



Dual Thurston norm of Euler classes of foliations on negatively curved 3-manifolds

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Abstract. In this paper we give an upper bound on the dual Thurston norm of the Euler class of an arbitrary smooth foliation \mathcal{F} of codimension one defined on a closed oriented 3-manifold M^3 of negative curvature, which depends on the constants bounding the injectivity radius $\text{inj}(M^3)$, the volume $\text{Vol}(M^3)$, the sectional curvature of M^3 and the modulus of the mean curvature of leaves of \mathcal{F} .

Анотація. В статті отримано верхню оцінку подвійної норми Терстона класу Ейлера довільного гладкого шарування \mathcal{F} розмірності один, визначеного на замкненому тривимірному орієнтованому многовиді M^3 від'ємної кривини, яка залежить від констант, обмежуючих радіус ін'єктивності $\text{inj}(M^3)$, об'єм $\text{Vol}(M^3)$, секційну кривину многовиду M^3 та модуль середньої кривини шарів шарування \mathcal{F} .

1. INTRODUCTION

Let (M^3, g) be a closed oriented Riemannian 3-manifold and \mathcal{F} be a transversely oriented C^∞ -smooth codimension one foliation on M^3 . Recall that a foliation \mathcal{F} is *taut* if its leaves are minimal submanifolds of M^3 for some Riemannian metric on M^3 . Sullivan [19] proved that a foliation is taut if and only if each leaf of \mathcal{F} is intersected by a transverse closed curve, which is equivalent to the condition that \mathcal{F} does not contain generalized Reeb components. Recall that a saturated (consisting of the leaves of \mathcal{F}) set A of a compact orientable foliated 3-manifold M^3 is called a *generalized Reeb component* if A is a connected 3-manifold with a boundary ∂A and any vector field transverse to \mathcal{F} and restricted to ∂A is directed either everywhere inwards or everywhere outwards from the generalized Reeb component A . In particular, the Reeb component (see [20]), i.e. a foliated solid torus, all of whose leaves except for the boundary are homeomorphic to the plane (see Fig. 3.1), is a generalized Reeb component. It is clear that ∂A consists of a finite set of compact leaves of the foliation \mathcal{F} . It is not difficult to show that ∂A is a family of tori, see [11].

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Recall that a 3-manifold M^3 is called *irreducible* if each embedded sphere bounds a ball in M^3 . In particular, $\pi_2(M^3) = 0$. Note that if M^3 admits a taut foliation, then M^3 is irreducible [14].

Thurston has proved [21] that for each closed embedded orientable surface $M^2 \subset M^3$ distinct from S^2 , the value of the Euler class $e(T\mathcal{F})$ on the class $[M^2]$ in the case of a taut \mathcal{F} satisfies the following estimate:

$$|e(T\mathcal{F})[M^2]| \leq -\chi(M^2). \tag{1.1}$$

Since any integer homology class in $H_2(M^3; \mathbb{Z})$ can be represented by a closed oriented surface (see subsection 2.7.1), the inequality above gives a bound for the possible values of the cohomology class $e(T\mathcal{F})$ on the generators of $H_2(M^3; \mathbb{Z})$. Therefore, the number of cohomology classes in $H^2(M^3; \mathbb{Z})$ realized as Euler classes $e(T\mathcal{F})$ of the tangent distribution to \mathcal{F} is finite. The inequality (1.1) means that the dual Thurston norm (see subsection 2.7.2) of the Euler class $e(T\mathcal{F})$ of a taut foliation \mathcal{F} on a closed oriented irreducible atoroidal (containing no incompressible tori) 3-manifold M^3 satisfies the following inequality:

$$\|e(T\mathcal{F})\|_{Th}^* \leq 1. \tag{1.2}$$

Remark 1.1. A closed 3-manifold M^3 of negative sectional curvature is an example of an irreducible atoroidal 3-manifold. Note that atoroidality follows from the fact that $\pi_1(M^3)$ is hyperbolic in the sense of Gromov and, therefore, does not contain a free abelian subgroup of rank 2 (see [12]).

Remark 1.2. Taut foliations play an important role in three-dimensional Seiberg-Witten theory. Namely, the Euler class $e(T\mathcal{F})$ of such a foliation can be interpreted as a monopole class [13].

In [2] we proved the following result.

Theorem 1.3. *Let $V_0 > 0$, $i_0 > 0$, $K_0 \geq 0$ be fixed constants, and let M^3 be a closed oriented Riemannian 3-manifold with the following properties:*

- (1) *the volume $\text{Vol}(M^3) \leq V_0$;*
- (2) *the sectional curvature K of M satisfies the inequality $K \leq K_0$;*
- (3) *$\min\{\text{inj}(M^3), \frac{\pi}{2\sqrt{K_0}}\} \geq i_0$, where $\text{inj}(M^3)$ is the injectivity radius of M^3 .*

Denote

$$H_0 = \begin{cases} \min \left\{ \frac{2\sqrt{3}i_0^2}{V_0}, \sqrt[3]{\frac{2\sqrt{3}}{V_0}} \right\}, & \text{if } K_0 = 0, \\ \min \left\{ \frac{2\sqrt{3}i_0^2}{V_0}, x_0 \right\}, & \text{if } K_0 > 0, \end{cases}$$

where x_0 is the root of the equation

$$\frac{1}{K_0} \operatorname{arccot}^2 \frac{x}{\sqrt{K_0}} - \frac{V_0}{2\sqrt{3}}x = 0.$$

Then any smooth transversely oriented codimension one foliation \mathcal{F} on M^3 , such that the modulus of the mean curvature H of its leaves satisfies the inequality $|H| < H_0$, should be taut. In particular, such a foliation \mathcal{F} has minimal leaves for some Riemannian metric on M^3 .

The constant H_0 in Theorem 1.3 may not be optimal. If we allow the constant H_0 to take an arbitrary positive real value, then the foliation \mathcal{F} , whose leaves satisfy the mean curvature condition $|H| \leq H_0$ may already contain (generalized) Reeb components. Assuming that the constant K_0 , which bounds the sectional curvature K of M^3 from above, can take arbitrary real values, the number of Reeb components of the foliation \mathcal{F} is bounded above as follows (see [4, 3]):

$$\{\text{number of Reeb components of } \mathcal{F}\} \leq \frac{4H_0V_0}{\sqrt{3}C_0^2},$$

where

$$C_0 = \begin{cases} 2 \min\{i_0, \frac{1}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}}\}, & \text{if } K_0 > 0 \\ 2 \min\{i_0, \frac{1}{H_0}\}, & \text{if } K_0 = 0 \\ 2 \min\{i_0, \frac{1}{\sqrt{-K_0}} \operatorname{arccoth} \frac{H_0}{\sqrt{-K_0}}\}, & \text{if } K_0 < 0 \text{ and } H_0 > \sqrt{-K_0} \\ 2i_0, & \text{if } K_0 < 0 \text{ and } H_0 \leq \sqrt{-K_0} \end{cases} \quad (1.3)$$

If the foliation \mathcal{F} has at least one non-simply connected leaf, then the constant C_0 has the following geometric meaning:

$$\operatorname{sys}(\mathcal{F}) \geq C_0,$$

where $\operatorname{sys}(\mathcal{F})$ (foliated systole) is the length of the shortest *integral loop* (i.e. a loop belonging to some leaf of the foliation \mathcal{F}) among all leaf-wise non-contractible (*essential*) integral loops. Such a loop exists and is a geodesic in the leaf (see [3]).

Remark 1.4. It follows from both the Reeb stability theorem (see [16]) and Rosenberg's theorem (see [17]) that, except for the following two cases below, the foliation \mathcal{F} contains a non-simply connected leaf:

- either \mathcal{F} is a fibration of $S^2 \times S^1$ by spheres $S^2 \times \{*\}$ (clearly, in this case M^3 is not irreducible);
- or \mathcal{F} is a foliation by planes (in this case $M^3 \cong T^3$, in particular, M^3 is not atoroidal).

In our recent work [4], we gave an upper bound on the L^2 -norm of the real Euler class $e(T\mathcal{F})$ of an arbitrary transversely oriented codimension one foliation \mathcal{F} , defined on a closed irreducible orientable Riemannian 3-manifold M^3 . Namely, the following theorem is proved:

Theorem 1.5 ([4]). *Let $V_0 > 0$, $i_0 > 0$, $H_0 > 0$, $k_0 \leq K_0$ be fixed constants such that $k_0 \leq 0$. Suppose (M^3, \mathcal{F}) is a closed oriented irreducible Riemannian 3-manifold equipped with a two-dimensional transversely oriented foliation \mathcal{F} , whose leaves have the modulus of mean curvature H bounded above by the constant H_0 . Assume also that M^3 satisfies the following conditions:*

- (1) *the volume $\text{Vol}(M^3) \leq V_0$;*
- (2) *the sectional curvature K of M satisfies the inequality $k_0 \leq K \leq K_0$;*
- (3) $\left\{ \begin{array}{ll} \min\{\text{inj}(M^3), \frac{\pi}{2\sqrt{K_0}}\}, & \text{if } K_0 > 0 \\ \text{inj}(M^3) & \text{if } K_0 \leq 0 \end{array} \right\} \geq i_0$, *where $\text{inj}(M^3)$ is the injectivity radius of M^3 .*

Then there exist a constant $\Lambda(k_0, V_0, i_0)$ such that

$$\|e(T\mathcal{F})\|_{L^2} \leq -\frac{3}{\pi}k_0\sqrt{V_0} + \frac{32H_0^2V_0^{\frac{3}{2}}}{3C_0^3}\Lambda, \tag{1.4}$$

where C_0 is given by the formula (1.3).

Remark 1.6. The constant $\Lambda(k_0, V_0, i_0)$ is taken from the inequality (2.6) below.

Remark 1.7. The latest version of the preprint [5] of article [4] corrects some inaccuracies. Specifically, in Section 3.2 of that preprint, we have slightly changed the presentation of the material, and in Section 3.3 (see Definition 3.10), we indicated that in the manifold N^2 obtained as a result of surgery, we discard the connected components homologous to zero.

In this paper we present the following result.

Theorem 1.8 (Main theorem). *Let (M^3, \mathcal{F}) be a closed oriented Riemannian 3-manifold equipped by a two-dimensional transversely oriented foliation \mathcal{F} , whose leaves have the modulus of mean curvature H bounded above by some constant $H_0 \geq 0$.*

Suppose also that M^3 satisfies the following conditions for some fixed constants $V_0 > 0$, $i_0 > 0$, $H_0 > 0$, and $k_0 \leq K_0 < 0$ bounding the volume $\text{Vol}(M^3)$, the sectional curvature K of M^3 and the injectivity radius $\text{inj}(M^3)$:

- (1) $\text{Vol}(M^3) \leq V_0$;
- (2) $k_0 \leq K \leq K_0$;
- (3) $\text{inj}(M^3) \geq i_0$.

Then the following statements hold.

- (a) *If $H_0 \leq \sqrt{-K_0}$, then \mathcal{F} is taut and $\|e(T\mathcal{F})\|_{T_h}^* \leq 1$.*

(b) If $H_0 > \sqrt{-K_0}$, then the dual Thurston norm is estimated from above as follows:

$$\|e(T\mathcal{F})\|_{Th}^* \leq 6 \frac{k_0}{K_0} \Lambda - \frac{64\pi H_0^2 V_0}{3K_0 C_0^3} \Lambda^2, \quad (1.5)$$

where Λ and C_0 are as in Theorem 1.5.

(c) If the metric on M^3 is hyperbolic ($k_0 = K_0 \equiv -1$), then

$$\|e(T\mathcal{F})\|_{Th}^* \leq 30 \frac{\sqrt{V_0}}{\sqrt{i_0}} + \frac{1600\pi H_0^2 V_0^2}{3C_0^3 i_0},$$

where

$$C_0 = \begin{cases} 2 \min\{i_0, \operatorname{arccoth} H_0\}, & \text{if } H_0 > 1 \\ 2i_0, & \text{if } H_0 \leq 1 \end{cases}$$

Corollary 1.9. If \mathcal{F} contains a compact leaf \mathcal{K} of genus $g > 1$, then

$$\frac{|\chi(\mathcal{K})|}{\|[\mathcal{K}]\|_{Th}} \leq 6 \frac{k_0}{K_0} \Lambda - \frac{64\pi H_0^2 V_0}{3K_0 C_0^3} \Lambda^2.$$

Remark 1.10. As follows from Remark 1.1 if \mathcal{K} is a torus, then it must be homologous to zero. Note also that \mathcal{K} cannot be a sphere, since in this case, by the Reeb stability theorem, $M^3 \cong S^2 \times S^1$ and \mathcal{F} is a foliation by spheres $S^2 \times \{*\}$. In particular, M^3 is not irreducible and does not admit a Riemannian metric of negative sectional curvature.

2. BACKGROUND MATERIAL

2.1. Comparison inequalities for Mean curvature. Recall the following comparison theorem for the normal curvatures.

Theorem 2.2 ([7, 22.3.2]). *Let M be a Riemannian manifold, $p \in M$ a point and $\beta: [0, r] \rightarrow M$ a radial geodesic of the ball $B(p, r)$ of radius r centered at p . Let also $\beta(r)$ be a point not conjugate with p along β . Suppose also that the radius r is such that there are no conjugate points in the space of constant curvature K_0 within the radius of length r . If at each point $\beta(t)$ the sectional curvatures K of the manifold M do not exceed K_0 , then the normal curvature k_n^S of the sphere $S(p, r)$ at the point $\beta(r)$ with respect to the normal $-\beta'$ is not less than the normal curvature k_n^0 of the sphere of radius r in the space of constant curvature K_0 .*

Let M^3 be a 3-manifold satisfying the conditions of Theorem 1.8. Note that all normal curvatures of the sphere $S(r) \subset M^3$ of radius r are positive, provided that $r < i_0$ and the normal to the sphere $S(r)$ is directed inside the ball $B(r)$ which it bounds¹. We will call such a normal *inward*.

¹The sphere $S(r)$ indeed bounds the ball, since by definition $r < \operatorname{inj}(M^3)$.

Definition 2.3. We will say that a hypersurface $S \subset M^3$ of the Riemannian manifold M^3 is a **supporting hypersurface** for the subset $A \subset M^3$ at the point $p \in \partial A \cap S$ with respect to the normal $n_p \perp T_p S$, if S cuts some spherical neighborhood B_p of the point p into two components, and $A \cap B_p$ is contained in the component to which the normal n_p is directed. We will say that the sphere $S(r) \subset M^3$ ($r < i_0$) is a **supporting sphere** for the set $A \subset M^3$ at the point $q \in A \cap S(r)$ if it is a supporting hypersurface for A at the point q with respect to the inward normal.

The following lemma is obvious.

Lemma 2.4 ([2, Lemma 4]). Assume that the sphere $S(r_0)$ ($r_0 < i_0$) is a supporting sphere for the surface $F \subset M^3$ at the point q . Then

$$k_n^S(v) \leq k_n^F(v) \quad \forall v \in T_q S(r_0),$$

where $k_n^S(v)$ and $k_n^F(v)$ denote corresponding normal curvatures of $S(r_0)$ and F at the point q in the direction v .

As a consequence of Lemma 2.4 and Theorem 2.2 we obtain the following inequalities at the touching point q :

$$0 < H_r^0 \leq H_r(q) \leq H(q), \tag{2.1}$$

where H_r^0 and H_r are the mean curvatures of the spheres $S(r)$ bounding the ball of radius r , $r < i_0$, in the space of constant curvature K_0 and the manifold M^3 respectively, and H is the mean curvature of the surface F .

2.5. Novikov’s theorem and vanishing cycle. Let (M^3, \mathcal{F}) be a foliated closed 3-manifold. An integral loop $\alpha: S^1 \rightarrow M^3$ is a **vanishing cycle** if there exists a homotopy $A: S^1 \times I \rightarrow M^3$ through integral loops $A_t := A|_{S^1 \times t}$ for \mathcal{F} such that $A_0 = \alpha$ and A_t is inessential (i.e. is contractible in the supporting leaf) for $0 < t \leq 1$. A vanishing cycle α is **non-trivial** if α is essential. The following well-known theorem by Novikov provides a topological criterion for the existence of a Reeb component in a foliation \mathcal{F} .

Theorem 2.6 ([14]). For a closed orientable smooth 3-manifold M^3 and a transversely orientable C^2 -smooth codimension one foliation \mathcal{F} on M^3 , the following conditions are equivalent.

- (1) The foliation \mathcal{F} has a Reeb component.
- (2) There is a leaf \mathcal{L} of \mathcal{F} that is not π_1 -injective. That is, the inclusion $i: \mathcal{L} \rightarrow M^3$ induces a homomorphism $i_*: \pi_1(\mathcal{L}) \rightarrow \pi_1(M^3)$ with nontrivial kernel.
- (3) Some leaf of \mathcal{F} contains a non-trivial vanishing cycle.

The support of the non-trivial vanishing cycle is a torus bounding a Reeb component.

2.7. Norms on cohomology.

2.7.1. *L²-norm on cohomology and harmonic forms.* Let M^3 be a closed oriented Riemannian 3-manifold. Recall that

$$H^1(M^3; \mathbb{Z}) \cong [M^3, S^1] \text{ (homotopy classes of mappings to the circle)}$$

and each cohomology class

$$a \in H^1(M^3; \mathbb{Z}) \tag{2.2}$$

can be obtained as an image of the generator

$$[S^1]^* \in H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$$

under the homomorphism

$$f^*: H^1(S^1; \mathbb{Z}) \rightarrow H^1(M^3; \mathbb{Z})$$

induced by the mapping $f: M^3 \rightarrow S^1$ uniquely defined up to homotopy. Recall that the group

$$H_2(M^3; \mathbb{Z}) \stackrel{PD}{\cong} H^1(M^3; \mathbb{Z})$$

does not contain torsion and we can identify $H^1(M^3; \mathbb{Z})$ with the integer lattice

$$H^1(M^3; \mathbb{Z})_{\mathbb{R}} \subset H^1(M^3; \mathbb{R})$$

and $H_2(M^3; \mathbb{Z})$ with the integer lattice

$$H_2(M^3; \mathbb{Z})_{\mathbb{R}} \subset H_2(M^3; \mathbb{R})$$

respectively. Observe that the Poincaré duality

$$H^1(M^3; \mathbb{R}) \stackrel{PD}{\cong} H_2(M^3; \mathbb{R})$$

induces the Poincaré duality of integer lattices

$$H^1(M^3; \mathbb{Z})_{\mathbb{R}} \stackrel{PD}{\cong} H_2(M^3; \mathbb{Z})_{\mathbb{R}}.$$

Let us identify S^1 with the unit length circle \mathbb{R}/\mathbb{Z} equipped with the natural parameter θ . If f is a smooth function, then the preimage $f^{-1}(\theta)$ of a regular value $\theta \in S^1$ is a smooth (not necessarily connected) oriented submanifold $M^2 \subset M^3$, representing the class $[M^2] \in H_2(M^3; \mathbb{Z})_{\mathbb{R}}$ dual to the cohomology class a (see (2.2)). Let us use the isomorphism between singular cohomology with real coefficients and de Rham cohomology. Recall that each homotopy class in $[M^3, S^1]$ can be represented by a harmonic mapping (see [9]). Let

$$u: M^3 \rightarrow S^1$$

be a harmonic map representing the nontrivial class

$$[u] \in [M^3, S^1] \cong H^1(M^3; \mathbb{Z}).$$

Observe that

$$\alpha = u^*d\theta, \quad \theta \in S^1,$$

is a harmonic 1-form ($d\alpha = \delta\alpha = 0$) on M^3 corresponding to the integer lattice class $[u] \in H^1(M^3; \mathbb{Z})_{\mathbb{R}}$. Indeed, locally, the Riemannian Laplacian $\Delta_g u = d\delta u + \delta du = \delta du$ of the function u is zero iff $du = u^*d\theta$ is a harmonic form.

On the space of differential k -forms $\Omega^k(M^3)$, $k \in \{0, 1, 2, 3\}$, one can introduce L^2 -norm:

$$\|\alpha\|_{L^2} = \sqrt{\int_{M^3} \alpha \wedge *\alpha} = \sqrt{\int_{M^3} |\alpha|^2}, \quad (2.3)$$

where $*$ denotes the Hodge star operator and

$$|\alpha_p| = \sqrt{*(\alpha_p \wedge *\alpha_p)}, p \in M^3.$$

Note that in 3-dimensional vector space $T_p M^3$ each k -form α_p is simple and $|\alpha_p|$ coincides with the comass norm:

$$|\alpha_p| = \max \alpha_p(e_1, \dots, e_k),$$

where the maximum is taken over all orthogonal frames of vectors (e_1, \dots, e_k) in $T_p M^3$.

We will also use the L^∞ -norm on $\Omega^*(M^3)$, which is defined as follows:

$$\|\alpha\|_{L^\infty} := \max_{p \in M^3} |\alpha_p|.$$

The norm (2.3) induces the L^2 -norm on the de Rham cohomology of M^3 as follows. For $a \in H^k(M^3; \mathbb{R})$ let

$$\|a\|_{L^2} :=$$

$$\inf_{\alpha} \{\|\alpha\|_{L^2} : \alpha \in \Omega^k(M^3) \text{ is a smooth closed } k\text{-form representing } a\}.$$

Denote by $\mathcal{H}^1(M^3)$ the subspace of harmonic 1-forms in $\Omega^1(M^3)$. From de Rham–Hodge theory it follows that $\|a\|_{L^2} = \|\alpha\|_{L^2}$, where $\alpha \in \mathcal{H}^1(M^3)$ is the unique harmonic form representing the class $a \in H^1(M^3; \mathbb{R})$.

Using Poincaré duality $H_i(M^3; \mathbb{R}) \stackrel{PD}{\cong} H^{3-i}(M^3; \mathbb{R})$ we can introduce the L^2 -norm on $H_2(M^3; \mathbb{R})$ by setting

$$\|b\|_{L^2} := \|PD(b)\|_{L^2}, b \in H_2(M^3; \mathbb{R}).$$

On the other hand, the non-degenerate Kronecker pairing

$$\langle \cdot, \cdot \rangle: H^k(M^3; \mathbb{R}) \times H_k(M^3; \mathbb{R}) \rightarrow \mathbb{R}$$

induced by integration of closed forms over cycles, allows us to define the L^2 -norm $\|\cdot\|_{L^2}^*$ on $H_k(M^3; \mathbb{R}) \cong (H^k(M^3; \mathbb{R}))^*$ dual to the L^2 -norm $\|\cdot\|_{L^2}$ on $H^k(M^3; \mathbb{R})$. As was shown in [1]

$$PD: (H^i(M^3; \mathbb{R}), \|\cdot\|_{L^2}) \rightarrow (H_{3-i}(M^3; \mathbb{R}), \|\cdot\|_{L^2}^*)$$

is an isometry for $i = 1, 2$.

Note that

$$PD([\alpha \wedge \beta]) = PD([\beta \wedge \alpha]) = \langle [\alpha], PD([\beta]) \rangle = \langle [\beta], PD([\alpha]) \rangle,$$

where $\alpha \in \Omega^1(M^3)$ and $\beta \in \Omega^2(M^3)$ are closed forms. Since the set of integer-directed rays from $0 \in H^1(M^3; \mathbb{R})$ is everywhere dense in $H^1(M^3; \mathbb{R})$, we have that

$$\|b\|_{L^2} = \|PD(b)\|_{L^2}^* = \sup_{a \neq 0} \frac{\langle a, PD(b) \rangle}{\|a\|_{L^2}} = \sup_{[\Sigma] \neq 0} \frac{\langle b, [\Sigma] \rangle}{\|[\Sigma]\|_{L^2}}, \quad (2.4)$$

where $b \in H^2(M^3; \mathbb{R})$, $a \in H^1(M^3; \mathbb{Z})_{\mathbb{R}}$ and Σ is a compact oriented surface embedded in M^3 such that $PD(a) = [\Sigma]$.

Let us recall the following inequality (see [15, 7.1.13, 7.1.17, 9.2.7, 9.2.8]). If α is a harmonic 1-form on a closed Riemannian manifold M^n , then

$$\|\alpha\|_{L^\infty} \leq \Lambda_n(k, D) \|\alpha\|_2. \quad (2.5)$$

Here $\|\alpha\|_2 = \frac{\|\alpha\|_{L^2}}{\sqrt{\text{Vol}(M^n)}}$, $D > 0$ is a constant satisfying the inequality $\text{Diam}(M^n) \leq D$ and $k \leq 0$ is a constant satisfying the inequality

$$\text{Ric}(M^n) \geq (n-1)k.$$

Croke in [8] gave an estimate for the diameter of a closed Riemannian manifold, which for the 3-dimensional case can be adapted as follows:

$$\text{Diam}(M^3) \leq \frac{27\pi \text{Vol}(M^3)}{8 \text{inj}(M^3)^2}.$$

In particular, if M^3 satisfies the conditions of Theorem 1.8 we can take

$$D = \frac{27}{8} \pi \frac{V_0}{i_0^2}.$$

Moreover, we can put² $k = k_0$. Then $\Lambda_3(k, D) = \Lambda(V_0, i_0, k_0)$, and we can rewrite the inequality (2.5) in our case as follows:

$$\|\alpha\|_{L^\infty} \leq \frac{\Lambda(V_0, i_0, k_0)}{\sqrt{\text{Vol}(M^3)}} \|\alpha\|_{L^2}, \alpha \in \mathcal{H}^1(M^3). \quad (2.6)$$

²Recall that $k_0 < 0$ by the condition of Theorem 1.8.

In the case where M^3 is hyperbolic, Brock and Dunfield [6] established the following inequality:

$$\|\alpha\|_{L^\infty} \leq \frac{5}{\sqrt{\text{inj}(M^3)}} \|\alpha\|_{L^2}, \alpha \in \mathcal{H}^1(M^3). \tag{2.7}$$

2.7.2. *Thurston norm.* The Thurston norm on $H_2(M^3; \mathbb{Z})$ is defined in [21] as follows:

$$\|a\|_{Th} := \inf_{\Sigma} \{\chi_-(\Sigma) : \Sigma \text{ is an embedded surface representing } a \in H_2(M^3; \mathbb{Z})\},$$

where $\chi_-(\Sigma) = \max\{-\chi(\Sigma), 0\}$. Recall that $\chi(\Sigma) = 2 - 2g$ denotes the Euler characteristic of a surface Σ of genus g . When Σ is not connected, define $\chi_-(\Sigma)$ to be the sum

$$\chi_-(\Sigma_1) + \dots + \chi_-(\Sigma_k),$$

where $\Sigma_i, i = 1, \dots, k$ are the connected components of Σ . As Thurston showed, the Thurston norm can be extended in a unique way to the seminorm in $H_2(M^3, \mathbb{R})$.

Suppose that M^3 is an irreducible atoroidal oriented 3-manifold, e.g. a closed oriented 3-manifold of negative curvature. Then $\|\cdot\|_{Th}$ is a norm. In this case we can define the dual Thurston norm as follows:

$$\|\alpha\|_{Th}^* = \sup_{\Sigma} \frac{\langle \alpha, [\Sigma] \rangle}{\|[\Sigma]\|_{Th}},$$

where $\alpha \in H^2(M^3, \mathbb{R})$ and the supremum being taken over all compact oriented surfaces Σ embedded in M^3 such that $0 \neq [\Sigma] \in H_2(M^3; \mathbb{Z})$.

Thurston proved that the convex hull of the Euler classes of taut foliations on M^3 is the unit ball for the dual Thurston norm. In particular, the dual Thurston norm $\|e(T\mathcal{F})\|_{Th}^* \leq 1$ for the taut foliation \mathcal{F} .

Define the Thurston norm on $H^1(M^3; \mathbb{R})$ by setting

$$\|a\|_{Th} := \|PD(a)\|_{Th}, \quad a \in H^1(M^3; \mathbb{R}).$$

Let $u: M^3 \rightarrow S^1$ be a harmonic map representing the nontrivial class

$$[u] \in [M^3, S^1] \cong H^1(M^3; \mathbb{Z}) \stackrel{PD}{\cong} H_2(M^3; \mathbb{Z}).$$

In [18] Stern estimates an average Euler characteristic of a surface dual to the harmonic mapping of M^3 into the circle, which made it possible to estimate Thurston’s norm from above:

$$\|\alpha\|_{Th} \leq \int_{\theta \in S^1} -\chi(\Sigma_\theta) \leq \frac{1}{4\pi} \|\alpha\|_{L^2} \|R^-\|_{L^2},$$

where $R^- := \min\{0, R\}$ is a negative part of the scalar curvature R and $\alpha = u^*d\theta$.

Remark 2.8. A similar estimate was obtained by Kronheimer and Mrowka in [13] for the dual Thurston norm on the space of 2-dimensional cohomology $H^2(M^3; \mathbb{R})$ of an irreducible atoroidal 3-manifold M^3 :

$$\|e\|_{Th}^* \geq 4\pi \frac{\|e\|_{L^2}}{\|R\|_{L^2}}, \quad e \in H^2(M^3; \mathbb{R}).$$

Moreover, they proved, that

$$\|e\|_{Th}^* = \sup_{g \in \mathcal{G}} 4\pi \frac{\|e\|_{L^2}}{\|R\|_{L^2}},$$

where \mathcal{G} is the space of all Riemannian metrics on M^3 .

Kronheimer and Mrowka also showed that if $e \in H^2(M^3; \mathbb{R})$ is a monopole class arising in 3-dimensional Seiberg–Witten theory, then

$$4\pi \frac{\|e\|_{L^2}}{\|R\|_{L^2}} \leq 1$$

for all Riemannian metrics on M^3 . In particular, $\|e\|_{Th}^* \leq 1$.

In the case, where M^3 is hyperbolic, the relation between the Thurston's norm and the L^2 -norm on $H^1(M^3; \mathbb{R})$ was obtained by Brock and Dunfield [6]:

$$\frac{\pi}{\sqrt{\text{Vol}(M^3)}} \|*\|_{Th} \leq \|*\|_{L^2} \leq \frac{10\pi}{\sqrt{\text{inj}(M^3)}} \|*\|_{Th}. \quad (2.8)$$

3. PROOF OF MAIN THEOREM

Proof of Part (a). According to (1.2), it is enough to prove that \mathcal{F} is taut if $H_0 \leq \sqrt{-K_0}$.

Suppose \mathcal{F} is not taut. Then it contains a generalized Reeb component A , which is bounded by tori (see section 1). Since a closed negative curvature 3-manifold has a hyperbolic fundamental group, it does not contain \mathbb{Z}^2 as a subgroup and hence does not contain an incompressible torus. Thus, by Theorem 2.6 \mathcal{F} contains a Reeb component \mathcal{R} .

Let us consider the universal covering $p: \widetilde{M}^3 \rightarrow M^3$ with the pull back Riemannian metric on \widetilde{M}^3 . By the Cartan–Hadamard theorem, the exponential map

$$\exp: T_x \widetilde{M}^3 \rightarrow \widetilde{M}^3$$

is a diffeomorphism.

Consider the following commutative diagram describing the embedding $i: \mathcal{R} \rightarrow M^3$ and its lift to the universal coverings:

$$\begin{array}{ccc} \tilde{\mathcal{R}} & \xrightarrow{\tilde{i}} & \tilde{M}^3 \\ p \downarrow & & \downarrow p \\ \mathcal{R} & \xrightarrow{i} & M^3 \end{array}$$

Let also $\bar{\mathcal{R}} := \tilde{i}(\tilde{\mathcal{R}})$ be the image of that lift.

Case 1. The lift $\bar{\mathcal{R}}$ of \mathcal{R} to \tilde{M}^3 is homeomorphic to the solid torus $D^2 \times S^1$. In this case we can find a ball $B(r) \subset \tilde{M}^3$ of radius r containing $\bar{\mathcal{R}}$ such that its bounding sphere $S(r) = \partial B(r)$ touches the torus $T^2 = \partial \bar{\mathcal{R}}$ at some point q . It is clear that $S(r)$ is a supporting sphere for T^2 at q . By (2.1) we have that

$$H_r(q) \leq H(q),$$

where $H_r(q)$ and $H(q)$ are mean curvatures of $S(r)$ and T^2 at the point q . But for all $r > 0$ we have that

$$H_r(q) \geq H_r^0 = \sqrt{-K_0} \coth(r\sqrt{-K_0}) > \sqrt{-K_0},$$

where H_r^0 is the mean curvature of the round sphere $S(r)$ of radius r in the space of constant curvature K_0 (see (2.1)).

This contradicts the assumption on H_0 in the Part (a).

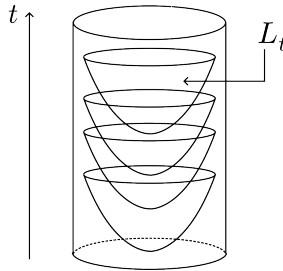


FIGURE 3.1. The universal covering $\tilde{\mathcal{R}}$

Case 2. The lift $\bar{\mathcal{R}}$ of \mathcal{R} to \tilde{M}^3 is homeomorphic to the cylinder $D^2 \times \mathbb{R}$. Consider the disk $D \subset \bar{\mathcal{R}}$ corresponding to the disk

$$D^2 \times 0 \subset D^2 \times \mathbb{R}$$

by the homeomorphism $\bar{\mathcal{R}} \simeq D^2 \times \mathbb{R}$. Let also

$$B(r) \subset \tilde{M}^3$$

be a ball such that $D \subset B(r)$ and its boundary $S(r) = \partial B(r)$ is in general position with respect to the $\partial \bar{\mathcal{R}}$. In this case the intersection $S(r) \cap \partial \bar{\mathcal{R}}$

is the family of circles $C = \{C_i\}$. Let A be the connected component of the compact $B(r) \cap \bar{\mathcal{R}}$ containing D . Note that the intersection $\partial A \cap S(r)$ is not necessarily a connected surface S , whose boundary consists of the circles in the family C . Since $B(r) \cap \bar{\mathcal{R}}$ is compact, we have a non-trivial decomposition $S = S_1 \sqcup S_2$ into the two disjoint not necessarily connected compact surfaces $\{S_1, S_2\}$ belonging to the different connected components of $\bar{\mathcal{R}} \setminus D$.

Observe that the lifted foliation of

$$\text{int } \bar{\mathcal{R}} \cong \mathbb{R}^2 \times \mathbb{R}$$

is a direct product foliation of $\mathbb{R}^2 \times \mathbb{R}$ by planes $L_t \simeq \mathbb{R}^2 \times t$. Without loss of generality, we assume that a leaf L_t goes to infinity if $t \rightarrow +\infty$ (see Fig. 3.1). Then we can find the parameters t_1, t_2 such that $L_{t_i} \cap S_i \neq \emptyset$, $i = 1, 2$, and $L_t \cap S_i = \emptyset$ for $t > t_i$, $i = 1, 2$. From the definition of S_1 and S_2 we conclude that one of the leaves L_{t_i} , $i = 1, 2$, for instance L_{t_1} , must intersect D and $S(r)$ is a supporting sphere for it at the touch points $S_1 \cap L_{t_1}$. The same reasoning as in Case 1 proves that the mean curvature of the leaf L_{t_1} should be greater than $\sqrt{-K_0}$ at the touch points, which leads to a contradiction. This completes the proof of Part (a). \square

Proof of Part (b). Let $0 \neq \alpha \in \mathcal{H}^1(M^3)$. Choose a surface Σ that is dual to $[\alpha] \in H^1(M^3, \mathbb{R})$, incompressible, and realizing the Thurston norm, i.e.

$$\|[\alpha]\|_{Th} = -\chi(\Sigma).$$

Since Σ is incompressible, we can assume by [10] that Σ has least area in its isotopy class and hence it is a stable minimal surface.

Lemma 3.1. *The following inequality holds:*

$$\text{Area}(\Sigma) \leq \frac{2\pi\chi(\Sigma)}{K_0}. \quad (3.1)$$

Proof. Since $K_\Sigma \leq K|_\Sigma$ for a minimal surface, where K_Σ denotes the Gauss curvature of Σ , the Gauss-Bonnet theorem yields

$$-K_0 \text{Area}(\Sigma) \leq \int_\Sigma -K_\Sigma = -2\pi\chi(\Sigma). \quad \square$$

Lemma 3.2. $\|\alpha\|_{L^2} \leq -\frac{2\pi\Lambda}{K_0\sqrt{\text{Vol}(M^3)}}\|\alpha\|_{Th}$.

Proof. We have that

$$\begin{aligned} \|\alpha\|_{L^2}^2 &= \int_{M^3} \alpha \wedge *\alpha \stackrel{PD}{=} \int_{\Sigma} *\alpha \\ &\leq \int_{\Sigma} |*\alpha| dA = \int_S |\alpha| dA \\ &\leq \int_{\Sigma} \|\alpha\|_{\infty} dA \leq \|\alpha\|_{\infty} \text{Area}(\Sigma) \\ &\stackrel{(3.1)+(2.6)}{\leq} \frac{2\pi\chi(\Sigma)\Lambda}{K_0\sqrt{\text{Vol}(M^3)}} \|\alpha\|_{L^2}. \end{aligned}$$

Recalling that $\|[\alpha]\|_{Th} := \|[\Sigma]\|_{Th} = -\chi(\Sigma)$ the result follows. □

Taking $V_0 = \text{Vol}(M^3)$, from (1.4) and Lemma 3.2 we obtain that:

$$\begin{aligned} \|e(T\mathcal{F})\|_{Th}^* &= \sup_{[\Sigma] \neq 0} \frac{\langle e(T\mathcal{F}), [\Sigma] \rangle}{\|[\Sigma]\|_{Th}} \\ &\leq \sup_{[\Sigma] \neq 0} -\frac{2\pi\Lambda}{K_0\sqrt{\text{Vol}(M^3)}} \frac{\langle e(T\mathcal{F}), [\Sigma] \rangle}{\|[\Sigma]\|_{L^2}} \\ &\leq 6\frac{k_0}{K_0}\Lambda - \frac{64\pi H_0^2 \text{Vol}(M^3)}{3K_0C_0^3}\Lambda^2. \end{aligned}$$

Now (1.5) follows for arbitrary $V_0 \geq \text{Vol}(M^3)$. This proves Part (b) of Theorem 1.8. □

Proof of Part (c). In [4] the following estimate was obtained:

$$|e(T\mathcal{F})([\Sigma])| \leq \frac{1}{2\pi} \|\alpha\|_{L^2} \|R^-\|_{L^2} + \frac{32H_0^2V_0^2}{3C_0^3} \|\alpha\|_{L^\infty}. \tag{3.2}$$

Applying (2.8) and (2.7) to (3.2) and, recalling that $R^- = -6$, we obtain:

$$\|e(T\mathcal{F})\|_{Th}^* \leq 30\frac{\sqrt{\text{Vol}(M)}}{\sqrt{\text{inj}(M)}} + \frac{H_0^2V_0^2}{3C_0^3} \frac{1600\pi}{\text{inj}(M)}.$$

that yields Part (c) of Theorem 1.8. □

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