{s-1}_{r-s+1}\}, \quad 0 < a < s;$$

$$E^b_{rn} = \{x : \; x = \Delta^{r_s}_{\underbrace{r\ldots r}_{n}b\beta_1\beta_2\ldots\beta_k\ldots}, \beta_k \in N^{s-1}_{r-s+1}\}, \quad r-s < b < r;$$

$$\overline{E}^a_{0n} \equiv E^a_{0n}\backslash D, \quad \overline{E}^b_{rn} \equiv E^b_{rn}\backslash D.$$

Then the set E has the following structure:

$$E = \bigcup_{n=0}^{\infty}\left[\bigcup_{a=1}^{s-1}\overline{E}^a_{0n} \cup \bigcup_{b=r-s+1}^{r-1}\overline{E}^b_{rn}\right].$$

All sets E^j_{in} with fixed n are isometric to each other and are self-similar. Each of them is similar to the set

$$C = \{x : \; x = \Delta^{r_s}_{\beta_1\beta_2\ldots\beta_n\ldots}, \beta_n \in N^{s-1}_{r-s+1}\},$$

which consists of numbers representable in the classical base-s numeral system using digits from the set N^{s-1}_{r-s+1}. This set is a self-similar Cantor-type set meeting the open set condition [6]. Therefore, its similarity dimension $\log_s(2s-r-1)$ coincides with its Hausdorff–Besicovitch dimension $\alpha_0(C)$.

The Hausdorff–Besicovitch dimensions of the sets E and C coincide due to the countable stability property of the dimension:

$$\alpha_0\Big(\bigcup_i F_i\Big) = \sup_i \alpha_0(F_i). \qquad \square$$

Remark 3.10. For the case $r = s$, the obtained formula coincides with the result given in [1, Theorem 6.1(c)]. It is also a particular case of the general formula for the Hausdorff–Besicovitch dimension of the set of numbers with a unique representation established in [10, Theorems 1.2 and 1.3], for all alphabets $\{0, 1, \dots, r\}$ and all real bases $q > 1$, where the dimension is determined by the topological entropy. An advantage of our presentation is that it provides an independent, elementary derivation that does not rely on the theory of topological entropy.

4. The set of numbers having a continuum of representations

Theorem 4.1. *A number x_0 for which some r_s-representation $\Delta^{r_s}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ has no period (0) or (r), but contains infinitely many digits from the set*

$$A_r \backslash N^{s-1}_{r-s+1} = \{0, 1, 2, \dots, r-s,\ s, s+1, \dots, r-1, r\},$$

admits a continuum of distinct r_s-representations.

Proof. The conditions of the theorem allow for the following cases:

(1) The number contains infinitely many zeros.
(2) The number contains infinitely many digits equal to r.
(3) The number contains only finitely many digits equal to 0 or r.

We consider these cases separately.

Case 1. Since the given r_s-representation of the number x_0 does not have the period (0), the sequence (α_k) contains infinitely many pairs $(a, 0)$ with $a \neq 0$ occurring as consecutive digits in the r_s-representation of x_0. However, each such pair admits an alternative replacement $(a-1, s)$. The existence of infinitely many such alternative replacements implies that the set of distinct r_s-representations of the number x_0 has the cardinality of the continuum.

Case 2. An analogous situation occurs in the second case. Since the r_s-representation of the number x_0 does not have the period (r), there exist infinitely many pairs (a, r) with $a \neq r$ occurring as consecutive digits in the r_s-representation of x_0. Since this pair admits an alternative replacement, namely the pair $(a+1, r-s)$, we again obtain a continuum of distinct r_s-representations of the number x_0.

Case 3. Suppose that the given r_s-representation of the number x_0 contains only finitely many digits equal to 0 and r. Then there exists a digit c belonging to one of the sets $\{1, 2, \ldots, r-s\}$ or $\{s, s+1, \ldots, r-1, r\}$ that occurs infinitely many times in the r_s-representation.

If the r_s-representation of x_0 is periodic with period (c), where $c \neq 0$ and $c \neq r$, then, by Lemma 2.2, the number x_0 has a continuum of distinct r_s-representations. On the other hand, if the r_s-representation of x_0 is not periodic, then the sequence (α_k) contains infinitely many pairs (b, c) of consecutive digits such that $c \neq b \neq 0$ and $b \neq r$.

If c belongs to the set $\{1, 2, \ldots, r-s\}$, then the pair (b, c) admits the alternative replacement $(b-1, c+s)$. If c belongs to $\{s, s+1, \ldots, r-1\}$, then the alternative replacement of (b, c) is $(b+1, c-s)$. In both cases, the existence of infinitely many alternative replacements implies that the set of distinct r_s-representations of the number x_0 is a continuum. □

Theorem 4.2. *Almost all numbers with respect to Lebesgue measure on the interval $(0, \frac{r}{s-1})$ have a continuum of distinct r_s-representations; that is, the set of such numbers has a Lebesgue measure equal to $\frac{r}{s-1}$.*

Proof. Let B denote the set of all numbers for which at least one r_s-representation does not satisfy the conditions of the preceding theorem; that is, among the digits of such an r_s-representation, only finitely many belong to the set $A_r \backslash N_{r-s+1}^{s-1}$, while all the remaining digits belong to the set

$$\{r-s+1, r-s+2, \ldots, s-1\}.$$

Denote by B_n the set of all numbers whose r_s-representations contain no digits from the set $A_r \backslash N_{r-s+1}^{s-1}$ starting from the n-th position, where $n \in \mathbb{N}$.

Consider the set C of numbers whose r_s-representation uses only digits from the set $\{r-s+1, r-s+2, \ldots, s-1\}$ and, therefore, does not use the digit 0 or redundant digits. It is known that in the classical base-s representation this set is a Cantor-type set, i.e., a nowhere dense set of Lebesgue measure zero.

The set

$$D_n = \left\{ x : x = \sum_{k=1}^{n} \frac{\alpha_k}{s^k}, (\alpha_1, \alpha_2, \ldots, \alpha_n) \in A_r^n \right\}$$

is finite.

The set B_n is the union of a finite number of sets congruent (isometric) to C, since B_n is the arithmetic sum of the sets D_n and $\frac{1}{s^n}C$. Therefore, it has Lebesgue measure zero.

Obviously,

$$B_1 \subset B_2 \subset \cdots \subset B_n \subset B_{n+1} \subset \ldots$$

and

$$B = \bigcup_{n=1}^{\infty} B_n = \lim_{n\to\infty} B_n.$$

Hence, for the Lebesgue measure we have

$$\lambda(B) = \lambda\big(\bigcup_{n=1}^{\infty} B_n\big) \leqslant \sum_{n=1}^{\infty} \lambda(B_n) = 0.$$

Therefore, the set of numbers whose r_s-representations satisfy the conditions of the theorem has full Lebesgue measure. □

Remark 4.3. This theorem parallels the theorem of N. Sidorov [21], according to which almost every number (in the sense of Lebesgue measure) has a continuum of representations with the alphabet $\{0, 1\}$ for bases $\beta \in (1, 2)$ (β-expansions).

Corollary 4.4. *The set of numbers with finitely or countably many distinct r_s-representations is of Lebesgue measure zero.*

Remark 4.5. The property for a number $x_0 \in [0, \frac{r}{s-1}]$ to have a continuum of distinct representations is typical.

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Mykola Pratsiovytyi
Institute of Mathematics of NAS of Ukraine, Dragomanov Ukrainian State University
Email: prats4444@gmail.com
ORCID: 0000-0001-6130-9413

Oleh Vynnyshyn
Institute of Mathematics of NAS of Ukraine
Email: oleh.vynnyshyn@imath.kiev.ua
ORCID: 0009-0006-7113-1501