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Remarks on equivariant asymptotics for complexified toric eigenfunctions

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Abstract. The eigenfunctions of the Laplace–Beltrami operator on a real-analytic Riemannian manifold admit a simultaneous holomorphic extension to a sufficiently small Grauert tube. If the manifold is endowed with an isometric action of a compact Lie group, one can decompose each eigenspace into isotypical components associated to the irreducible representations of the group. The local asymptotics of the complexified eigenfunctions in a fixed isotypical component and (heuristically speaking) belonging to a spectral band drifting to infinity have been studied recently in [arXiv:2409.04753]. In this note, we illustrate these results in the special case where the base manifold is a d-dimensional torus with the standard metric, acted upon by a proper subtorus.

Анотація. Власні функції оператора Лапласа–Бельтрамі на дійсно-аналітичному рімановому многовиді допускають одночасне голоморфне продовження на достатньо малу трубку Грауерта. Якщо на многовиді задано ізометричну дію компактної групи Лі, то кожен власний простір можна розкласти на ізотипічні компоненти, що відповідають незвідним зображенням цієї групи. Локальні асимптотики комплексифікованих власних функцій у фіксованій ізотипічній компоненті, які (евристично кажучи) належать до спектральної смуги, що зміщується до нескінченності, були нещодавно досліджені у роботі [arXiv:2409.04753]. В даній статті ми ілюструємо ці результати для окремого випадку, коли на d-вимірному торі зі стандартною метрикою діє власний підтор.

Let (M, κ) be a closed and connected real-analytic Riemannian manifold, with non-negative Laplacian Δ. Let

$$0 = \mu_0^2 < \mu_1^2 < \mu_2^2 < \cdots$$

be the distinct eigenvalues of Δ, where $\mu_j \geqslant 0$; for every j, let $V_j \subset \mathcal{C}^\infty(M)$ be the eigenspace of μ_j^2, and let $(\varphi_{jk})_k$ be an L^2-orthonormal basis of V_j.

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Furthermore, let $\tilde{M}$ denote the Bruhat–Whitney complexification of M (a complex manifold in which M sits as a totally real submanifold). Then every real-analytic function $f\colon M \to \mathbb{R}$ admits a holomorphic extension $\tilde{f}$ to some open neighbourhood of M in $\tilde{M}$. As was first pointed out by Boutet de Monvel [1], there exists an open neighbourhood of M in $\tilde{M}$ to which every φ_{jk} can be holomorphically extended (see also [14, 9] for discussions and different proofs).

On the other hand, on a sufficiently small open neighbourhood of M in $\tilde{M}$ there is a canonically induced non-negative strictly plurisubharmonic function ρ, with zero-locus M and enjoying several nice properties making its study of considerable interest from the points of view of Riemannian, symplectic and complex geometry.

For instance $\sqrt{\rho}$, which is smooth on $\tilde{M}\backslash M$, satisfies the homogeneous complex Monge–Ampère equation, its Hamiltonian flow is closely related to the homogeneous geodesic flow, and the corresponding Monge–Ampère foliation is essentially a manifestation of the Riemannian foliation on the tangent bundle of M. One key to interpreting these statements is that κ induces a symplectomorphism from a tubular neighbourhood of the zero section in (TM, Ω_{can}) to $(\tilde{M}, \Omega)$, where Ω_{can} is the canonical symplectic structure on $TM \cong T^\vee M$, $\Omega := \imath\partial\overline{\partial}\rho$, and the tangent and cotangent bundles TM and $T^\vee M$ are identified by κ (see [10, 7, 8]).

Therefore, for any sufficiently small $\tau > 0$, the level sets

$$X^\tau := \rho^{-1}(\tau^2)$$

are boundaries of strictly pseudoconvex domains $M^\tau \subset \tilde{M}$, and are endowed with a CR structure, a Hardy space $H(X^\tau) \subset L^2(X^\tau)$, and a Szegő projector $\Pi^\tau\colon L^2(X^\tau) \to H(X^\tau)$. In particular,

$$\varphi^\tau_{jk} := \tilde{\varphi}|_{jk} \in H(X^\tau),$$

and the relation between φ_{jk} and φ^τ_{jk} is mediated by a Fourier integral operator $\mathfrak{P}^\tau$ with very interesting microlocal properties (see, e.g., [14, 15, 6]).

These considerations lead one to study, for a fixed sufficiently small $\tau > 0$, certain Toeplitz operator kernels on X^τ related to *complexified wave-Poisson* operators on X^τ defined in terms of $\mathfrak{P}^\tau$. These kernels may be regarded as heuristic holomorphic analogues of the smoothed spectral projectors classically used in the spectral theory of pseudodifferential operators; in particular, they encapsulate information on the asymptotic growth and concentration properties of the complexified eigenfunctions $\tilde{\varphi}_{jk}$. It turns out that these operator kernels asymptotically concentrate along the graph of the homogeneous geodesic flow, and satisfy along the latter certain scaling asymptotics of *Heisenberg type* ([3, 2, 12, 4]).

In addition, given the smooth isometric action on (M, κ) of a compact Lie group G, this analysis can be refined to the equivariant setting. That is, given a fixed irreducible representation of G, and under appropriate hypothesis on the cotangent moment map, one considers the asymptotics of the associated isotypical components of these kernels. Such issues have been addressed in [4]; with respect to the action-free case, one additional feature is that the equivariant kernels asymptotically concentrate along the zero locus of the moment map.

We won't provide here in detail the precise statements given in [4] (see Theorem 7, 29). The goal of this note is rather to illustrate the general result by describing what it implies in a specific situation. Furthermore, in order to avoid introducing too many technicalities and terminology, we shall confine our discussion to the diagonal setting.

Let

$$T^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d) \cong (S^1)^d$$

be a d-dimensional compact torus, with projection

$$\pi\colon \boldsymbol{\vartheta} \in \mathbb{R}^d \mapsto e^{\imath\boldsymbol{\vartheta}} \in T^d,$$

and let κ denote the standard metric on it (induced from the standard Euclidean metric on $\mathbb{R}^d$). Let $T^r \leqslant T^d$ denote a proper r-dimensional subtorus; thus, $r < d$ and T^r acts freely on T^d by translations; this action determines in a standard manner a unitary representation of T^r on $L^2(T^d)$. We shall let $\mathfrak{t}^d$ and $\mathfrak{t}^r \leqslant \mathfrak{t}^d$ denote the Lie algebras of T^d and T^r, respectively. Using κ, we shall identify the Lie algebras and the corresponding coalgebras, and correspondingly, for any vector subspace $V \subseteq \mathfrak{t}^d$, its orthocomplement in $V^\perp \subseteq \mathfrak{t}^d$ with its annihilator subspace in $V^0 \subseteq (\mathfrak{t}^d)^\vee$.

The eigenfunctions of the Laplacian Δ of (T^d, κ) are the one-dimensional characters of T^d, and they are in one-to-one correspondence with the multiindexes in $\mathbb{Z}^d$. Given $\boldsymbol{\lambda} \in \mathbb{Z}^d$, the corresponding L^2-normalized eigenfunction is

$$\varphi_{\boldsymbol{\lambda}}(e^{\imath\boldsymbol{\vartheta}}) := c_d e^{-\imath\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}, \quad c_d := (2\pi)^{-d/2}, \tag{1}$$

and the corresponding eigenvalue is $\|\boldsymbol{\lambda}\|^2$.

On the other hand, by the theorem of Peter and Weyl, $L^2(T^d)$ also decomposes unitarily as a direct sum of isotypical components for T^r, each such component being associated to an irreducible representation, hence to an irreducible character, of T^r.

If $\boldsymbol{\lambda} \in \mathbb{Z}^d$, then $\varphi_{\boldsymbol{\lambda}}$ restricts on T^r to the character $\psi_{\boldsymbol{\lambda}'}$, where $\boldsymbol{\lambda}'$ is the weight on T^r induced by $\boldsymbol{\lambda}$. Every weight of T^r is the restriction of a weight of T^d, and given $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{Z}^d$ we have $\boldsymbol{\lambda}_1' = \boldsymbol{\lambda}_2'$ if and only if

$\boldsymbol{\lambda}_1' - \boldsymbol{\lambda}_2' \in (\mathfrak{t}^r)^\perp \cap \mathbb{Z}^d$. Thus, if $\boldsymbol{\nu}$ is a weight for T^r, and $\tilde{\boldsymbol{\nu}} \in \mathbb{Z}^d$ is any choice of a lift of $\boldsymbol{\nu}$, then the weights of T^d restricting to $\boldsymbol{\nu}$ on T^r are precisely those of the form $\boldsymbol{\lambda} := \tilde{\boldsymbol{\nu}} + \boldsymbol{\eta}$ with $\boldsymbol{\eta} \in (\mathfrak{t}^r)^\perp \cap \mathbb{Z}^d$.

Therefore, the T^r-isotypical component of $L^2(T^d)$ corresponding to $\boldsymbol{\nu}$ is

$$L^2(T^d)_{\boldsymbol{\nu}} := \bigoplus_{\boldsymbol{\eta} \in (\mathfrak{t}^r)^\perp \cap \mathbb{Z}^d} \mathbb{C} \cdot \varphi_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}; \tag{2}$$

here $\varphi_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}$ is an eigenfunction of Δ, with eigenvalue $\|\tilde{\boldsymbol{\nu}} + \boldsymbol{\eta}\|^2$.

Heuristically, for a given fixed $\tau > 0$, we are interested in the asymptotic concentration of the complexification of those of the eigenfunctions appearing in (2) that pertain to a spectral band of $\sqrt{\Delta}$ drifting to infinity (in other words, with $\|\tilde{\boldsymbol{\nu}} + \boldsymbol{\eta}\|$ ranging in an interval of fixed length that is being translated to infinity). Thus, the asymptotic concentration is considered at a fixed distance τ from the real locus and letting the spectral parameter travel to infinity. In practice, we shall primarily deal with an appropriately *smoothed* version of this problem.

The complexification $\tilde{T}^d$ can be identified with $(\mathbb{C}^*)^d \cong \mathbb{C}^d/(2\pi\mathbb{Z}^d)$, the projection being $\tilde{\pi} \colon \mathbf{z} \mapsto e^{\imath \mathbf{z}}$. Furthermore,

$$X^\tau \cong T^d \times S^{d-1}(\tau) \subseteq \tilde{T}^d$$

is the projection of

$$\mathbb{R}^d \times \imath S^{d-1}(\tau) \subseteq \mathbb{C}^d,$$

where $S^{d-1}(\tau) \subseteq \mathbb{R}^d$ is the sphere of radius τ centered at the origin (when $\tau = 1$ we shall set $S^{d-1} := S^{d-1}(1)$). In this case (as for any compact Lie group with a bi-invariant metric) $\tau \in (0, +\infty)$, see [13].

The holomorphically extended eigenfunction $\tilde{\varphi}_{\boldsymbol{\lambda}}$ is obtained by replacing $\boldsymbol{\vartheta} \in \mathbb{R}^d$ in (1) by $\mathbf{z} \in \mathbb{C}^d$. Thus if we write $\mathbf{z} = \boldsymbol{\vartheta} - \imath \mathbf{r}$, with $\boldsymbol{\vartheta}, \mathbf{r} \in \mathbb{R}^d$, then

$$\tilde{\varphi}_{\boldsymbol{\lambda}}\left(e^{\imath \mathbf{z}}\right) := c_d e^{-\imath \langle \boldsymbol{\lambda}, \boldsymbol{\vartheta} \rangle - \langle \boldsymbol{\lambda}, \mathbf{r} \rangle}.$$

The restriction φ^τ is obtained by taking $\mathbf{r} = \tau \boldsymbol{\omega}$, with $\boldsymbol{\omega} \in S^{d-1}$.

The tempered *holomorphic equivariant projector kernel*

$$P^\tau_{\boldsymbol{\nu},j} \in \mathcal{C}^\infty(X^\tau \times X^\tau)$$

associated to the eigenvalue μ_j of $\sqrt{\Delta}$ is then defined to be

$$P^\tau_{\boldsymbol{\nu},j}(x,y) := e^{-2\tau\mu_j} \sum^{(j)}_{\boldsymbol{\eta} \in (\mathfrak{t}^r)^\perp \cap \mathbb{Z}^d} \varphi^\tau_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}(x) \overline{\varphi^\tau_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}(y)},$$

where $\sum^{(j)}$ means that we are summing only over those $\boldsymbol{\eta}$'s for which

$$\|\tilde{\boldsymbol{\nu}} + \boldsymbol{\eta}\|^2 = \mu_j^2.$$

The exponential factor $e^{-2\tau\mu_j}$ tempers the exponential growth of the complexified eigenfunctions at distance τ from the real locus.

Since the complexified eigenfunctions $\varphi^{\tau}_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}$ do not form an orthonormal system in $H(X^\tau)$ (i.e., orthonormality is generally lost through holomorphic extension to X^τ), $P^{\tau}_{\boldsymbol{\nu},j}$ is not an orthogonal projector kernel. Nonetheless, it encapsulates information on the asymptotic growth and concentration properties of the $\varphi^{\tau}_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}$'s. In particular, its diagonal restriction is

$$P^{\tau}_{\boldsymbol{\nu},j}(x,x) := e^{-2\tau\mu_j} \sum^{(j)}_{\boldsymbol{\eta}\in(\mathfrak{t}^r)^\perp\cap\mathbb{Z}^d} \left|\varphi^{\tau}_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}(x)\right|^2,$$

so that any asymptotic pointwise estimate on $P^{\tau}_{\boldsymbol{\nu},j}(x,x)$ determines an estimate on the growth rate of the complexified eigenfunctions along X^τ.

Since we are interested in the asymptotics related to a traveling spectral band, following a classical procedure in spectral theory (see e.g. [5]) we shall consider a *smoothed version* of the tempered holomorphic projector kernel associated to an interval drifting to infinity. To this end, let us fix $\chi \in \mathcal{C}^\infty_0(-\epsilon,\epsilon)$ for some sufficiently small $\epsilon > 0$, and for $\lambda \in \mathbb{R}$ consider the smoothing kernel

$$P^{\tau}_{\boldsymbol{\nu},\lambda} := \sum_j \hat{\chi}(\lambda-\mu_j) P^{\tau}_{\boldsymbol{\nu},j},$$

which can be regarded as a smooth approximation of the sum of the $P^{\tau}_{\boldsymbol{\nu},j}$'s pertaining to an interval centered at λ and drifting to infinity with $\lambda \to +\infty$.

We may equivalently rewrite $P^{\tau}_{\boldsymbol{\nu},\lambda}$ in the form

$$P^{\tau}_{\boldsymbol{\nu},\lambda}(x,y) = \sum_{\boldsymbol{\eta}\in(\mathfrak{t}^r)^\perp\cap\mathbb{Z}^d} \hat{\chi}\big(\lambda - \|\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}\|\big) e^{-2\tau\|\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}\|}\, \varphi^{\tau}_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}(x)\, \overline{\varphi^{\tau}_{\tilde{\boldsymbol{\nu}}+\boldsymbol{\eta}}(y)}. \tag{3}$$

which may be regarded as a tempered holomorphic analogue of the smoothed projection kernels used in classical spectral theory [5] and in Kähler geometric quantization (see for instance, with no pretense of completeness, the discussions in [11, 16, 17]).

For the sake of simplicity, we shall restrict here to diagonal asymptotics, and refer to [4] for more general statements. Let us consider $x \in X^\tau$, and write as above $x = e^{\imath\mathbf{z}}$, where $\mathbf{z} = \boldsymbol{\vartheta} - \imath\tau\boldsymbol{\omega}$, where $\boldsymbol{\omega} \in S^{d-1}$. The first conclusion that we can draw from the theory in [4] is that

$$P^{\tau}_{\boldsymbol{\nu},\lambda}(x,x) = O\left(\lambda^{-\infty}\right) \quad \text{as} \quad \lambda \to +\infty, \tag{4}$$

unless $\boldsymbol{\omega} \in S^{d-1} \cap (\mathfrak{t}^r)^\perp$. In fact, a stronger statement holds: for any choice of constants $C, \epsilon > 0$, (4) holds uniformly for

$$\mathrm{dist}\big(\boldsymbol{\omega}, (\mathfrak{t}^r)^\perp\big) \geqslant C\lambda^{\epsilon-1/2}.$$

Thus we need to restrict to shrinking neighbourhoods of

$$Z^\tau := T^d \times \imath\big(S^{d-1}(\tau) \cap (\mathfrak{t}^r)^\perp\big)$$

in order to find non-trivial asymptotics.

To get a heuristic glimpse of why this is true in the special case at hand, let us consider for simplicity the case of the trivial representation, so that we may assume $\tilde{\boldsymbol{\nu}} = 0$, and a point $x = e^{\imath \mathbf{z}} \in X^\tau$ with $\boldsymbol{\omega} \notin (\mathfrak{t}^r)^\perp$. Then for every non-zero $\boldsymbol{\eta} \in (\mathfrak{t}^r)^\perp$ we have

$$\begin{aligned} e^{-2\tau\|\boldsymbol{\eta}\|} \left|\varphi^\tau_{\boldsymbol{\eta}}(x)\right|^2 &= e^{-2\tau\|\boldsymbol{\eta}\|} c_d e^{-2\tau\langle \boldsymbol{\eta}, \boldsymbol{\omega}\rangle} \\ &= c_d e^{2\tau\|\boldsymbol{\eta}\|\cdot[\langle \boldsymbol{\eta}/\|\boldsymbol{\eta}\|, -\boldsymbol{\omega}\rangle - 1]}. \end{aligned} \tag{5}$$

The expression

$$\langle \boldsymbol{\eta}/\|\boldsymbol{\eta}\|, -\boldsymbol{\omega}\rangle - 1$$

remains negative and bounded from above when $\boldsymbol{\eta} \in (\mathfrak{t}^r)^\perp \backslash \{0\}$ and $\boldsymbol{\omega} \in S^{d-1} \backslash (\mathfrak{t}^r)^\perp$ is fixed, hence (5) is rapidly decreasing with $\|\boldsymbol{\eta}\| \to +\infty$. On the other hand, (roughly speaking) the non-negligible contributions to (3) come from those $\boldsymbol{\eta}$ with $\|\boldsymbol{\eta}\| \sim \lambda$.

Let us refine the previous remark, and consider instead a moving point $x \in X^\tau$ approaching Z^τ at a slow pace with λ, that is, we assume that x is at a variable distance $\sim C\lambda^{\epsilon - 1/2}$, for some sufficiently small $\epsilon > 0$. Thus, $\tau\boldsymbol{\omega}$ is at a distance $\geqslant C\lambda^{\epsilon-1/2}$ from $(\mathfrak{t}^r)^\perp$. If $\tau\boldsymbol{\omega}_0$ is the point of $S^{d-1}(\tau) \cap (\mathfrak{t}^r)^\perp$ closest to $\tau\boldsymbol{\omega}$, we can write to first order

$$\tau\boldsymbol{\omega} = \tau\boldsymbol{\omega}_0 + \frac{1}{\sqrt{\lambda}}\mathbf{n} + \cdots .$$

where $\mathbf{n} \in \mathfrak{t}^r$ has norm $\sim C\lambda^\epsilon$, for some sufficiently small $\epsilon > 0$. Taking into account that $\boldsymbol{\omega}, \boldsymbol{\omega}_0 \in S^{d-1}$, we have

$$\boldsymbol{\omega} = \sqrt{1 - \left\|\frac{1}{\tau\sqrt{\lambda}}\mathbf{n}\right\|^2}\boldsymbol{\omega}_0 + \frac{1}{\tau\sqrt{\lambda}}\mathbf{n}.$$

Hence, for every non-zero $\boldsymbol{\eta} \in (\mathfrak{t}^r)^\perp$ we have

$$\begin{aligned}
2\tau\|\boldsymbol{\eta}\|\cdot &\left[\left\langle \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}, -\boldsymbol{\omega} \right\rangle - 1\right] \\
&= 2\tau\|\boldsymbol{\eta}\| \cdot \left[-\sqrt{1 - \left\|\frac{1}{\tau\sqrt{\lambda}}\mathbf{n}\right\|^2} \left\langle \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|}, \boldsymbol{\omega}_0 \right\rangle - 1\right] \\
&\leqslant 2\tau\|\boldsymbol{\eta}\| \cdot \left[-\frac{1}{2\tau^2\lambda}\|\mathbf{n}\|^2 + R_3\left(\frac{1}{\tau\sqrt{\lambda}}\mathbf{n}\right)\right] \\
&= -\frac{1}{\tau}\frac{\|\boldsymbol{\eta}\|}{\lambda}\|\mathbf{n}\|^2 + R_3\left(\frac{1}{\tau\sqrt{\lambda}}\mathbf{n}\right),
\end{aligned}$$

where R_3 vanishes to third order at the origin. Thus, in the regime where $\lambda \sim \|\boldsymbol{\eta}\|$, we obtain an exponential decrease in $\lambda \to +\infty$.

Let us fix $x \in Z^\tau$, and consider the asymptotics of $P^\tau_{\boldsymbol{\nu},\lambda}(x,x)$ for $\lambda \to +\infty$. In this case, the results in [4] imply that there exists a full asymptotic expansion in descending powers of λ:

$$P^\tau_{\boldsymbol{\nu},\lambda}(x,x) \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2^{d-1}\pi^r}\left(\frac{\lambda}{\pi\tau}\right)^{(d-1-r)/2}\left[\chi(0) + \sum_{j=1}^{\infty}\lambda^{-j}a_{\boldsymbol{\nu},j}(x)\right]. \quad (6)$$

One is then led to look for an analogue of (6) when $x \to Z^\tau$ in X^τ at a sufficiently fast pace (if x approaches Z^τ slowly enough, we already know that $P^\tau_{\boldsymbol{\nu},\lambda}(x,x)$ is rapidly decreasing). A convenient way to formulate the problem is then to consider a small displacement from a given $x \in Z^\tau$ in a direction normal to Z^τ; using an additive notation to be interpreted in an appropriate system of local coordinates centered at x, we shall denote such small displacement by $x + \mathbf{n}/\sqrt{\lambda} \in X^\tau$, where $\|\mathbf{n}\| \leqslant C\lambda^\epsilon$ for some suitably small $\epsilon > 0$. The previous discussion suggests that there should be an exponential Gaussian type factor mediating between the asymptotic expansion (6) valid on Z^τ, and the rapid decrease holding at a fixed pair (x_0, x_0) with $x_0 \notin Z^\tau$. In place of (6), one has

$$\begin{aligned}
P^\tau_{\boldsymbol{\nu},\lambda}&\left(x + \frac{\mathbf{n}}{\sqrt{\lambda}}, x + \frac{\mathbf{n}}{\sqrt{\lambda}}\right) \sim \\
&\sim \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{2}{\tau}\|\mathbf{n}\|_1^2}}{2^{d-1}\pi^r}\left(\frac{\lambda}{\pi\tau}\right)^{(d-1-r)/2}\left[\chi(0) + \sum_{j=1}^{\infty}\lambda^{-j}a_{\boldsymbol{\nu},j}(x,\mathbf{n})\right],
\end{aligned}$$

where, in agreement to conventions for which we refer to [4], we have set $\|\cdot\|_1^2 := \|\cdot\|^2/2$, and $a_{\boldsymbol{\nu},j}(x,\cdot)$ is a polynomial (of degree $\leqslant 6j$).

Finally, let us dwell on what the previous expansion looks like in the action-free case (thus with T^r the trivial subgroup). Then $r = 0$, $Z^\tau = X^\tau$

and we obtain that, uniformly on X^τ,

$$\sum_{\boldsymbol{\eta}\in\mathbb{Z}^d} \hat{\chi}\big(\lambda - \|\boldsymbol{\eta}\|\big) e^{-2\tau\|\boldsymbol{\eta}\|} \left|\varphi^\tau_{\boldsymbol{\eta}}(x)\right|^2 \sim \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2^{d-1}} \left(\frac{\lambda}{\pi\tau}\right)^{(d-1)/2} \left[\chi(0) + \sum_{j=1}^{\infty} \lambda^{-j} a_j(x)\right].$$

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