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From Maxwell's equations to relativistic Schrödinger equations via Schwartz Linear Algebra and Killing frames on the two-sphere

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Abstract. In this work, we develop a rigorous framework partially unifying classical electromagnetism and relativistic quantum mechanics through the language of Schwartz Linear Algebra and via continuous symmetries of the 2-sphere. Building upon our recent analysis of Killing vector fields on the two-sphere and the associated orthonormal right-handed frames, we construct explicit correspondences between two fundamental distribution spaces:

$$V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}), \qquad W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3),$$

where $\mathbb{M}_4$ is Minkowski space-time. The space V hosts scalar tempered wave distributions (the de Broglie basis of relativistic quantum mechanics), while W hosts vector-valued tempered fields with natural Maxwellian structure. For each unit vector $e \in \mathbb{S}^2$, we introduce a canonical orthonormal tangent frame f_e arising from the Killing field K_e on the sphere (canonically associated with e) and we define the associated families of de Broglie waves η^e and Maxwell-de Broglie fields w^e. By these families we construct two continuous and Schwartz-linear operators:

$$\begin{aligned} \mathcal{F}_e &\colon V_e \to W_e, & \qquad \Psi_e &\colon W_e \to V_e \\ \psi &\mapsto \int_{\mathbb{M}_4^*} (\psi)_\eta w^e, & \qquad F &\mapsto \int_{\mathbb{M}_4^*} (F)_{w^e} \eta^e \end{aligned}$$

which are shown to be inverse isomorphisms of each other. The operator $\mathcal{F}_e$ transforms a wave distribution ψ into a Maxwellian field F_ψ, while the projection Ψ_e recovers the wave distribution from an electromagnetic-like field. We prove that these isomorphisms are dynamically compatible: a scalar distribution $\psi \in V_e$ solves the relativistic Schrödinger equation

$$i\hbar\partial_t\psi = \hat{H}_{m_0}\psi,$$

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if and only if its Maxwellian counterpart $F_\psi = \mathcal{F}_e(\psi)$ solves the corresponding generalized Maxwell-Schrödinger equation in W_e:

$$i\hbar\partial_t F_\psi = \widehat{H}^e_{m_0} F_\psi.$$

This duality holds in both the massless (photonic) case — where $\widehat{H}^e_0$ reduces to the curl operator multiplied times the Planck's constant and the speed of light — and in the massive case — where $\widehat{H}^e_{m_0}$ is the spectral principal square root of the Schwartz diagonalizable and (strictly) positive operator

$$c^2(\hbar\nabla\times)^2 + (m_0c^2)^2\mathbb{I}_{W^e}.$$

The construction provides a precise mathematical mechanism to pass from Maxwell theory to the relativistic Schrödinger picture, mediated by Killing frames on the two-sphere. Beyond its intrinsic geometric interest, this approach suggests new perspectives in the analysis of wave propagation, polarization, and spectral synthesis, as well as possible applications to inverse problems in electromagnetism and the inclusion of curvature from general relativity.

Анотація. В роботі розроблено строгий математичний апарат, що частково об'єднує класичну електродинаміку та релятивістську квантову механіку на мові лінійної алгебри Шварца та за допомогою неперервних симетрій 2-сфери. Спираючись на недавній аналіз автора векторних полів Кіллінга на двовимірній сфері та пов'язаних з ними ортонормованих правих реперів, побудовано явні відповідності між двома фундаментальними просторами розподілів (узагальнених функцій):

$$V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}), \qquad W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3),$$

де $\mathbb{M}_4$ – простір-час Мінковського. Простір V містить скалярні хвильові розподіли повільного росту (базис де Бройля релятивістської квантової механіки), тоді як W містить векторнозначні поля повільного росту з природною максвеллівською структурою. Для кожного одиничного вектора $e \in \mathbb{S}^2$ введено канонічний ортонормований дотичний репер f_e, що виникає з поля Кіллінга K_e на сфері (канонічно пов'язаного з e), і визначено відповідні сім'ї хвиль де Бройля η^e та полів Максвелла-де Бройля w^e. За допомогою цих сімей побудовано два неперервні та лінійні за Шварцом оператори:

$$\mathcal{F}_e \colon V_e \to W_e, \qquad\qquad \Psi_e \colon W_e \to V_e$$
$$\psi \mapsto \int_{\mathbb{M}_4^*} (\psi)_\eta w^e, \qquad\qquad F \mapsto \int_{\mathbb{M}_4^*} (F)_{w^e}\eta^e$$

які, як доведено, є взаємно оберненими ізоморфізмами.Оператор $\mathcal{F}_e$ перетворює хвильовий розподіл ψ на максвеллівське поле F_ψ, тоді як проекція Ψ_e відновлює хвильовий розподіл з електромагнітоподібного поля. Доведено, що ці ізоморфізми є динамічно сумісними:

скалярний розподіл $\psi \in V_e$ є розв'язком релятивістського рівняння Шредінгера

$$i\hbar\partial_t\psi = \hat{H}_{m_0}\psi,$$

тоді й лише тоді, коли його максвеллівський відповідник $F_\psi = \mathcal{F}_e(\psi)$ є розв'язком відповідного узагальненого рівняння Максвелла-Шредінгера в W_e:

$$\hbar\partial_t F_\psi = \widehat{H}^e_{m_0} F_\psi.$$

Ця двоїстість зберігається як у безмасовому (фотонному) випадку – де $\widehat{H}^e_0$ зводиться до оператора ротора (curl), помноженого на сталу Планка та швидкість світла, – так і в випадку ненульової маси – де $\widehat{H}^e_{m_0}$ є спектральним головним квадратним коренем з діагоналізовного за Шварцом і (строго) додатного оператора

$$c^2(\hbar\nabla\times)^2 + (m_0c^2)^2\mathbb{I}_{W^e}.$$

Ця побудова надає точний математичний механізм для переходу від теорії Максвелла до релятивістської картини Шредінгера за допомогою реперів Кіллінга на двовимірній сфері. Окрім свого внутрішнього геометричного інтересу, цей підхід пропонує нові перспективи в аналізі поширення хвиль, поляризації та спектрального синтезу, а також можливі застосування до обернених задач в електродинаміці та врахування кривини із загальної теорії відносності.

1. Introduction

The unification of quantum mechanics and classical field theory has long been a central theme in both mathematics and physics. On the one hand, Maxwell's equations provide the fundamental description of electromagnetic radiation, with solutions that include plane waves, polarization states, and geometric symmetries. On the other hand, the relativistic Schrödinger equation governs the evolution of scalar quantum states, encoding the dynamics of both massless and massive particles.

Traditionally these two pictures are treated separately: Maxwell's theory as a real vectorial field theory, Schrödinger's equation as a complex scalar wave mechanics.

In this article we propose a unifying perspective: both can be realized within the rigorous framework of complex Schwartz Linear Algebra, and, more importantly, they are related by explicit complex topological isomorphisms.

The correspondence we develop allows us to *project* Maxwellian fields onto Schrödinger wave distributions and to *embed* Schrödinger distributions into Maxwell-type fields, in such a way that **the two dynamical pictures become equivalent**.

1.1. **Geometric origin: Killing vector fields on the two-sphere.** The geometric foundation of our construction lies in the classical symmetries of the two-sphere $\mathbb{S}^2$. The rotation group $SO(3)$ acts isometrically on $\mathbb{S}^2$, with infinitesimal generators given by Killing vector fields.

In [10], we studied the isomorphism

$$K: \mathbb{R}^3 \to \mathcal{K}(\mathbb{S}^2), \qquad v \mapsto K_v : q \in \mathbb{S}^2 \mapsto v \times q,$$

which identifies each vector in $\mathbb{R}^3$ with a Killing field on $\mathbb{S}^2$.

Fixing a point $e \in \mathbb{S}^2$, we construct a canonical orthonormal tangent frame

$$f_e(u) = \big(\widehat{K}_e(u), u \times \widehat{K}_e(u)\big), \qquad u \in \mathbb{S}^2 \backslash \{\pm e\},$$

which is smooth, orthonormal, and right-handed. We proved that every smooth orthonormal right-handed frame on the pierced sphere $\mathbb{S}^2\backslash\{\pm e\}$ is necessarily obtained from f_e by a smooth rotation field

$$\rho: \mathbb{S}^2\backslash\{\pm e\} \to SO(3).$$

Thus the Killing frame f_e is, in a precise sense, canonical. By homogeneity, this frame extends to the dual of Minkowski space-time $\mathbb{M}_4^*$, giving a polarization structure valid for all wavevectors.

1.2. **From geometry to electromagnetic solutions.** Canonical frames f_e allow us to associate with each wavevector k a pair of orthogonal polarization vectors

$$f_e(k) = (r_e(k), s_e(k)),$$

tangent to the unit sphere at $\vec{k}/|\vec{k}|$. Combining this geometric frame with de Broglie plane waves

$$\eta_k = e^{i\langle k,x\rangle}, \qquad \langle k, x\rangle = \vec{k} \cdot \vec{x} - k^0 ct,$$

where k is any four wave-vector and x is the identity coordinate of the vector space-time $\mathbb{M}_4$, we obtain the Maxwell-de Broglie fields

$$w_k^e = e^{i\langle k,x\rangle}\big(r_e(k) + i s_e(k)\big).$$

In [10] we showed that, for light-like k, these fields satisfy the Maxwell equations in free space, representing circularly polarized plane-wave solutions. Varying the rest mass parameter m_0, the same construction yields generalized Maxwellian solutions with dispersion relations consistent with Einstein's formula

$$E^2 = |\vec{p}|^2 c^2 + m_0^2 c^4.$$

Thus, from the geometry of the two-sphere, one derives a complete family of plane-wave solutions for both electromagnetic radiation and its massive generalizations.

1.3. **Schwartz linear algebra and distribution spaces.** The analytic framework for these constructions is *Schwartz Linear Algebra*, developed in [7]. This theory provides a rigorous language for working with bases and superpositions in distribution spaces.

In our context, the relevant spaces are:

$$V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}), \qquad W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3),$$

where V hosts scalar tempered wave distributions and W hosts vector-valued tempered fields.

The ordered families

$$\eta^e = (\eta_k)_{k\in\mathbb{M}_4^*\setminus\Pi_e}, \qquad \Pi_e = \mathrm{span}(e_0, e), \qquad w^e = (w_k^e)_{k\in\mathbb{M}_4^*\setminus\Pi_e}$$

form Schwartz bases of two large subspaces $V_e \subset V$ and $W_e \subset W$, respectively.

Every element of V_e is a superposition of de Broglie waves with tempered coefficients, while every element of W_e is an analogous superposition of Maxwell-de Broglie fields.

1.4. **The new correspondence: projection and embedding.** The central contribution of the present work is the construction of explicit inverse isomorphisms between V_e and W_e. For each $e \in \mathbb{S}^2$ we define:

$$\mathcal{F}_e \colon V_e \to W_e, \qquad \Psi_e \colon W_e \to V_e,$$

where $\mathcal{F}_e$ is the *embedding* that maps a wave distribution ψ into the corresponding Maxwell field F_ψ, and Ψ_e is the *projection* that recovers ψ from F_ψ. We prove that

$$\Psi_e \circ \mathcal{F}_e = \mathbb{I}_{V_e}, \qquad \mathcal{F}_e \circ \Psi_e = \mathbb{I}_{W_e}.$$

Thus the spaces V_e and W_e are not merely analogous, but actually isomorphic under Schwartz-linear maps.

1.5. **Compatibility with dynamics.** The power of this construction lies in its dynamical compatibility. In the scalar space V, the relativistic Schrödinger equation reads

$$i\hbar\partial_t\psi = \hat{H}_{m_0}\psi, \qquad \hat{H}_{m_0} = \sqrt{c^2|\vec{P}|^2 + (m_0c^2)^2\mathbb{I}_V}.$$

In the vectorial space W, we define the generalized Maxwell-Schrödinger equation

$$i\hbar\partial_t F = \widehat{H}_{m_0}F, \qquad \widehat{H}_{m_0} = \sqrt{c^2(\hbar\nabla\times)^2 + (m_0c^2)^2\mathbb{I}_W}.$$

On the basis fields w_k^e, both operators act spectrally with identical eigenvalues. It follows that: *ψ solves the m_0-Schrödinger equation in V_e iff $F_\psi = \mathcal{F}_e(\psi)$ solves the m_0-Maxwell–Schrödinger equation in W_e.* This result establishes a precise duality: Maxwell's theory and Schrödinger's picture are two sides of the same coin, connected by the isomorphisms $\mathcal{F}_e$ and Ψ_e.

1.6. **Perspective and applications.** The framework we develop brings together geometry (Killing fields, frame bundles), analysis (Schwartz distributions, spectral operators), and physics (Maxwell and Schrödinger dynamics). It provides a mathematically rigorous mechanism to translate between scalar quantum states and electromagnetic-type vector fields. Beyond its foundational importance, this perspective opens new directions for applications:

- In *optics and photonics*, polarization states can be analyzed directly through Killing frames.
- In *inverse problems*, symmetry properties of frames on $\mathbb{S}^2$ may enhance reconstruction of electromagnetic fields.
- In *quantum field theory*, the identification of state spaces suggests new models for field quantization.
- In *general relativity*, the construction remains valid under metric deformations g, yielding a gravitationally-aware Maxwell-Schrödinger operator.

In summary, this article provides a comprehensive passage from Maxwell's equations to the relativistic Schrödinger equation, via Schwartz Linear Algebra and Killing frames on the two-sphere. The key novelty lies in the explicit isomorphisms $\mathcal{F}_e$ and Ψ_e, which demonstrate that the two theories are mathematically equivalent on vast distribution subspaces, and dynamically interchangeable.

2. Background and preliminaries

2.1. **Minkowski space-time and its dual.** We denote by

$$\mathbb{M}_4 = \mathbb{R} \times \mathbb{R}^3$$

the flat Minkowski space-time, with canonical coordinates $x = (ct, \vec{x})$ and dual coordinates $k = (k^0, \vec{k}) \in \mathbb{M}_4^*$. The Minkowski bilinear form is written as

$$\langle k, x \rangle = \vec{k} \cdot \vec{x} - k^0 ct.$$

This pairing governs the spectral decomposition of tempered distributions on $\mathbb{M}_4$ via the Fourier transform. The light cone

$$\langle k, k \rangle = 0$$

corresponds to massless excitations (photons), while the hyperboloid

$$\langle k, k \rangle = -\frac{m_0^2 c^2}{\hbar^2}$$

corresponds to a massive particle of rest mass m_0.

2.2. **Schwartz distribution spaces.** Let $\mathcal{S}(\mathbb{M}_4, \mathbb{C})$ denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{M}_4$, and let

$$V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$$

denote its dual, the space of complex tempered distributions. Elements $\psi \in V$ are interpreted as scalar wave distributions, forming the natural habitat for relativistic Schrödinger dynamics.

Analogously, let

$$W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3),$$

the space of tempered $\mathbb{C}^3$-valued distributions. This is the natural space for Maxwell-type fields $\vec{F} = \vec{E} + ic\vec{B}$, where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively.

The essential philosophy of *Schwartz Linear Algebra* [7] is that these distribution spaces admit continuous bases, called $\mathcal{S}$-bases, with respect to the fundamental operation of Schwartz superposition.

Pettis–Bourbaki integrals allow us to rigorously perform continuous superpositions of the form

$$u = \int_I \mu v : \phi \mapsto \mu(\langle v, \phi \rangle),$$

with coefficient systems given by tempered measures

$$\mu \colon C_c(I) \to \mathbb{C}$$

and superposing families

$$v \colon I \to \mathcal{S}'$$

(of distributions v_i indexed by a continuous set I) which are scalarly of class C_c.

On the other hand, Schwartz superpositions extend the coefficient system domain to the entire tempered distribution spaces (based upon any finite dimensional Euclidean space $\mathbb{R}^m$), restricting the vector families $v \colon \mathbb{R}^m \to \mathcal{S}'$ to the so-called Schwartz families, which are distribution-valued Schwartz functions (in the weak scalarly sense), defined on finite dimensional Euclidean spaces.

In this language, the de Broglie plane waves and the Maxwell-de Broglie fields constitute $\mathcal{S}$-bases of vast subspaces of V and W, respectively.

2.3. **The de Broglie basis.** For each $k \in \mathbb{M}_4^*$ we define the plane wave

$$\eta_k = e^{i\langle k, \cdot \rangle}.$$

These functions constitute the *de Broglie basis* $\eta = (\eta_k)_{k \in \mathbb{M}_4^*}$ of the space V. More precisely, every tempered distribution $\psi \in V$ admits a representation

$$\psi = \int_{\mathbb{M}_4^*} (\psi)_\eta \eta,$$

where $(\psi)_\eta$ is a tempered coefficient distribution in momentum space. This Fourier-Schwartz representation is the analytic backbone of our framework.

2.4. **Killing vector fields on the sphere.** Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere. For each vector $v \in \mathbb{R}^3$, define a Killing vector field K_v acting as follows

$$K_v(q) = v \times q, \quad q \in \mathbb{S}^2.$$

The map $v \mapsto K_v$ is a linear isomorphism

$$\mathbb{R}^3 \cong \mathcal{K}(\mathbb{S}^2),$$

where $\mathcal{K}(\mathbb{S}^2)$ is the Lie algebra of Killing fields of $\mathbb{S}^2$, isomorphic to $\mathfrak{so}(3)$. Thus the geometry of $\mathbb{S}^2$ encodes directly the algebra of rotations.

Fixing a point $e \in \mathbb{S}^2$, one defines the unit Killing field

$$\widehat{K}_e(u) = \frac{e \times u}{|e \times u|}, \quad u \in \mathbb{S}^2 \backslash \{\pm e\},$$

and the associated *Killing frame*

$$f_e(u) = \big(\widehat{K}_e(u), u \times \widehat{K}_e(u)\big).$$

In [10] we proved that:

- The vectors in $f_e(u)$ form a smooth orthonormal basis of $T_u\mathbb{S}^2$ for all $u \neq \pm e$.
- The frame is right-handed: $\widehat{K}_e(u) \times (u \times \widehat{K}_e(u)) = u$.
- Every smooth orthonormal right-handed sphere-tangent frame on the pierced sphere $\mathbb{S}^2 \backslash \{\pm e\}$ is obtained from f_e by a smooth rotation field ρ, with $\rho(u) \in SO(3)$.

Thus, up to rotations, f_e is the canonical orthonormal frame on the pierced sphere.

2.5. **Homogeneous extension to Minkowski dual space.** The Killing frame f_e can be extended homogeneously to the dual Minkowski space:

$$\tilde{f}_e(k) = f_e\left(\frac{\vec{k}}{|\vec{k}|}\right), \quad k \in \mathbb{M}_4^* \backslash \Pi_e,$$

where Π_e is the singular plane spanned by $(1,0,0,0)$ and $(0,e)$. For each k, this provides a polarization pair

$$(r_e(k), s_e(k)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

orthonormal and tangent to the sphere at $\vec{k}/|\vec{k}|$.

2.6. **Maxwell-de Broglie basis.** By using this extension, we can define the family

$$w_k^e = e^{i\langle k,x\rangle}(r_e(k) + is_e(k)), \quad k \in \mathbb{M}_4^* \backslash \Pi_e.$$

These fields w_k^e are circularly polarized plane waves oriented according to the Killing frame f_e. For light-like k they solve Maxwell's equations in free space:

$$\nabla \times w_k^e = |\vec{k}| w_k^e, \qquad i\partial_t w_k^e = c|\vec{k}| w_k^e.$$

The family $w^e = (w_k^e)$ forms a Schwartz basis of a subspace $W_e \subset W$; we call w^e the *Maxwell–de Broglie basis of north pole e*.

2.7. **Summary.** We have introduced:

- The scalar space $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ with de Broglie basis η.
- The vector space $W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$ with basis $\eta \otimes \mathbf{e}$;
- The subspaces $W_e = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)_e$ with bases w^e (a framed subspace (W_e, w^e) for any $e \in \mathbb{S}^2$).
- The canonical Killing frames f_e that determine the polarization structure of w_k^e.

The stage is thus set for the construction of the isomorphisms

$$\mathcal{F}_e \colon V_e \to W_e, \qquad \Psi_e \colon W_e \to V_e,$$

which will be the subject of the next section.

3. The embedding and the projection: isomorphisms between V_e and W_e

We now construct the operators

$$\mathcal{F}_e \colon V_e \to W_e, \qquad \Psi_e \colon W_e \to V_e,$$

and prove that they are continuous Schwartz-linear isomorphisms inverse to one another. This establishes a precise correspondence between scalar distributional waves and Maxwellian vector fields, with respect to a prechosen pole $e \in \mathbb{S}^2$.

3.1. **Spectral synthesis in the scalar space V_e.** Let

$$\eta^e = (\eta_k^e)_{k \in \mathbb{M}_4^* \backslash \Pi_e} = (\eta_k)_{k \in \mathbb{M}_4^* \backslash \Pi_e}$$

denote the restriction of the de Broglie basis η to the domain of f_e, that is, a subbasis of η adapted to the frame f_e. Every scalar wave distribution $\psi \in V_e :=^{\mathcal{S}} \operatorname{span}(\eta^e)$ can be represented, by definition, as

$$\psi = \int_{\mathbb{M}_4^* \backslash \Pi_e} (\psi)_{\eta^e} \eta^e,$$

where $(\psi)_{\eta^e}$ is a tempered coefficient distribution in momentum space which vanishes — by definition — about the plane Π_e. Therefore, V_e is the subspace of all tempered distributions whose representation in the whole of the basis η vanishes in a neighborhood of the above plane, by our definition of the superposition above.

This representation is unique, since η is a basis of the entire space V and, consequently, η^e is a Schwartz basis of V_e.

In other terms, we define V_e as the subspace of all ψ whose momentum representation ψ_η vanishes about the plane Π_e; then, we put

$$\int_{\mathbb{M}_4^* \backslash \Pi_e} (\psi)_{\eta^e} \eta^e := \int_{\mathbb{M}_4^*} (\psi)_\eta \eta, \; (\psi)_{\eta^e} := (\psi)_\eta,$$

for every $\psi \in V_e$.

3.2. **Spectral synthesis in the vector space W_e.** Analogously, the Maxwell–de Broglie family

$$w^e = (w_k^e)_{k \in \mathbb{M}_4^* \backslash \Pi_e}, \qquad w_k^e = e^{i\langle k, \cdot \rangle}(r_e(k) + i s_e(k)),$$

forms a Schwartz basis of the subspace $W_e \subset W$. Thus every $F \in W_e$ has a unique representation

$$F = \int_{\mathbb{M}_4^* \backslash \Pi_e} (F)_w w^e,$$

with tempered coefficient distribution $(F)_w$.

The definition of superposition in this case is different from that of η^e, in which we simply recover the local superpositions from the global superpositions of η. The family w^e is not the subfamily of a wider Schwartz basis of the entire tempered distribution field space, so we cannot define the superpositions of w^e by means of the superpositions of any Schwartz family defined upon a complete Euclidean Space.

We define any local superposition

$$\int\limits_{\mathbb{M}_4^*\backslash\Pi_e} a w^e,$$

where $a \in S'(\mathbb{M}_4^*)$ is an everywhere defined tempered coefficient distribution vanishing in an open neighborhood U_a of the plane Π_e, by means of the global superposition

$$\int\limits_{\mathbb{M}_4^*\backslash\Pi_e} a w^e := \int_{\mathbb{M}_4^*} a\tilde{w}^e,$$

where $\tilde{w}^e$ is a Schwartz family obtained by continuously extending any convenient local family $\chi \cdot w^e$, product of a smooth Urysohn's mapping separating any convenient pair

$$(\bar{U}, \mathbb{M}_4^*\backslash U_a).$$

In detail, the everywhere defined family $\tilde{w}^e$ is obtained by:

(1) multiplying w^e by a convenient multiplier function χ, specifically, a smooth function such that:
   - χ is equal to 1 in the complement of the open neighborhood U_a;
   - χ is equal to 0 on the whole of a smaller open neighborhood U of Π_e, whose closure is contained in U_a (and thus χ equals 0 on Π_e);

(2) extending $\chi \cdot w^e$, continuously, to the entire dual of the Minkowski spacetime.

3.3. **Definition of the embedding $\mathcal{F}_e$.** We define the embedding operator $\mathcal{F}_e \colon V_e \to W_e$ by the rule

$$\mathcal{F}_e(\eta_k^e) := w_k^e,$$

extended linearly and continuously to the whole space V_e by spectral synthesis. Explicitly, for $\psi \in V_e$ with Fourier coefficients $(\psi)_\eta$, we have

$$\mathcal{F}_e(\psi) = \int\limits_{\mathbb{M}_4^*\backslash\Pi_e} (\psi)_\eta w^e.$$

Thus $\mathcal{F}_e$ maps the scalar wave ψ to its electromagnetic-type counterpart F_ψ.

3.4. **Definition of the projection Ψ_e.** Dually, we define the projection operator $\Psi_e\colon W_e \to V_e$ by linearly (and continuously) extending

$$\Psi_e(w_k^e) := \eta_k^e.$$

Hence, for $F \in W_e$ with coefficients $(F)_{w^e}$, we have

$$\Psi_e(F) = \int_{\mathbb{M}_4^* \backslash \Pi_e} (F)_{w^e} \eta^e.$$

This recovers a scalar distribution from its Maxwellian expansion.

3.5. **Isomorphism theorem.**

Theorem 3.5.1. *For every $e \in \mathbb{S}^2$, the operators $\mathcal{F}_e$ and Ψ_e are continuous Schwartz-linear isomorphisms inverse to one another:*

$$\Psi_e \circ \mathcal{F}_e = \mathbb{I}_{V_e}, \qquad \mathcal{F}_e \circ \Psi_e = \mathbb{I}_{W_e}.$$

Proof. By construction, $\mathcal{F}_e$ maps the basis element η_k^e to w_k^e, and Ψ_e maps w_k^e back to η_k^e. Thus, for all $k \in \mathbb{M}_4^* \backslash \Pi_e$,

$$\begin{aligned}\Psi_e(\mathcal{F}_e(\eta_k^e)) &= \Psi_e(w_k^e)\\ &= \eta_k^e,\end{aligned} \quad \text{and similarly} \quad \begin{aligned}\mathcal{F}_e(\Psi_e(w_k^e)) &= \mathcal{F}_e(\eta_k^e)\\ &= w_k^e.\end{aligned}$$

Since both operators are Schwartz-linear, the equalities extend by linearity and continuity to all superpositions, i.e. to all $\psi \in V_e$ and $F \in W_e$. Therefore

$$\Psi_e \circ \mathcal{F}_e = \mathbb{I}_{V_e} \qquad \text{and } \mathcal{F}_e \circ \Psi_e = \mathbb{I}_{W_e}.$$ □

3.6. **Interpretation.** The operators $\mathcal{F}_e$ and Ψ_e formalize the passage:
- from a scalar wave distribution ψ in the Schrödinger picture,
- to an electromagnetic-type field F_ψ in the Maxwell picture,
- and back, with no loss of information.

This construction shows that the two spaces V_e and W_e are mathematically equivalent, once the polarization structure determined by the Killing frame f_e has been fixed.

In the next section we will prove that this equivalence is compatible with the respective dynamics: Schrödinger in V_e, Maxwell–Schrödinger in W_e.

4. Dynamics compatibility: Maxwell–Schrödinger equivalence

In this section we prove that the isomorphisms

$$\mathcal{F}_e\colon V_e \to W_e, \qquad \Psi_e\colon W_e \to V_e,$$

intertwine the relativistic Schrödinger evolution in V_e with the Maxwell evolution in W_e, both in the massless and in the massive cases.

4.1. **Operators on V and W.** We work on tempered distributions throughout. On the scalar space $V = \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ we consider the standard quantum observables

$$\hat{E} := i\hbar\partial_t, \qquad \vec{P} := -i\hbar\nabla, \qquad |\vec{P}| := \hbar|\nabla|,$$

so that, on de Broglie waves $\eta_k = e^{i\langle k,\cdot\rangle}$,

$$\hat{E}\eta_k = c(\hbar k^0)\eta_k, \qquad \vec{P}\eta_k = (\hbar\vec{k})\eta_k, \qquad |\vec{P}|\eta_k = (\hbar|\vec{k}|)\eta_k.$$

On the vector space $W = \mathcal{S}'(\mathbb{M}_4, \mathbb{C}^3)$ we use

$$\begin{aligned} \widehat{E} &:= i\hbar\partial_t \text{ (componentwise)}, \\ \mathrm{curl} &:= \nabla\times \text{ (acting on the spatial components)}, \end{aligned}$$

and the spectral square-root Hamiltonians introduced below.

On the Maxwell-de Broglie basis defined by

$$w_k^e = e^{i\langle k,\cdot\rangle}\big(r_e(k) + is_e(k)\big),$$

with (r_e, s_e) sphere-tangent, right-handed and orthonormal, we see the key eigen-relations

$$\mathrm{curl}\, w_k^e = |\vec{k}|w_k^e, \qquad \widehat{E}w_k^e = c(\hbar k^0)w_k^e, \qquad \text{and } \nabla\cdot w_k^e = 0,$$

which is the fundamental first Maxwell condition: transversality (divergence-free condition).

4.2. **Dynamical generators.**

4.2.1. *Massless quantum-particles.* Consider the massless relativistic Schrödinger equation in V_e:

$$i\hbar\partial_t\psi = c|\vec{P}|\psi.$$

In this case, the relativistic Hamiltonian operator is

$$\hat{H}_0 := c|\vec{P}|$$

with dispersion relation

$$E = c|\vec{p}|.$$

On W_e, we know the Maxwell-Schrödinger equation

$$i\hbar\partial_t F = c\hbar\,\mathrm{curl}\, F,$$

whose spectral action (eigenvalue) on the electromagnetic field w_k^e is $\hbar c|\vec{k}|$.

4.2.2. *Massive spin-0 quantum field.* On V_e the relativistic Hamiltonian is the spectrally defined operator

$$\hat{H}_{m_0} := \sqrt{c^2|\vec{P}|^2 + (m_0c^2)^2\mathbb{I}_V},$$

so that

$$i\hbar\partial_t\psi = \hat{H}_{m_0}\psi.$$

On W_e, we define spectrally the Hamiltonian operator

$$\widehat{H}^e_{m_0} \;:=\; \sqrt{c^2(\hbar\,\mathrm{curl})^2 + (m_0c^2)^2\mathbb{I}_{W_e}},$$

generating the equation

$$i\hbar\partial_t F = \widehat{H}^e_{m_0}F.$$

All the above operators and equations are Schwartz-linear; the operators are kind of von Neumann (Schwartz) linear extensions, via spectral calculus (with multipliers and coefficients in the same spaces). It is also possible to prove, with a little work, that the Schrödinger equation is the corresponding von Neumann type Schwartz-linear extension of the equation governing the dynamics in the finite manifold η of momentum "certainty", which is essentially a Hamilton–Jacobi equation.

4.3. **Curl-momentum magnitude spectral coincidence on W_e.**

Lemma 4.3.1 (Spectral identification on W_e)**.** *On the subspace*

$$W_e = {}^{\mathcal{S}}\,\mathrm{span}(w^e)$$

we have, on basis w^e,

$$(\hbar\,\mathrm{curl})w^e_k = \hbar|\vec{k}|w^e_k, \qquad\qquad \widehat{H}^e_{m_0}w^e_k = H_{m_0}(\hbar|\vec{k}|)w^e_k;$$

hence $\widehat{H}^e_{m_0}$ agrees spectrally with $\hat{H}_{m_0}$ under the identification

$$|\vec{P}_W| = \hbar\,\mathrm{curl}$$

on the subspace W_e.

Proof. The first identity is the defining eigen-relation of w^e_k. For the second, apply functional calculus to the eigenvalue $\hbar|\vec{k}|$ of $\hbar\,\mathrm{curl}$:

$$\begin{aligned}\widehat{H}^e_{m_0}w^e_k &= \sqrt{c^2(\hbar|\vec{k}|)^2 + (m_0c^2)^2}\;w^e_k\\ &= H_{m_0}(\hbar|\vec{k}|)w^e_k\\ &= H_{m_0}(\vec{P}_W)w^e_k\\ &= H^W_{m_0}w^e_k,\end{aligned}$$

as we claimed. □

Remark. *It is important to note that the rightful spatial momentum operator on W, the vector differential operator*

$$\vec{P}_W = (-\imath\hbar\partial_j)_{j=1}^3,$$

is defined everywhere on W.

On the other hand, the mod (magnitude) of this vector operator is defined on the subspace of W constituted by those vector fields F whose representations $(F)_b$ — in the global tensor product basis

$$b = \eta \otimes \vec{e} = (\eta_p \otimes \vec{e}_j)_{p\in\mathbb{M}_4^*, j\in\underline{3}},$$

of the de Broglie basis η times the canonical basis $\vec{e}$ of $\mathbb{C}^3$ — vanish in a neighborhood of the straightline $\mathbb{M}_4^*\backslash\operatorname{span}(e_0)$. The operator $|\vec{P}_W|$ is spectrally defined on the tensor product basis $b = \eta \otimes \vec{e}$, of the de Broglie basis times the canonical basis of $\mathbb{C}^3$ by

$$|\vec{P}_W|(\eta_p \otimes \vec{e}_j) = |\vec{p}|(\eta_p \otimes \vec{e}_j),$$

for every non-zero $\vec{p}$; in form of superposition we have:

$$|\vec{P}_W|F = \int_{\mathbb{M}_4^*\times\underline{3}} (F)_{(\eta\otimes\vec{e})}.|\vec{\mathrm{p}}|(\eta\otimes\vec{e}),$$

for every vector field F whose representation $(F)_b$ vanishes at the straightline $\mathbb{M}_4^*\backslash\operatorname{span}(e_0)$.

Indeed, the necessity for w^e is required for the functional calculus involving $\nabla\times$, because b is an eigenbasis of $|P|$ but not of the operator curl: those two operators coincide on every subspace W_e, for every $e \in \mathbb{S}^2$, not everywhere on W.

An analogous remark could be observed for the operators $\hat{H}_{m_0}^W = H_{m_0}(\vec{P})$ and $\hat{H}_{m_0}^e$, which is defined spectrally by w^e: for the Quantum mechanics-like operators on W we need only the much simpler tensorial eigenbasis b, on the other hand, for the Maxwell operators on W, based upon the curl operator, a functional calculus requires the more elaborated Maxwell–Schwartz eigenbases w^e. The union of all W_e is a vector bundle endowed with the bundle-frame $e \mapsto w^e$.

4.4. **Intertwining of observables diagonal on the de Broglie basis.**

Proposition 4.4.1 (Observable compatibility)**.** *Let $G_V = g(|\vec{P}|) \in \mathcal{L}(V)$ be any $\mathcal{O}_M$ function of $|\vec{P}|$ (including $|\vec{P}|$, $\hat{H}_{m_0}$, etc.), and let*

$$G_W = g(\hbar\,\mathrm{curl})$$

be the corresponding operator on W_e, spectrally defined by w^e. Then

$$\mathcal{F}_e \circ G_V = G_W \circ \mathcal{F}_e, \qquad G_V \circ \Psi_e = \Psi_e \circ G_W.$$

Proof. Both equalities hold on basis elements because $\mathcal{F}_e(\eta_k^e) = w_k^e$ and f acts by the same spectral value $f(\hbar|\vec{k}|)$. Extend by linearity and continuity to superpositions. □

4.5. **Dynamical equivalence theorems.**

Theorem 4.5.1 (Massless dynamical equivalence)**.** *For $\psi \in V_e$ set*

$$F_\psi := \mathcal{F}_e(\psi).$$

Then

$$i\hbar\partial_t\psi = c|\vec{P}|\psi \iff i\hbar\partial_t F_\psi = c\hbar\nabla \times F_\psi.$$

Equivalently,

$$\psi(t) = e^{-\frac{i}{\hbar}tc|\vec{P}|}\psi_0 \iff F_\psi(t) = e^{-\frac{i}{\hbar}tc(\hbar\nabla\times)}F_{\psi_0}$$

with $F_{\psi_0} = \mathcal{F}_e(\psi_0)$.

Proof. Use the observable compatibility with $G_V = c|\vec{P}|$ and $G_W = c\hbar\nabla\times$, we obtain

$$\mathcal{F}_e\Big(c|\vec{P}|\psi\Big) = c\hbar\nabla \times \mathcal{F}_e(\psi).$$

Thus $\mathcal{F}_e$ intertwines the generators; hence it intertwines the unitary groups and solutions. □

Theorem 4.5.2 (Massive dynamical equivalence)**.** *For $\psi \in V_e$ set*

$$F_\psi := \mathcal{F}_e(\psi).$$

Then

$$i\hbar\partial_t\psi = \hat{H}_{m_0}\psi \qquad \iff \qquad i\hbar\partial_t F_\psi = \widehat{H}^e_{m_0}F_\psi.$$

Proof. Apply the observable compatibility with

$$G_V = \hat{H}_{m_0} = \sqrt{c^2|\vec{P}|^2 + (m_0c^2)^2\mathbb{I}_V}$$

and

$$G_W = \widehat{H}^e_{m_0} = \sqrt{c^2(\hbar\,\mathrm{curl})^2 + (m_0c^2)^2\mathbb{I}_W}$$

both defined spectrally. The intertwining of generators yields the equivalence of solutions. □

4.6. **Consequences.**

Corollary 4.6.1 (Equivalence of Cauchy problems)**.** *For every initial datum $\psi_0 \in V_e$ there is a unique $F_0 = \mathcal{F}_e(\psi_0) \in W_e$ such that the corresponding Cauchy problems are equivalent under $\mathcal{F}_e$ and Ψ_e in both massless and massive cases.*

Remark (Energy, momentum and polarization)**.** The intertwining shows that spectral data (energy E, momentum magnitude $|\vec{p}|$) coincide in the two pictures. Polarization is carried geometrically by the Killing frame (r_e, s_e); transversality and circular polarization are encoded in the basis w^e, while the scalar phase lies in the basis η^e. The pair (ψ, F_ψ) thus gives a complete and non-redundant dual description, as soon as we fix the unit vector e.

5. Discussion

In this section we collect several refinements, complementary constructions, and broader perspectives. Some of these ideas were introduced in our AGMA 2025 presentation, and they enrich the central results of this paper.

5.1. **On Schwartz-linearity and domains.** It is worth emphasizing that all operators considered here (Fourier synthesis, $\mathcal{F}_e$, Ψ_e, $|\vec{P}|$, $\hbar\nabla\times$, and the spectral square-root Hamiltonians) are *Schwartz-linear and Schwartz diagonalizable*. That is, they are defined by multiplication of tempered Fourier coefficients against smooth and Schwartz distribution-valued functions of momentum, followed by resynthesis (recomposition via superpositions).

In particular, all our constructions act continuously on Schwartz tempered distributions of coefficients vanishing in a neighborhood of the singular plane Π_e. This ensures that no singularities are introduced and that the intertwining theorems hold on vast dense subspaces of V and W.

5.2. **Position-space embeddings.** Besides the momentum-space embeddings $\mathcal{F}_e$, one may also define position-space embeddings I_j using the Dirac basis δ

$$\delta\colon x \mapsto \delta_x \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C}), \quad x \in \mathbb{M}_4.$$

For example, one may construct the embedding

$$I_j\colon V \to W\colon \psi \mapsto \int_{\mathbb{M}_4} \psi(\delta \otimes \vec{f}),$$

where $\vec{f}$ is a complex 3-dimensional smooth frame field of components f_j. These embeddings preserve position eigenstates and demonstrate a duality between frequency- and position-space formulations of the theory. In this way, the distributional formalism admits multiple geometric realizations.

5.3. **Geometric example: Frenet frames along curves.** To illustrate the flexibility of the construction, consider a smooth spatial curve

$$\gamma \colon \mathbb{R} \to \mathbb{R}^3.$$

Let $f \circ \gamma$ be its Frenet frame. Then the field

$$t \mapsto \delta_{\gamma(t)} \cdot f(\gamma(t))$$

is supported along γ and polarized according to its geometry. This gives a localized electromagnetic-like field whose distributional support encodes the trajectory γ itself. Such examples suggest applications in optics, where wavefronts can be encoded with geometric signatures.

5.4. **Delta distributions and photons.** A particularly interesting family of solutions are the delta distributions

$$\psi(t, \vec{x}) = \delta_0(x \mp ct),$$

which solve the massless relativistic Schrödinger equation with spectral support on the positive or negative light cone. They correspond to right-moving and left-moving photons, respectively. Under the embedding $\mathcal{F}_e$, these give rise to electromagnetic pulses F_ψ supported on the corresponding null hyperplanes.

5.5. **Relativistic mass of photons.** In our spectral setting, the relation

$$m = \frac{\hbar|\vec{k}|}{c}$$

emerges naturally, associating to each monochromatic photon mode a relativistic mass depending on its wavevector. This relation is consistent with the spectral identity $E = \hbar c|\vec{k}|$ and provides a bridge between wave content and relativistic mass-energy equivalence.

5.6. **Extension to curved space-time.** The Maxwell–Schrödinger correspondence remains valid under deformations of the Minkowski metric $\langle \cdot, \cdot \rangle$ to a pseudo-Riemannian metric g. In this case, the curl operator is replaced by curl_g, and the Hamiltonian takes the form

$$\widehat{H}^g_{m_0} = \sqrt{c^2(\hbar \ \mathrm{curl}_g)^2 + (m_0 c^2)^2 \mathbb{I}_W} \, .$$

This observation suggests that the framework provides a gravitationally-aware extension of the Maxwell-Schrödinger duality.

5.7. **General remarks.** Several remarks are in order:

- The isomorphisms $\mathcal{F}_e$ and Ψ_e identify two seemingly different worlds: scalar relativistic quantum mechanics and vectorial Maxwell electromagnetism. This equivalence is not approximate but exact on the Schwartz spans V_e and W_e.
- The geometric role of Killing frames is essential: they anchor the polarization structure in a canonical way, linking the algebra of rotations $SO(3)$ to the analysis of wave distributions.
- The framework unifies three fundamental layers:
  (i) the relativistic geometry of $\mathbb{M}_4$ and $\mathbb{S}^2$,
  (ii) the analytic theory of tempered distributions,
  (iii) the physics of electromagnetic and quantum dynamics.
- The construction is functorial: embeddings $J(\beta, f)$ can be built from arbitrary Schwartz bases β and smooth frame fields f, and they preserve the action of all observables diagonal in β. This shows that the passage from scalar to vectorial representations is not an isolated fact, but part of a more general algebraic architecture.

In conclusion, the discussion shows that the Maxwell-Schrödinger correspondence is robust, geometrically natural, and analytically rigorous. It opens paths toward applications in optics, inverse problems, quantum field theory, and relativity.

6. Conclusions

In this work we have established a rigorous passage from Maxwell's equations to the relativistic Schrödinger equation by means of Schwartz Linear Algebra and Killing frames on the two-sphere.

The central result is the construction of two continuous, Schwartz-linear operators

$$\mathcal{F}_e \colon V_e \to W_e, \qquad\qquad \Psi_e \colon W_e \to V_e,$$

which are inverse isomorphisms, between the scalar distributional space V_e (generated by de Broglie waves) and the vectorial Maxwellian space W_e (generated by Maxwell-de Broglie fields).

We proved that these operators intertwine the dynamical generators of both pictures. A scalar distribution ψ satisfies the relativistic Schrödinger equation if and only if its Maxwellian counterpart $F_\psi = \mathcal{F}_e(\psi)$ satisfies the Maxwell–Schrödinger equation. This duality holds in the massless case, where $|\vec{P}|$ and $\hbar\,\nabla\times$ coincide spectrally, and in the massive case, where the spectral square-root Hamiltonians $\widehat{H}_{m_0}$ and $\widehat{H}^W_{m_0}$ agree on corresponding bases.

Geometrically, the construction is rooted in the canonical orthonormal Killing frames on $\mathbb{S}^2$, which provide natural polarization directions associated with each momentum. Analytically, it relies on the theory of Schwartz bases and tempered superpositions, developed by the author of this paper in the last 30 years.

Physically, it reveals that Maxwell's fields and Schrödinger's wave distributions are two sides of the same mathematical structure, linked by a precise isomorphism and equivalent dynamics.

The broader applicative implications are multiple. In quantum optics and photonics, the framework better clarifies the role of polarization and circular modes. In electromagnetic field analysis, it suggests new tools for inverse problems involving spherical symmetries.

In mathematical physics, it points toward a unified spectral treatment of Maxwell, Schrödinger, Klein-Gordon, Pauli, Proca and even Dirac type equations, within the same distributional setting.

Finally, in Einstein's general relativity, the method adapts naturally to curved space-times, offering a possible route toward gravitational extensions of the Maxwell-Schrödinger theory.

In conclusion, fixing a certain spatial direction e, the equivalence mediated by the isomorphisms $\mathcal{F}_e$ and Ψ_e shows that (as soon as we restrict our analysis to the vast twin subspaces V_e and W_e) the scalar and vectorial pictures of relativistic wave phenomena are not merely parallel, but are rigorously isomorphic at the distributional level.

Just to understand how vast are the twin spaces associated with a certain spatial direction e, consider that:

- in momentum representation, the Maxwell field space W_e covers all possible four-momenta p whose spatial part $\vec{p}$ is different from the null vector (so involving all plane-wave fields with an effective direction of propagation $\vec{u}$) with the negligible exception of the one direction e, but including all possible momentum representations $(F)_{w^e}$ vanishing at the negligible plane Π_e.
- this amounts to saying (in Schrödinger momentum space $V_\eta = \mathcal{S}'(\mathbb{M}_4^*, \mathbb{C})$) that all the possible Schwartz linear superpositions

$$\int_{\mathbb{M}_4^*} a(\delta_p)_{p\in\mathbb{M}_4^*\setminus\Pi_e},$$

  belong to our analysis, as soon as we define the improper superposition above for arbitrary distribution coefficients a vanishing at Π_e.

This duality provides a solid analytic and geometric foundation for future explorations in the intersection of Functional Analysis, Schwartz distribution theory, differential geometry, Lie algebra and group analysis, electromagnetism, Relativity and quantum field theory.

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