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Deformations of unduloid with stationary Ricci tensor

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Abstract. In this paper we consider first-order infinitesimal deformations of simply connected regular surfaces in three-dimensional Euclidean space with a stationary Ricci tensor. The search for the vector field of this deformation in the general case is reduced to the study and solution of a system of seven equations, including differential equations, with respect to seven unknown functions. It is proved that every regular surface of non-zero Gaussian and mean curvatures admits a first-order nonlinear deformation with a stationary Ricci tensor. The methods of tensor analysis, the theory of differential equations and their boundary value problems are used to solve the stated problems.

Анотація. В роботі розглядаються нескінченно малі деформації першого порядку однозв'язних регулярних поверхонь у тривимірному евклідовому просторі зі стаціонарним тензором Річчі. Пошук векторного поля цієї деформації в загальному випадку зводиться до дослідження та розв'язування системи семи рівнянь, серед яких є і диференціальні рівняння, відносно семи невідомих функцій. Доведено, що кожна регулярна поверхня ненульової гаусової та середньої кривин допускає н. м. деформацію першого порядку зі стаціонарним тензором Річчі. Для сформульованих задач використовуються методи тензорного аналізу, теорія диференціальних рівнянь та їх крайові задачі.

Introduction

Infinitesimal deformations of surfaces have been the subject of research by many scientists. The analysis of infinitesimal deformations allows us to create analytical solutions to problems of equilibrium, study the stability of systems, and formulate boundary conditions and criteria for material failure. In combination with experimental methods, this theory provides a deeper understanding of the behaviour of surfaces under stress.

In this paper, to avoid uncertainty, the following restrictions are imposed on a given surface: the Ricci tensor is preserved everywhere on the surface.

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To solve the problem, a mathematical model was created in the form of a system of seven equations with respect to seven unknown functions. It is shown that each solution of the obtained system of equations determines the displacement field of a first-order deformation of a surface of non-zero Gaussian curvature, which is well defined (up to a constant vector).

It has been substantiated that a first-order infinitesimal deformations with a stationary Ricci tensor exist on each regular surface of non-zero Gaussian and mean curvatures. We also explicitly describe tensor fields (depending on two functions), which are solutions of a linear inhomogeneous differential equation of the second order with partial derivatives.

The class of rigid surfaces with respect to the specified infinitesimal deformations is defined. Assuming that one of the functions is predefined, we obtain an inhomogeneous differential equation of hyperbolic type. It should be noted that in the case when the predefined function is a certain characteristic function (being a solution of the Weingarten equation), the problem is reduced to finding solutions of a homogeneous partial differential equation.

1. The mathematical model of the problem and its mechanical meaning

Let S be a simply connected regular surface of class C^3 in $\mathbb{R}^3$ with radius vector

$$\bar{r} = \bar{r}(x^1, x^2), \tag{1.1}$$

so r is a homeomorphism some domain G of the plane $\mathbb{R}^2$ onto S.

An infinitesimal deformation of a surface without any constraints is called a *general (infinitesimal) deformation.*

Consider the following first-order general deformation of S with a displacement vector $\bar{y}(x^1, x^2) \in C^3$, whose partial derivatives are given by:

$$\bar{y}_i = \left(c_{i\alpha}T^{\alpha\beta} - \mu\delta_i^\beta\right)\bar{r}_\beta + c_{i\alpha}T^\alpha\bar{n}, \tag{1.2}$$

where $c_{i\alpha}$ is the discriminant tensor:

$$c_{11} = c_{22} = 0, \qquad c_{12} = -c_{21} = \sqrt{g}, \qquad g = g_{11}g_{22} - g_{12}^2,$$

g_{ij} is a metric tensor, $\bar{r}_\alpha = \dfrac{\partial\bar{r}}{\partial x^\alpha}$, $\bar{n}$ (orthonormal on S) are basis vectors, and

$$\delta_i^\beta = \begin{cases} 1, & \text{if } i = \beta \\ 0, & \text{if } i \neq \beta. \end{cases}$$

Then the functions $T^{\alpha\beta}, T^{\alpha}, \mu(x^1, x^2)$ constitute the solution of the following system of equations [2, 12]:

$$\begin{cases} T^{\alpha\beta}_{,\alpha} = b^{\beta}_{\alpha} T^{\alpha} + \mu_{\alpha} c^{\beta\alpha}, \\ b_{\alpha\beta} T^{\alpha\beta} + T^{\alpha}_{,\alpha} = 0, \\ c_{\beta\alpha} T^{\alpha\beta} = 0, \end{cases} \tag{1.3}$$

where $b^{\beta}_{\alpha} = g^{\beta i} b_{i\alpha}$, $b_{i\alpha}$ are the coefficients of the second fundamental form of S, g^{ij} are the elements of the matrix inverse to the matrix $\|g_{ij}\|$, and $c^{ij} = c_{\alpha\beta} g^{\alpha i} g^{\beta j}$. The comma denotes the covariant differentiation based on g_{ij}, while the indices take the values $1, 2$ everywhere.

The system of equations (1.3) is called *basic equations* of a general infinitesimal first-order deformation of S. It contains four equations with respect to seven unknown functions $T^{\alpha\beta}, T^{\alpha}, \mu$.

To avoid uncertainty, we will introduce the following additional restrictions. Namely, we will require that the Ricci tensor is stationary under this deformation, i.e., its first variation is zero:

$$\delta R_{ij} = 0.$$

Each geometric quantity characterising a particular property of a surface receives a certain increment under deformation, which in the regular case can be decomposed into powers of some small parameter. The coefficients of this expansion are called respectively the *first, second*, and so on, *variations of the corresponding value.*

Theorem 1.1. *A first-order inifinitesimal deformation* (1.2) *of a surface S of nonzero Gaussian curvature* ($K \neq 0$) *preserves its Ricci tensor if and only if the following equalities are satisfied:*

$$\left(c_{i\alpha} g_{j\beta} + c_{j\alpha} g_{i\beta}\right) T^{\alpha\beta} + g_{ij} c_{\gamma\alpha} d^{\gamma\beta} T^{\alpha}_{,\beta} = 0, \tag{1.4}$$

where $d^{\alpha\beta} = \dfrac{1}{K} c^{\alpha i} c^{\beta j} b_{ij}$.

Proof. First note that $R_{ij} = -K g_{ij}$. Then varying these equalities, we obtain the variation of the Ricci tensor

$$\delta R_{ij} = -g_{ij} \delta K - 2K \epsilon_{ij}. \tag{1.5}$$

Using the expressions of the variations of δg_{ij} and δK via the tensor fields $T^{\alpha\beta}$, T^{α}, and the function μ from [3, 16], we obtain that

$$2\epsilon_{ij} = \delta g_{ij} = \left(c_{i\alpha} g_{j\beta} + c_{j\alpha} g_{i\beta}\right) T^{\alpha\beta} - 2\mu g_{ij},$$

$$\delta K = K c_{i\alpha} d^{i\beta} T^{\alpha}_{,\beta} + 2K\mu.$$

Then (1.5) can be rewritten as follows:

$$\delta R_{ij} = -K\left(c_{i\alpha}g_{j\beta} + c_{j\alpha}g_{i\beta}\right)T^{\alpha\beta} - Kg_{ij}c_{\gamma\alpha}d^{\gamma\beta}T^{\alpha}_{,\beta}. \tag{1.6}$$

Now, since $K \neq 0$, we get from (1.6) that the Ricci tensor is preserved, i.e. δR_{ij}, if and only if (1.4) holds. □

Corollary 1.2. *Suppose that a first-order inifinitesimal deformation* (1.2) *of a surface of nonzero Gaussian curvature preserves the Ricci tensor. Then*

$$c_{\gamma\alpha}d^{\gamma\beta}T^{\alpha}_{,\beta} = 0. \tag{1.7}$$

Proof. Wrap (1.4) with g^{ij}. Since $T^{\alpha\beta}$ is a symmetric tensor and $g_{ij}g^{ij} = 2$, we obtain (1.7). □

Corollary 1.3. *If a first-order inifinitesimal deformation* (1.2) *of a surface of nonzero Gaussian curvature preserves the Ricci tensor, then*

$$\delta K = 2K\mu. \tag{1.8}$$

Proof. Set the right hand side of (1.5) to zero and multiply by g^{ij}. Taking into account that $g^{ij}\epsilon_{ij} = -2\mu$, we obtain (1.8). □

Thus, the mathematical model of the problem will be the system of equations (1.3), (1.4) containing seven equations with respect to seven unknowns: the symmetric tensor $T^{\alpha\beta}$, the contravariant vector T^{α} and the function $\mu(x^1, x^2) \in C^2$. For any set of its solutions the displacement vector $\bar{y}(x^1, x^2)$ according to (1.2) will be of the form:

$$\bar{y}(M) = \int\limits_{M_0 M} c_{i\alpha}\Big(\big(T^{\alpha\beta} + \mu c^{\alpha\beta}\big)\bar{r}_{\beta} + T^{\alpha}\bar{n}\Big)\,dx^i + \bar{y}_0, \tag{1.9}$$

where the integral is calculated along any straight curve contained in S and connecting any fixed point M_0 with the variable point M, and $\bar{y}_0$ is a constant vector. Since S is simply connected, the curvilinear integral (1.9) does not depend on the integration path [8, 15]. Thus, the vector field $\bar{y}(x^1, x^2)$ is well-defined (up to a constant vector).

If $\bar{y} = const$, then the surface S is called *rigid* with respect to a given infinitesimal strain.

Comparing the basic system of equations of equilibrium of a loaded shell in a momentless stress state [14, 17]:

$$\tilde{T}^{\alpha\beta}_{,\alpha} + X^{\beta} = 0, \qquad b_{\alpha\beta}\tilde{T}^{\alpha\beta} + Z = 0, \qquad c_{\alpha\beta}\tilde{T}^{\alpha\beta} = 0$$

with the system of equations (1.3), we see that they coincide whenever the following conditions hold:

$$\tilde{T}^{\alpha i} = T^{\alpha i}, \qquad X^i = -b^i_{\alpha}T^{\alpha} + \mu_{\alpha}c^{\alpha i}, \qquad Z = T^{\alpha}_{,\alpha}. \tag{1.10}$$

This indicates that any infinitesimal deformation of S with $K \neq 0$ can be interpreted as a certain moment-free stress state of equilibrium of a loaded shell with a force tensor $T^{\alpha\beta}$ and a surface load

$$X = \left(\mu_\alpha c^{\alpha i} - b^i_\alpha T^\alpha\right) \bar{r}_i + Z\bar{n},$$

where Z is its normal component and T^α is the shear force.

1.4. **Study of the system of equations (1.3),(1.4).** In the previous section we shown that the problem of existence of infinitesimal deformations of surfaces of nonzero Gaussian curvature with a stationary Ricci tensor is reduced to finding solutions to the system of equations (1.3), (1.4). So we can formulate the following statement.

Theorem 1.5. *Any surface $S \in C^4$ of non-zero gaussian ($K \neq 0$) and mean ($H \neq 0$) curvatures admits an uniaxial deformation with a stationary Ricci tensor. The corresponding tensor fields of such a deformation will have the following representations:*

$$T^{\alpha\beta} = \varphi g^{\alpha\beta}, \tag{1.11}$$

$$T^k = \varphi_\alpha d^{\alpha k} + \mu_\alpha c^{\alpha\beta} d^k_\beta, \tag{1.12}$$

where the functions $\mu(x^1, x^2)$ and $\varphi(x^1, x^2)$ are of class C^3 and satisfy the following identity:

$$\left(d^{\alpha\beta}\varphi_\alpha\right)_{,\beta} + 2H\varphi = -\left(\mu_{\alpha,k} c^{\alpha\beta} d^k_\beta + \mu_\alpha c^{\alpha\beta}\left(d^k_\beta\right)_{,k}\right), \tag{1.13}$$

with $d^k_\beta = d^{k\alpha} g_{\alpha\beta}$.

If the functions $\mu(x^1, x^2)$ and $\varphi(x^1, x^2)$ are zero then the surface S is rigid.

Proof. Let S be a surface with $K \neq 0$ and $H \neq 0$. Let also (1.2) be an infinitesimal deformation of S with a displacement field $\bar{y}(x^1, x^2)$ such that the Ricci tensor of S is preserved.

Let us refer the surface S to the curvature lines ($b_{12} = g_{12} = 0$). Then the system of equations (1.3), (1.4), taking into account (1.7) will be written as follows:

$$\begin{cases} g_{11}T^{11} - g_{22}T^{22} = 0, \\ b_{11}T^{11} + b_{22}T^{22} = -T^\alpha_{,\alpha}, \\ T^{12} = T^{21} = 0. \end{cases}$$

Then, by Kramer's rule, its solution can be represented in tensor form as follows:

$$T^{\alpha\beta} = -\frac{T^\alpha_{,\alpha}}{2H} g^{\alpha\beta}.$$

Consider the following function

$$\varphi = -\frac{T^{\alpha}_{,\alpha}}{2H}. \tag{1.14}$$

Then the deformation tensor $T^{\alpha\beta}$ will take the form (1.11). Obviously, $T^{\alpha\beta}$ must satisfy the system of equations (1.3). From (1.11), we find the covariant derivative $T^{\alpha\beta}_{,\alpha}$ [4, 10] and substitute it into (1.3):

$$g^{\alpha\beta}\varphi_{\alpha} = b^{\beta}_{\alpha}T^{\alpha} + \mu_{\alpha}c^{\beta\alpha}.$$

Multiply these equalities by d^{k}_{β}. Since

$$b^{\beta}_{\alpha}d^{k}_{\beta} = \delta^{k}_{\alpha}, \qquad\qquad g^{\alpha\beta}d^{k}_{\beta} = d^{\alpha k},$$

we obtain the expression for T^{α} of the form (1.12).

Substitutin the expression (1.12) for $T^{\alpha}_{,\alpha}$ into (1.14) we get one second-order partial differential equation of the form (1.13) with respect to the unknown functions $\mu(x^1, x^2)$ and $\varphi(x^1, x^2)$.

Put

$$\mu = 0, \qquad \varphi = 0.$$

Then, according to (1.11) and (1.12), the displacement field $\bar{y} = const.$ This means that in this case the surface will be rigid with respect to the specified infinitesimal deformations. □

Thus, the search for solutions to the system of equations (1.3), (1.4) is reduced to the study and solution of a single differential equation of the form (1.13).

It should be noted that (1.13) is the differential equation with respect to the function $\varphi(x^1, x^2)$ under assumption that $\mu(x^1, x^2)$ is a predefined function, which was studied in [11, 13].

1.6. **Study of the equation** (1.13) **with respect to the function** $\mu(x^1, x^2)$**.** Assume that $\varphi(x^1, x^2)$ is a predefined function of a point of a surface of class C^3. Then the equation (1.13) in general form is a nonhomogeneous partial differential equation of second order with respect to the function $\mu(x^1, x^2)$:

$$\mu_{\alpha,k}c^{\alpha\beta}d^{k}_{\beta} + \mu_{\alpha}c^{\alpha\beta}\left(d^{k}_{\beta}\right)_{,k} = F(\varphi), \tag{1.15}$$

where

$$F(\varphi) = -\left(\left(d^{\alpha\beta}\varphi_{\alpha}\right)_{,\beta} + 2H\varphi\right).$$

Using the expressions for

$$\rho^{\alpha\beta} = \tfrac{1}{2}(c^{\alpha i}b^{\beta}_{i} + c^{\beta i}b^{\alpha}_{i})$$

and $d^{\alpha\beta}$, the previous equation can be represented in a different form:

$$\mu_{\alpha,\beta}\left(\rho^{\alpha\beta}+Hc^{\alpha\beta}\right)-K\mu_{\alpha}c^{\alpha\beta}\left(d^{k}_{\beta}\right)_{,k}=-KF(\varphi).$$

Consider the tensor $\rho^{\alpha\beta}+Hc^{\alpha\beta}$. Multiplying it by $c_{\alpha\beta}$ we get

$$c_{\alpha\beta}(\rho^{\alpha\beta}+Hc^{\alpha\beta})=2H\neq 0.$$

From this we can see that it is not symmetric. Then the equality is true:

$$\begin{aligned}(\rho^{\alpha\beta}+Hc^{\alpha\beta})\mu_{\alpha,\beta}&=\frac{1}{2}\big((\rho^{\alpha\beta}+Hc^{\alpha\beta})\mu_{\alpha,\beta}+(\rho^{\beta\alpha}+Hc^{\beta\alpha})\mu_{\beta,\alpha}\big)\\&=\rho^{\alpha\beta}\mu_{\alpha,\beta}.\end{aligned}$$

Using the definition of the covariant derivative [9], the previous equation will take the form

$$\mu_{\alpha\beta}\rho^{\alpha\beta}-(\Gamma^{s}_{\alpha\beta}\rho^{\alpha\beta}+Kc^{s\beta}\big(d^{k}_{\beta}\big)_{,k})\mu_{s}=-KF(\varphi). \tag{1.16}$$

Let us find the discriminant of this equation:

$$\frac{1}{2g}c_{\alpha\lambda}c_{\beta\mu}\rho^{\alpha\beta}\rho^{\lambda\mu}=-\frac{E}{g}<0,$$

where

$$E=H^{2}-K=\frac{(k_{2}-k_{1})^{2}}{4}>0$$

is the Euler difference, k_1, k_2 is the principal curvatures [3].

It implies that the equation (1.15) is a hyperbolic equation.

In particular, if $\varphi(x^1,x^2)$ is a certain characteristic function (i.e., a solution of the homogeneous Weingarten equation), then the equation (1.15) will become a homogeneous partial differential equation of second order of hyperbolic type with respect to the function $\mu(x^1,x^2)$:

$$\mu_{\alpha\beta}\rho^{\alpha\beta}-(\Gamma^{s}_{\alpha\beta}\rho^{\alpha\beta}+Kc^{s\beta}\big(d^{k}_{\beta}\big)_{,k})\mu_{s}=0. \tag{1.17}$$

1.7. **Geometric meaning of the functions $\mu(x^1,x^2)$ and $\varphi(x^1,x^2)$.** To find out the geometric meaning of the functions $\mu(x^1,x^2)$ and $\varphi(x^1,x^2)$ we will start from the expressions of variations of some geometric quantities under a common deformation. Note that at the first-order deformation, the variation of the normal is [15]

$$\delta\bar{n}=c_{\alpha\beta}T^{\alpha}\bar{r}^{\beta}.$$

Let us find $T^{\alpha}_{,\alpha}$ from here and substitute it into the equation (1.14):

$$\varphi=-\frac{c^{\alpha\beta}\bar{r}_{\alpha}(\delta\bar{n})_{,\beta}}{2H}. \tag{1.18}$$

From equation (1.8) we get that

$$\mu(x^1, x^2) = \frac{\delta K}{2K}. \tag{1.19}$$

Thus, the equalities (1.18) and (1.19) imply the geometric meaning of the functions $\mu(x^1, x^2)$ and $\varphi(x^1, x^2)$ respectively, when studying the problem.

1.8. **The Cauchy problem for an inhomogeneous differential equation with respect to the function $\mu(x^1, x^2)$.** Let S be a surface of class C^5 of nonzero gaussian and mean curvature. Then the second-order partial differential equation is a hyperbolic type equation with respect to the function $\mu(x^1, x^2)$, and the function $\varphi(x^1, x^2)$ is predefined. Provided that the surface S is related to the lines of curvature, the equation (1.16) will take the canonical form:

$$\mu_{12} + t\mu_1 + s\mu_2 = -\frac{K}{2\rho^{12}} F(\varphi), \tag{1.20}$$

where

$$t = -\frac{\left(\Gamma^1_{\alpha\beta}\rho^{\alpha\beta} + K\frac{1}{\sqrt{g}}(d^2_2)_{,2}\right)}{2\rho^{12}}, \qquad s = -\frac{\left(\Gamma^2_{\alpha\beta}\rho^{\alpha\beta} - K\frac{1}{\sqrt{g}}(d^1_1)_{,1}\right)}{2\rho^{12}}.$$

Let $J = (a; b) \subset \mathbb{R}$ be some open interval, $g\colon J \to \mathbb{R}$ be some C^1 function, and $\gamma = \{(x^1, g(x^1)) \mid x^1 \in J\} \subset \mathbb{R}^2$ be its graph. Thus γ is given by the equation $x^2 = g(x^1)$. Note also that γ intersects each horizontal line $x^2 = \text{const}$ in at most one point.

Let us fix two functions $\omega_0, \omega_1 : \gamma \to \mathbb{R}$ such that $\omega_0 \in C^2$ and $\omega_1 \in C^1$ and consider the following Cauchy problem: *find a solution to the equation* (1.2) *in some neighbourhood of γ satisfying the boundary conditions:*

$$\mu|_{x^2=g(x^1)} = \omega_0(x^1), \qquad \left.\frac{\partial\mu}{\partial x^2}\right|_{x^2=g(x^1)} = \omega_1(x^1). \tag{1.21}$$

Since there exists a unique solution to this problem [5, 9], the following theorem holds true:

Theorem 1.9. *Let S be a surface of class C^5 having nonzero Gaussian and mean curvatures. Then, under the boundary conditions* (1.21)*, it admits deformations with a stationary Ricci tensor in the class of C^2 surfaces. The tensor fields of such deformations depend on two functions each of one variable and a predefined function $\varphi(x^1, x^2)$ of class C^3.*

1.10. **Mechanical meaning of $\mu(x^1, x^2)$ and $\varphi(x^1, x^2)$.** It follows from equation (1.14) that $\varphi(x^1, x^2)$ is normalised by the mean curvature invariant T^α_α (up to a constant). This invariant is interpreted as the normal component of the assumed surface load.

Taking into account the equality of (1.10) and using (1.14), we get the following equation

$$\varphi = -\frac{Z}{2H},$$

which reveals the mechanical meaning of the function $\varphi(x^1, x^2)$.

In the case of a simply connected surface, our geometric problem of finding a deformation with a stationary Ricci tensor of the surface, is equivalent to the mechanical problem of finding the momentless stress state equilibrium of a shell with a median surface, provided that the surface load of this shell is expressed in terms of two functions $\mu(x^1, x^2)$ and $\varphi(x^1, x^2)$ being are the solution of equation (1.16) in the form

$$X = (2\mu_\alpha c^{\alpha i} - \varphi_\beta g^{\beta i})\bar{r}_i - 2H\varphi\bar{n}. \tag{1.22}$$

It means that the infinitesimal deformation with a stationary Ricci tensor describes the momentless equilibrium state of the shell with a surface load, which is determined by the tensor field (1.22).

2. Infinite deformations with stationary Ricci tensor of unduloid

Recall that a *Delaunay surface* is a surface of rotation of constant mean curvature. With the exception of a sphere, they are formed by roulettes when they are rotated around a straight line. Roulettes are formed by the foci of a parabola, ellipse, and hyperbola, that rolls without slipping along a straight line being the axis of rotation of a surface. These surfaces include 5 surfaces: catenoids, unduloids, nodoids, spheres and straight cylindrical surfaces of rotation. Theey are used in gas dynamics when studying the surfaces of soap films and bubbles [1, 5, 7].

The first and second fundamental forms of unduloids are given by [6]

$$I = du^2 + \frac{1}{2}\left(a^2 + c^2 + (c^2 - a^2)\sin\frac{2u}{a+c}\right)dv^2,$$

$$II = \frac{(c-a)\left(c - a + (a+c)\sin\dfrac{2u}{a+c}\right)}{(a+c)\left(a^2 + c^2 + (c^2 - a^2)\sin\dfrac{2u}{a+c}\right)}du^2 + \\ + \frac{1}{2}\left(a + c + (c-a)\sin\frac{2u}{a+c}\right)dv^2.$$

The following theorem is true

Theorem 2.1 ([6]). *The mean and gaussian curvatures of the unduloid are given by the following formulas:*

$$H = \frac{1}{a+c}, \qquad K = \frac{1 - \left(\dfrac{ac}{z^2(u)}\right)^2}{(a+c)^2},$$

where $z(u) = \sqrt{m \sin \Theta u + n}$, *and*

$$\Theta = \frac{2}{a+c}, \qquad n = \frac{c^2+a^2}{2}, \qquad m = \frac{c^2-a^2}{2}.$$

Let us give a geometric interpretation of the appearing a and c.

Note that when $\sin \Theta u$ changes from -1 to 1, then z takes on values between a and c.

Since $\sin \Theta u = -1$ corresponds to its minimum $z(u)$, and $\sin \Theta u = 1$ to its maximum, this means that $c \geqslant a$.

In the case of $c = a$, the unduloid degenerates into a cylinder or acquires a sphere (the smoothness of the unduloid is destroyed).

Therefore, in the following, we will assume that $c > a$. It should be noted that a is the major semi-axis, c is the focal semi-distance of the rolling ellipse, u is the parameter of the roulette ($u \in R$), and v is the angle of rotation ($0 \leqslant v < 2\pi$).

Consider an unduloid for which $a = \frac{1}{2}$ and $c = \frac{3}{2}$. Then

$$I = du^2 + \frac{5 + 4\sin u}{4} dv^2,$$

$$II = \frac{1 + 2\sin u}{5 + 4\sin u} du^2 + \frac{2 + \sin u}{2} dv^2,$$

$$H = \tfrac{1}{2}, \qquad K = \frac{2(1 + 2\sin u)(2 + \sin u)}{(5 + 4\sin u)^2}.$$

Consider the first-order infinitesimal deformation of an unduloid with a stationary Ricci tensor provided that the function $\varphi(u,v) = 0$. Then equation (1.15) will take the form:

$$\frac{\partial^2 \mu}{\partial u \partial v} - \frac{2\cos u}{1 + 2\sin u} \frac{\partial \mu}{\partial v} = 0, \tag{2.1}$$

where $u = x^1$ and $v = x^2$.

Theorem 2.2. *An unduloid admits an infinitesimal deformation with a stationary Ricci tensor provided that* $\varphi(u,v) = 0$. *Then the tensor fields of* *such a deformation are represented explicitly*

$$T^{11} = 0, \qquad T^{12} = 0, \qquad T^{22} = 0, \tag{2.2}$$

$$T^1 = -2D'(v)\sqrt{5 + 4\sin u}, \qquad T^2 = \frac{2D(v)\cos u + G'(u)}{\sqrt{5 + 4\sin u}}, \tag{2.3}$$

where $D(v), G(u)$ are arbitrary functions of one variable.

Proof. Note that (2.1) can be written in the following way:

$$\left(\frac{\partial \mu}{\partial u} - \frac{2\cos u}{1 + 2\sin u}\mu\right)_v = 0.$$

Hence

$$\frac{\partial \mu}{\partial u} - \frac{2\cos u}{1 + 2\sin u}\mu = B(u),$$

where $B(u)$ is an arbitrary function of variable u. The latter equation can be rewritten as follows:

$$\left(\frac{\mu}{1 + 2\sin u}\right)_u = \frac{B(u)}{1 + 2\sin u}.$$

Therefore

$$\mu(u, v) = D(v)(1 + 2\sin u) + G(u),$$

where

$$G(u) = (1 + 2\sin u)\int \frac{B(u)}{1 + 2\sin u}du.$$

Substituting now the expression for μ into (1.11) and (1.12) we get the formulas (2.2) and (2.3). □

For any arbitrary surface, we can define the following closed region (Riemannian domain) $\bar{T} \subset G$, in which there always exists a nonzero solution of the homogeneous Weingarten equation.

Let us check that for any surface this solution is the function

$$\varphi = \bar{n}\bar{c}, \tag{2.4}$$

where $\bar{c}$ is a constant vector.

Obviously, a non-zero vector $\bar{c}$ can always be chosen in such a way that the function $\varphi = \bar{n}\bar{c} > 0$ (< 0), will be distinct from zero everywhere in some region $\bar{T} \subset G$. The size of the region $\bar{T}$ depends on the spherical mapping of the surface. In the domain $\bar{T}$, the vectors $\bar{n}$ and $\bar{c}$, form an acute (obtuse) angle everywhere.

Now let us use the derivative equations of surface theory

$$\bar{r}_{\alpha,\beta} = b_{\alpha\beta}\bar{n}, \qquad \bar{n}_k = -b_k^{\alpha}\bar{r}_{\alpha}.$$

Taking them into account we obtain that

$$F(\varphi) = -\Big(\big(d^{\alpha\beta}\varphi_\beta\big)_{,\alpha} + 2H\varphi\Big) = -\Big(\big(d^{\alpha k}\bar{n}_k\bar{c}\big)_{,\alpha} + 2H\bar{n}\bar{c}\Big)$$
$$= \big(d^{\alpha k}b_k^s\bar{r}_s\bar{c}\big)_{,\alpha} - 2H\bar{n}\bar{c} = \left(g^{\alpha s}\bar{r}_s\bar{c}\right)_{,\alpha} - 2H\bar{n}\bar{c}$$
$$= g^{\alpha s}b_{s\alpha}\bar{n}\bar{c} - 2H\bar{n}\bar{c} = 2H\bar{n}\bar{c} - 2H\bar{n}\bar{c} = 0.$$

So, the following statement holds true

Theorem 2.3. *An unduloid admits infinitesimal deformation with a stationary Ricci tensor provided that the function $\varphi \in C^3$ is predefined in the form* (2.4)*. The tensor fields of such deformations will have the following representation:*

$$T^{\alpha\beta} = \bar{n}\bar{c}g^{\alpha\beta},$$
$$T^1 = -\bar{r}_1\bar{c} - 2D'(v)\sqrt{5+4\sin u},$$
$$T^2 = -\frac{4}{5+4\sin u}\bar{r}_2\bar{c} + \frac{2D(v)\cos u + G'(u)}{\sqrt{5+4\sin u}}.$$

Conclusions

Cauchy problem considered in the paper led to the following result: any regular surface of non-zero Gaussian and mean curves under certain boundary conditions admits deformations of surfaces with stationary Ricci tensors. The deformation tensors depend on two functions (each of which in turn depends on one variable) and on a predefined function.

As an example, we consider the infinitesimal deformation with a stationary Ricci tensor on the unduloid surface. It is proved that, under certain conditions, unduloids can undergo infinitesimal deformations with a stationary Ricci tensor.

The obtained results can be used both in further scientific research, and in the momentless theory of thin elastic shells in the calculation of their equilibrium [18, 19]. The developed methods can be applied to the study of deformations of other surfaces that we encounter in many real-world problems from designing buildings and bridges to calculating the stress-strain state of of machine parts.

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