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On canonical 2F-planar mappings of the first type of pseudo-Riemannian spaces with f-structure

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Abstract. The article is devoted to the basic questions of the theory of $2F$-planar mappings of manifolds, which are endowed with a certain type affinor structure. We proved that the class of pseudo-Riemannian spaces with absolutely parallel f-structure is closed with respect to the considered mappings. In addition, under the condition of covariant constancy of the affinor f-structure in the mapped spaces, non-trivial $2F$-planar mappings can be of three types: complete and canonical of types I and II.

Previously, we studied complete $2F$-planar mappings in detail. This article considers the main issues for canonical $2F$-planar mappings of the first type.

Theorems have been proved that give a regular method that allows for any pseudo-Riemannian space with absolutely parallel f-structure (V_n, g_{ij}, F_i^h) to either find all spaces $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ onto which V_n admits a canonical $2F$-planar mapping of the first type, or to prove that there are no such spaces.

In particular, we have shown that a pseudo-Riemannian space with absolutely parallel f-structure, in which there is a concircular or quasi-concircular vector field, admits a non-trivial canonical $2F$-planar mapping of the first type.

Анотація. Стаття присвячена базовим питанням теорії $2F$-планарних відображень многовидів, які наділені афінорною структурою певного типу. Доведено, що клас псевдоріманових просторів з абсолютно паралельною f-структурою є замкненим відносно розглянутих відображень. Крім того, за умови коваріантної сталості афінора f-структури у відображуваних просторах, нетривіальні $2F$-планарні відображення можуть бути лише трьох видів: повні (основного типу) і канонічні типів I та II.

Раніше ми докладно вивчали повні $2F$-планарні відображення. У цій статті розглядаються основні питання для канонічних $2F$-планарних відображень типу I.

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Доведено теореми, які дають регулярний метод, що дозволяє для будь -якого псевдоріманового простору з абсолютно паралельною f-структурою (V_n, g_{ij}, F_i^h) або знайти всі простори $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$, на які V_n допускає канонічне $2F$-планарне відображення першого типу, або довести, що таких просторів немає. Зокрема, показано, що псевдорімановий простір з абсолютно паралельною f-структурою, в якому існує конциркулярне або квазіконциркулярне векторне поле, допускає нетривіальне канонічне $2F$-планарне відображення типу I.

1. Introduction

In [3, 4] we considered $2F$-planar mappings [2] which are a natural generalization of geodesic mappings [8] of affine-connected and Riemannian spaces, as well as holomorphically-projective and F-planar [1, 5, 6, 9] mappings of manifolds endowed with a certain type affinor structure.

In [2] the concept of $2F$-planar mapping ($2F$PM) of affinely connected and Riemannian spaces was introduced and it was shown that $2F$PM between the spaces (V_n, g_{ij}, F_i^h), $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ with metric tensors $g_{ij}, \overline{g}_{ij}$ and affinor structures $F_i^h, \overline{F}_i^h$, respectively, necessarily preserves the structure.

In other words, in the common with respect to the mapping coordinate system (x^i) we have

$$F_i^h(x) = \overline{F}_i^h(x), \quad h, i = 1, 2, \dots, n.$$

In [2], $2F$PMs of pseudo-Riemannian spaces with a structure of the form $F_\alpha^h F_\beta^\alpha F_i^\beta = \delta_i^h$ were also investigated.

Continuing the study of $2F$PM in [3], we found out that a pseudo-Riemannian space with an absolutely parallel f-structure is a product of two spaces, one of which is Kähler, and also that the class of pseudo-Riemannian spaces with an absolutely parallel f-structure is closed with respect to the considered mappings. It was also shown that under the condition of covariant constancy of the f-structure, non-trivial $2F$-planar mappings can be of three types: complete and canonical of type I or II.

In [3] it is proved that $2F$PM, depending on its type, induces on the product components, which represent the mapped spaces, a geodesic [8], holomorphic-projective [6] or affine [8] mapping.

In [4] we constructed geometric objects being invariant with respect to the considered mappings, identified classes of pseudo-Riemannian spaces with absolutely parallel f-structure that admit $2F$PM (of the main type and canonical) onto a flat space, and obtained their metrics in a special coordinate system.

In [11] we considered the basic questions of the theory of $2F$PM of the main type. It is developed a regular method allowing, for any pseudo-Riemannian space with absolutely parallel f-structure (V_n, g_{ij}, F_i^h), either to find all spaces $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ onto which V_n admits a $2F$-planar mapping of the main type, or to prove that there are no such spaces.

The study is carried out in tensor form, locally, in the class of real rather smooth, and in the general case analytic functions.

2. $2F$PMs of pseudo-Riemannian spaces with absolutely parallel f-structure and their regularities

Consider two pseudo-Riemannian spaces (V_n, g_{ij}, F_i^h) and $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$, in which affinor structures are given. In the common with respect to the mapping coordinate system (x^i) a $2F$-planar mapping

$$(V_n, g_{ij}, F_h^i) \xrightarrow{2F\mathrm{PM}} (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h),$$

is characterized by the fundamental equations [2]:

$$\begin{gathered} \overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}\delta_{j)}^h + \phi_{(i}\overset{1}{F}{}_{j)}^h + \sigma_{(i}\overset{2}{F}{}_{j)}^h, \\ F_i^h(x) = \overline{F}_i^h(x), \end{gathered} \tag{2.1}$$

where

$$\overset{1}{F}{}_i^h = F_i^h, \qquad \overset{2}{F}{}_i^h = F_\alpha^h F_i^\alpha.$$

Γ_{ij}^h, $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of $V_n, \overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$, $\sigma_i(x)$ are certain covectors; brackets (i,j) denote the symmetrization with respect to the corresponding indices; comma "," is a sign of the covariant derivative in respect to the connection of V_n.

$2F$PM is considered trivial when $\psi_i = \phi_i = \sigma_i = 0$.

We will assume that the affinor F_i^h defines a *f-structure* in (V_n, g_{ij}, F_i^h) [7], [10], i.e. the following conditions hold

$$\begin{gathered} F_\alpha^h F_\beta^\alpha F_i^\beta + F_i^h = 0, \quad i, h, \alpha, \beta, \dots = 1, 2, \dots, n, \\ Rg\|F_i^h\| = 2k, \quad (2k < n). \end{gathered} \tag{2.2}$$

Note that if

- the affinor agrees with the metrices on V_n and $\overline{V}_n$ in the sense that the following identities hold:

$$\begin{aligned} \overset{1}{F}_{ij} + \overset{1}{F}_{ji} &= 0, & \overset{1}{\overline{F}}_{ij} + \overset{1}{\overline{F}}_{ji} &= 0, \\ \overset{1}{F}_{ij} &= g_{i\alpha}\, \overset{1}{F}{}_j^\alpha, & \overset{1}{\overline{F}}_{ij} &= \overline{g}_{i\alpha}\, \overset{1}{F}{}_j^\alpha. \end{aligned} \tag{2.3}$$

- and the corresponding affinor structure is absolutely parallel in V_n, i.e.

$$\overset{1}{F}{}^h_{i,j} = 0, \tag{2.4}$$

then, as shown in [3], we will have that

$$\overset{1}{F}{}^h_{i|j} = 0, \tag{2.5}$$

where " , " and " | " are the signs of the covariant derivative in V_n and $\overline{V}_n$, respectively.

In [3] we also found out that under the conditions (2.2)-(2.5) between the vectors ψ_i, ϕ_i, σ_i in the fundamental equations (2.1) there is a dependence

$$\psi_\alpha \overset{1}{F}{}^\alpha_i = 0, \qquad \sigma_i = \psi_i - \phi_\alpha \overset{1}{F}{}^\alpha_i, \qquad \phi_i = \sigma_\alpha \overset{1}{F}{}^\alpha_i, \tag{2.6}$$

and also that the condition $\sigma_i = 0$ implies $\psi_i = \phi_i = 0$, i.e. in this case $2F$PM is trivial. Therefore, for nontrivial $2F$-planar mappings (V_n, g_{ij}, F^h_i) onto $(\overline{V}_n, \overline{g}_{ij}, \overline{F}{}^h_i)$ with the fundamental equations (2.1) one of the following options occurs:

(I) $\psi_i = 0$, $\phi_i \neq 0$, $\sigma_i \neq 0$;
(II) $\psi_i \neq 0$, $\phi_i = 0$, $\sigma_i \neq 0$;
(III) $\psi_i \neq 0$, $\phi_i \neq 0$, $\sigma_i \neq 0$.

We will call the $2F$-planar mapping *canonical* I(II) *type* and denote it by $2F$PM(I), $2F$PM(II) in cases I, II and $2F$PM *of fundamental type* in case III.

Further in this article, we consider only $2F$PM(I).

Let us define an operation of contraction with an affinor, which is called the *conjugation* with respect to the corresponding indices and is denoted as follows:

$$A^{\cdots}_{\bar{i}\ldots} = A^{\cdots}_{\alpha\ldots} \overset{1}{F}{}^\alpha_i, \qquad A^{\bar{i}\ldots}_{\cdots} = A^{\alpha\ldots}_{\cdots} \overset{1}{F}{}^i_\alpha,$$
$$A^{\cdots}_{\bar{\bar{i}}\ldots} = A^{\cdots}_{\alpha\ldots} \overset{2}{F}{}^\alpha_i, \qquad A^{\bar{\bar{i}}\ldots}_{\cdots} = A^{\alpha\ldots}_{\cdots} \overset{2}{F}{}^i_\alpha.$$

3. Linear equations of the theory of $2F$PM(I)

A pseudo-Riemannian space V_n with an absolutely parallel f-structure admits a non-trivial mapping

$$(V_n, g_{ij}, F^i_h) \xrightarrow{2F\mathrm{PM}(I)} (\overline{V}_n, \overline{g}_{ij}, \overline{F}{}^h_i),$$

onto the space $\overline{V}_n$, if and only if in the common with respect to the mapping coordinate system (x^i) the fundamental equations of the considered mapping have the form:

$$\begin{gathered}\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \phi_{(i} \overset{1}{F}{}^h_{j)} + \sigma_{(i} \overset{2}{F}{}^h_{j)}, \\ F^h_i(x) = \overline{F}^h_i(x),\end{gathered} \tag{3.1}$$

at the same time

$$\sigma_i = -\phi_\alpha \overset{1}{F}{}^\alpha_i, \qquad \phi_i = \sigma_\alpha \overset{1}{F}{}^\alpha_i. \tag{3.2}$$

It follows easily from here

$$\sigma_i = -\sigma_\alpha \overset{2}{F}{}^\alpha_i, \qquad \phi_i = -\phi_\alpha \overset{2}{F}{}^\alpha_i. \tag{3.3}$$

Relations (3.1) in (V_n, g_{ij}, F^h_i) are the system of nonlinear differential equations in partial derivatives of the first order with respect to the components of the tensor $\overline{g}_{ij}(x)$ and the vectors $\phi_i(x) \neq 0$, $\sigma_i(x) \neq 0$ under conditions (3.2), (3.3).

The modern theory of differential equations does not provide regular methods for investigating the conditions for the existence and uniqueness of solutions to this system.

Using the methods developed in the theory of geodesic mappings of Riemannian spaces [6, 8], we will now reduce the fundamental equations of $2F$PM(I) (3.1)–(3.3) to a form that allows for efficient investigation.

It follows from relations (3.1) and (3.2) that

$$\begin{gathered}\overline{\Gamma}^\alpha_{i\alpha}(x) = \Gamma^\alpha_{i\alpha}(x) - 2(k+1)\sigma_i(x), \\ Rg\|F^h_i\| = 2k \quad (2k < n).\end{gathered}$$

Therefore the vector σ_i is gradient, that is, there exists an invariant $\sigma(x)$ such that

$$\sigma_i = \frac{\partial \sigma(x)}{\partial x^i}. \tag{3.4}$$

Further, since $\overline{g}_{ij|k} = 0$ in $\overline{V}_n$, Eq. (2.1) can be written in an equivalent form

$$\overline{g}_{ij,k} = \phi_{(i} \overset{1}{\overline{F}}_{j)k} + 2\sigma_k \overset{2}{\overline{F}}_{ij} + \sigma_{(i} \overset{2}{\overline{F}}_{j)k}, \tag{3.5}$$

where

$$\begin{gathered}\overset{1}{\overline{F}}_{ij} = \overline{g}_{i\alpha} \overset{1}{F}{}^\alpha_j, \qquad \overset{2}{\overline{F}}_{ij} = \overline{g}_{i\alpha} \overset{2}{F}{}^\alpha_j, \\ \overset{2}{\overline{F}}_{ij} = \overset{2}{\overline{F}}_{ji}.\end{gathered}$$

Let us consider a non-degenerate tensor

$$\tilde{g}_{ij} = \overline{g}_{i\alpha}\left(a\delta^\alpha_j + c \overset{2}{F}{}^\alpha_j\right), \tag{3.6}$$

where a, c are some invariants.

After covariant differentiation (3.6) in V_n taking into account the conditions (3.5) we obtain:

$$\tilde{g}_{ij,k} = \phi_{(i}\ \overset{1}{\tilde{F}}_{j)k} + \sigma_{(i}\ \overset{2}{\tilde{F}}_{j)k} + T_{ijk}, \tag{3.7}$$

where

$$\overset{1}{\tilde{F}}_{ij} = \tilde{g}_{i\alpha}\ \overset{1}{F}{}^{\alpha}_{j}, \qquad \overset{2}{\tilde{F}}_{ij} = \tilde{g}_{i\alpha}\ \overset{2}{F}{}^{\alpha}_{j},$$

$$T_{ijk} = \overline{g}_{ij} a_{,k} + \overset{2}{\overline{F}}_{ij}\left(c_{,k} + 2(a-c)\sigma_k\right).$$

We choose the invariants a and c so that $T_{ijk} = 0$, i.e.

$$\overline{g}_{ij} a_{,k} + \overset{2}{\overline{F}}_{ij}\big(c_{,k} + 2(a-c)\sigma_k\big) = 0. \tag{3.8}$$

Contracting the obtained relations with $\overset{2}{F}{}^{j}_{k}$ by the index j and comparing the result with the original equality, we conclude that (3.8) are equivalent to a system of first-order partial differential equations

$$\begin{cases} a_{,k} = 0, \\ c_{,k} + 2(a-c)\sigma_k = 0. \end{cases}$$

This system is completely integratble and its general solution, taking into account (3.4), has the form

$$a = C_1, \qquad\qquad c = C_1 + C_2 e^{2\sigma(x)},$$

where C_1, C_2 are arbitrary constants. We need any partial solution for which

$$\tilde{g}_{ij}(x) \neq a\overline{g}_{ij}(x), \quad \det\|\tilde{g}_{ij}\| \neq 0,$$

and therefore we choose $C_1 = C_2 = 1$. Then

$$\tilde{g}_{ij} = \overline{g}_{i\alpha}\Big(\delta^{\alpha}_{j} + (1+e^{2\sigma})\ \overset{2}{F}{}^{\alpha}_{j}\Big). \tag{3.9}$$

The matrix $(\tilde{g}_{ij})$ is non-degenerate. It is easy to verify that the tensor

$$\tilde{g}^{ij} = \overline{g}^{i\alpha}\left(\delta^{j}_{\alpha} + (1+e^{-2\sigma})\ \overset{2}{F}{}^{j}_{\alpha}\right)$$

satisfies the condition

$$\tilde{g}_{i\alpha}\tilde{g}^{\alpha j} = \delta^{j}_{i}.$$

Let us differentiate this identity covariantly in V_n:

$$\tilde{g}_{i\alpha,k}\tilde{g}^{\alpha j} + \tilde{g}_{i\alpha}\tilde{g}^{\alpha j}_{,k} = 0.$$

Hence

$$\tilde{g}^{ij}_{,k} = -\tilde{g}_{\alpha\beta,k}\tilde{g}^{\alpha i}\tilde{g}^{\beta j}$$

and, according to (3.7), (3.8),

$$a_{ij,k} = \overset{1}{\lambda}_{(i}\, \overset{1}{F}_{j)k} + \overset{2}{\lambda}_{(i}\, \overset{2}{F}_{j)k}, \tag{3.10}$$

where

$$\begin{gathered} a_{ij} = \tilde{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \\ \overset{1}{\lambda}_i = -\phi_\alpha \tilde{g}^{\alpha\beta} g_{\beta j}, \qquad \overset{2}{\lambda}_i = -\sigma_\alpha \tilde{g}^{\alpha\beta} g_{\beta j}. \end{gathered} \tag{3.11}$$

In view of (3.2) the vectors $\overset{1}{\lambda}_i$, $\overset{2}{\lambda}_i$ are related as follows

$$\overset{2}{\lambda}_{\bar{i}} = \overset{1}{\lambda}_i, \qquad \overset{2}{\lambda}_i = -\overset{1}{\lambda}_{\bar{i}}. \tag{3.12}$$

It easily follows from (2.3), (3.11) that the tensor a_{ij} satisfies the conditions

$$a_{i\alpha}\, \overset{1}{F}{}^{\alpha}_{j} = -a_{j\alpha}\, \overset{1}{F}{}^{\alpha}_{i}, \qquad \det\|a_{ij}\| \neq 0. \tag{3.13}$$

Let us contract (3.10) with $\overset{2}{F}{}^{ij}$ by indices i, j. Then taking into account (3.12) it turns out that the vector $\overset{2}{\lambda}_i$ is gradient, since

$$a_{\alpha\beta,k}\, \overset{2}{F}{}^{\alpha\beta} = (a_{\alpha\beta}\, \overset{2}{F}{}^{\alpha\beta})_{,k} = 4\overset{2}{\lambda}_k.$$

Therefore, if a pseudo-Riemannian space (V_n, g_{ij}, F^h_i) with an absolutely parallel f-structure admits a nontrivial $2F$PM(I) onto the space $(\overline{V}_n, \overline{g}_{ij}, \overline{F}{}^h_i)$, i.e. it satisfies (3.5), (2.2), (2.3), (2.4), then there must exist in it a nonsingular symmetric tensor a_{ij} that satisfies (3.10), (3.12), (3.13) for some nonzero vectors $\overset{1}{\lambda}_i$, $\overset{2}{\lambda}_i$.

The converse is also true. In fact, if a_{ij}, $\overset{1}{\lambda}_i$, and $\overset{2}{\lambda}_i$ satisfy the equations (3.10), (3.12), (3.13), then for

$$\begin{gathered} \tilde{g}_{ij} = a^{\alpha\beta} g_{\alpha i} g_{\beta j}, \\ a_{i\alpha} a^{\alpha j} = \delta^j_i \end{gathered}$$

the identity (3.7) holds when

$$T_{ijk} = 0, \qquad \phi_i = -\overset{1}{\lambda}_\beta g^{\beta\alpha}\tilde{g}_{\alpha i}, \qquad \sigma_i = -\overset{2}{\lambda}_\beta g^{\beta\alpha}\tilde{g}_{\alpha i}.$$

Let $\tilde{\Gamma}^h_{ij}$ be the Christoffel symbols of the second kind derived from the tensor $\tilde{g}_{ij}$. Given that

$$2\tilde{\Gamma}^\alpha_{i\alpha} = \tilde{g}^{\alpha\beta}\frac{\partial \tilde{g}_{\alpha\beta}}{\partial x^i} = \tilde{g}^{\alpha\beta}(\tilde{g}_{\alpha\beta,i} + 2\tilde{g}_{\alpha\gamma}\Gamma^\gamma_{\beta i}),$$

and taking into account (2.6), (3.7) we find

$$-2\sigma_i = \tilde{\Gamma}^{\alpha}_{i\alpha} - \Gamma^{\alpha}_{i\alpha}$$

which indicates the gradient of the vector σ_i.

But then for the tensor

$$\overline{g}_{ij} = \tilde{g}_{i\alpha}\Big(\delta^{\alpha}_{j} + (1+e^{2\sigma})\overset{2}{F}{}^{\alpha}_{j}\Big)$$

conditions (3.5) follow from (3.7) and (3.9). Obviously, $\overline{g}_{ij}$ is a metric tensor of a pseudo-Riemannian space with a completely parallel f-structure $(\overline{V}_n, \overline{g}_{ij}, \overline{F}{}^{h}_{j})$.

Thus, we proved

Theorem 3.1. *A pseudo-Riemannian space with an absolutely parallel f-structure (V_n, g_{ij}, F^h_j) admits $2F$PM(I)s if and only if in this space there exists a non-singular, symmetric, doubly covariant tensor a_{ij} that satisfies the equations* (3.10), (3.13) *for some vectors $\overset{1}{\lambda}_i \neq 0$, $\overset{2}{\lambda}_i \neq 0$, related by the conditions* (3.12).

Equations (3.10) represent a linear form of the fundamental equations of the theory of canonical 2F-planar mappings of the first type of spaces with f-structure.

Let us deduce a useful corollary of Theorem 3.1. Before formulating it, recall that a Riemannian space (V_n, g_{ij}), in which there exists a vector field $\xi_i \neq 0$, satisfying the equation

$$\xi_{i,j} = \rho g_{ij} \tag{3.14}$$

is called *equidistant* [6,8]. Such vector fields are called *concircular* [6,8]. An equidistant space is considered to be of the fundamental type when $\rho \neq 0$ and of the special type when (3.10) holds.

Corollary 3.2. *Every pseudo-Riemannian space with an absolutely parallel f-structure (V_n, g_{ij}, F^i_j) in which there exists a concircular vector field ξ_i satisfying* (3.14) *with $\rho \neq 0$, admits a nontrivial* $2F$PM(I) *provided that $\xi_{\bar{\bar{i}}} \neq 0$.*

Proof. Suppose that there exists a vector field ξ_i in (V_n, g_{ij}, F^h_j) satisfying (3.14) and $\xi_{\bar{\bar{i}}} \neq 0$. Then for the tensor

$$a_{ij} = C_1 g_{ij} + C_2 \overset{2}{F}_{ij} + C_3(\xi_{\bar{i}}\xi_{\bar{j}} + \xi_{\bar{\bar{i}}}\xi_{\bar{\bar{j}}})$$

with constants $C_1, C_2, C_3 \neq 0$ such that a_{ij} will be non-singular, we will have that:

$$a_{ij,k} = \overset{1}{\lambda}_{(i} \overset{1}{F}_{j)k} + \overset{2}{\lambda}_{(i} \overset{2}{F}_{j)k},$$

where

$$\overset{1}{\lambda}_i = -C_3 \rho \xi_{\bar{i}}, \qquad \overset{2}{\lambda}_i = C_3 \rho \xi_{\bar{\bar{i}}}.$$

By direct verification, we will see that for a_{ij} and $\overset{1}{\lambda}_i \neq 0$, $\overset{2}{\lambda}_i \neq 0$, and the properties (3.12) and (3.13) hold. Hence, by Theorem 3.1 (V_n, g_{ij}, F^i_j) admits $2F$PM(I). □

Let us generalize the concept of a concircular vector field. To do this, we introduce into consideration vector fields satisfying the conditions

$$\zeta_{i,j} = \rho_1 g_{ij} + \rho_2 \overset{2}{F}_{ij}. \tag{3.15}$$

We will call them *quasiconcircular*.

Corollary 3.3. *Every pseudo-Riemannian space with an absolutely parallel f-structure (V_n, g_{ij}, F^i_j), in which there exists a quasi-concircular vector field ζ_i satisfying the equation* (3.15), *admits a* $2F$PM(I) *whenever $\rho_1 \neq \rho_2$, $\rho_1 \neq 0$, $\rho_2 \neq 0$, $\zeta_{\bar{\bar{i}}} \neq 0$.*

Suppose that is a space (V_n, g_{ij}, F^h_j) there exists a vector field ζ_i satisfying (3.15) with $\rho_1 \neq \rho_2$, $\rho_1 \neq 0$, $\rho_2 \neq 0$, $\zeta_{\bar{\bar{i}}} \neq 0$. Then, by direct calculation, we can verify that the vector field

$$(\rho_1 - \rho_2)\zeta_i - \rho_2 \zeta_{\bar{\bar{i}}}$$

is concircular and, therefore, in accordance with Corollary 3.2, our space admits $2F$PM(I).

4. Fundamental theorems of the theory of $2F$PM(I) of pseudo-Riemannian spaces with absolutely parallel f-structure

Let (V_n, g_{ij}, F^h_i) be a pseudo-Riemannian space with an absolutely parallel f-structure, so we know its metric tensor $g_{ij}(x)$ and the affinor of the f-structure $F^h_i(x)$. Then the problem of *whether this space (V_n, g_{ij}, F^h_i) allows the first type canonical $2F$-planar mappings* reduces to the study of equations (3.10) with respect to the tensor a_{ij} and the vectors $\overset{1}{\lambda}_i \neq 0$, $\overset{2}{\lambda}_i \neq 0$ satisfying the conditions (3.12), (3.13).

To solve that problem, consider the integrability conditions of the equations (3.10). Taking into account the Ricci identity, they will have the form:

$$a_{\alpha(j}R^{\alpha}_{i)kl} = Q_{(ij)[kl]}, \tag{4.1}$$

where

$$Q_{ijkl} = \overset{1}{\lambda}_{i,l}\,\overset{1}{F}_{jk} + \overset{2}{\lambda}_{i,l}\,\overset{2}{F}_{jk},$$

the square brackets denote the operation of alternation by the corresponding indices.

Hence, using suitable algebraic transformations, we find

$$2k\overset{1}{\lambda}_{i,l} = \nu\,\overset{1}{F}_{il} + \left(a_{\alpha\beta}R^{\alpha\beta}_{.i.\gamma} + a_{i\alpha}R^{\alpha}_{\gamma}\right)\overset{1}{F}{}_{l}^{\gamma} \tag{4.2}$$

where ν is some invariant.

The differential conditions on $\overset{2}{\lambda}_i$ are easily obtained from (3.12) and (4.2).

In equations (4.2) new unknown invariant ν has appeared, for which differential conditions must also be found. To do this, we write the integrability conditions of the last equation:

$$\begin{aligned} 2(k+1)\overset{1}{\lambda}_{\alpha}R^{\alpha}_{ilk} = {} & \left(\nu_{,[l} - \overset{1}{\lambda}_{\overline{\alpha}}R^{\alpha}_{[l}\right)\overset{1}{F}_{k]i} + \overset{1}{\lambda}_{\alpha}R^{\alpha}_{[l}\,\overset{2}{F}_{k]i} + \\ & + 2\overset{1}{\lambda}_{\bar{i}}R_{\bar{l}k} - a_{\alpha\beta}\left(R^{\alpha}_{\bar{i}lk,\,.}{}^{\beta} + \overset{1}{F}{}_{i}^{\beta}R^{\alpha\delta}_{..lk,\delta}\right). \end{aligned} \tag{4.3}$$

Hence, by certain algebraic transformations, we find that:

$$\nu_{,k} = 2\overset{1}{\lambda}_{\alpha}R^{\alpha}_{\bar{k}}. \tag{4.4}$$

Equations (3.10), (4.2) and (4.4) constitute a closed system of the first order partial differential equations of Cauchy type with respect to the unknown functions a_{ij}, $\overset{1}{\lambda}_i$, ν. Let us denote it by (A). In the theory of differential equations regular methods have been developed for such systems. Thus, we proved

Theorem 4.1. *A pseudo-Riemannian space* $(V_n, g_{ij}(x), F^h_i(x))$ *with an absolutely parallel f-structure admits a canonical $2F$-planar mapping of the first type, if and only if the system of differential equations* (A) *has a nontrivial solution*

$$a_{ij}(x)\ \left(= a_{ji}(x),\ \det\|a_{ij}(x)\| \neq 0\right), \qquad \overset{1}{\lambda}_i(x) \neq 0, \qquad \nu(x),$$

satisfying conditions (3.13).

From the theory of differential equations it is known that the system (A) has at most one solution for each set of Cauchy initial values

$$a_{ij}(x_\circ) = \underset{\circ}{a}_{ij}, \qquad \overset{1}{\lambda}_i(x_\circ) = \underset{\circ}{\overset{1}{\lambda}}_i, \qquad \nu(x_\circ) = \underset{\circ}{\nu},$$

therefore the number of arbitrary constants in the general solution of the equations (A) is limited. Note that this system is not always compatible and the existence of non-trivial solutions depends on whether the set of integrability conditions (A) and their differential extensions are compatible.

The integrality conditions of (3.10) are obtained from (4.1) after replacing the derivative of the vector $\overset{1}{\lambda}_i$ with the expressions (4.2) in the form:

$$a_{\alpha\beta} T^{\alpha\beta}_{ijkl} = 0, \tag{4.5}$$

where

$$T^{\alpha\beta}_{ijkl} = 2k(n-2k)\left(\delta^\beta_j R^\alpha_{.ikl} + \delta^\beta_i R^\alpha_{.jkl}\right) + \overset{1}{T}{}^{\alpha\beta}_{(ij)[kl]},$$
$$\overset{1}{T}{}^{\alpha\beta}_{ijkl} = \left(R^{\alpha}_{.i\gamma}{}^{\beta}_{.} - R^\alpha_\gamma \delta^\beta_i\right)\left((n-2k)\,\overset{1}{F}{}^{\gamma}_{l}\,\overset{1}{F}_{jk} + (2k\delta^\gamma_l + n\,\overset{2}{F}{}^{\gamma}_{l})\,\overset{2}{F}_{jk}\right).$$

We obtain the integrability conditions of (4.2) from (4.3) after replacing the derivatives of ν in them with the expressions (4.4) in the form:

$$a_{\alpha\beta} L^{\alpha\beta}_{ilk} + 2(k+1)\overset{1}{\lambda}_\alpha \tilde{Q}^{\alpha}_{il\bar{\bar{k}}} = 0,$$

where

$$L^{\alpha\beta}_{ilk} = R^{\alpha}_{\bar{i}lk,}{}^{\beta}_{.} + \overset{1}{F}{}^{\beta}_{i} R^{\alpha\delta}_{..lk,\delta},$$
$$\tilde{Q}^\alpha_{ilk} = g^{\alpha\beta} g_{i\gamma} Q^\gamma_{\beta lk},$$
$$Q^h_{ilk} = R^h_{ilk} - \tfrac{1}{2(k+1)}\left(\overset{1}{F}{}^{h}_{k} R_{i\bar{l}} - \overset{1}{F}{}^{h}_{l} R_{i\bar{k}} + 2\,\overset{1}{F}{}^{h}_{i} R_{k\bar{l}} + \overset{2}{F}{}^{h}_{k} R_{i\bar{\bar{l}}} - \overset{2}{F}{}^{h}_{l} R_{i\bar{\bar{k}}}\right), \tag{4.6}$$
$$k = \frac{1}{2} Rg\|F^h_i\| \neq 1.$$

Note that the tensor (4.6) is invariant under canonical $2F$PM of the first type. It was built by us in [4].

Finally, the integrability conditions of equations (4.4) are as follows:

$$a_{\alpha\beta}\left(R^{\alpha}_{\cdot\gamma[l}{}^{\beta}_{.} - \delta^\beta_\gamma R^\alpha_{[l}\right) R^{\overline{\gamma}}_{k]} + 2k\overset{1}{\lambda}_\alpha R^{\overline{\alpha}}_{[k,l]} = 0. \tag{4.7}$$

Let us denote the integrability conditions of the system (A), which we present in the form (4.5), (4.6)-(4.7), by (B), and their differential extensions by $(B_1), (B_2), (B_3), \ldots$. As we can see, $(B), (B_1), (B_2), (B_3), \ldots$ are a system of linear homogeneous algebraic equations with respect to $a_{ij}, \overset{1}{\lambda}_i$,

ν with coefficients from V_n. Since the number of unknown functions is limited, there will be a natural number s such that (B_s) and the following extensions will be the consequences of $(B), (B_1), (B_2), (B_3), \dots, (B_{s-1})$.

From the analytical theory of differential equations, for the system (A) there is a nontrivial solution in the neighborhood of the point $M_\circ$ if and only if the system of equations $(B), (B_1), (B_2), (B_3), \dots, (B_{s-1})$ have a nontrivial solution at this point.

The following theorem holds:

Theorem 4.2. *A pseudo-Riemannian space $(V_n, g_{ij}(x), F_i^h(x))$ with an absolutely parallel f-structure admits a canonical $2F$-planar mapping of the first type, if and only if the system of homogeneous algebraic equations $(B), (B_1), (B_2), (B_3), \dots, (B_{s-1})$ has in (V_n, g_{ij}, F_i^h) a nontrivial solution*

$$a_{ij}(x)\ (= a_{ji}(x),\ \det\|a_{ij}(x)\| \neq 0)\,, \qquad \overset{1}{\lambda}_i(x) \neq 0, \qquad \nu(x),$$

which satisfies the conditions (3.13)*.*

Theorems 4.1 and 4.2 can be regarded as the fundamental theorems of the theory of the first type canonical $2F$-planar mappings of pseudo-Riemannian spaces with absolutely parallel f-structure, since they give a regular method that allows for any such space (V_n, g_{ij}, F_i^h) either to find all pseudo-Riemannian spaces $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ onto which V_n admits $2F$PM(I)s, or to prove that there are no such spaces. However, it should be recognized that for large n directly solving of this problem may be technically quite difficult. Therefore, it is very important to develop other approaches to studying the problem of the existence of $2F$PM(I) and their features.

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