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Estimates of the functional with fixed points on hyperbola branches

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Abstract. In the paper an extremal problem of geometric function theory of a complex variable on the maximum of product of the inner radii on a system of n mutually non-overlapping multiply connected domains B_k containing the fixed points a_k, $k = \overline{1, n}$, located on hyperbola branches is investigated.

Анотація. В роботі досліджується екстремальна задача геометричної теорії функцій комплексної змінної про максимум добутку внутрішніх радіусів на системі n взаємно неперетинних багатозв'язних областей B_k відносно фіксованих точок a_k, $k = \overline{1, n}$, розташованих на гілках гіперболи.

1. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{C}$ be the complex plane, and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one point compactification.

Consider a simply connected domain $B \subset \overline{\mathbb{C}}$ and a finite point $a \in B$. Let $f(z)$ be a meromorphic function in the disk $|z| < 1$. Suppose that f univalently map that disk $|z| < 1$ onto the domain B such that $f(0) = a$. Then the quantity $R(B, a) = |f'(0)|$ is called the *conformal radius* of the domain B at the point a.

The concept of a conformal radius has the following geometric meaning. By Riemann mapping theorem, for every simply connected domain $B \subset \mathbb{C}$ and a point $a \in B$ there exists a unique regular function $f(z)$ that univalently maps the given domain onto an open disk centered at the origin such that $f(a) = 0$ and $f'(a) = 1$. The radius of the formed disk will be the conformal radius of the domain B at the point a.

An analogue of the concept of a conformal radius for a multiply connected domain is the concept of an inner radius.

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A harmonic function $G_B\colon B\backslash\{a\} \to \mathbb{R}$ is called a *Green's function* of the domain B with a pole at the point $a \in B$ whenever G_B tends to zero when z goes to the boundary of B, and for which the following asymptotic equality holds on some neighborhood of a:

$$G_B(z,a) = \begin{cases} -\ln|z-a| + \gamma + o(1), & z \to a, \quad a \text{ is a finite point}, \\ \ln|z| + \gamma + o(1), & z \to \infty, \quad a = \infty. \end{cases}$$

In this case the quantity

$$r(B,a) := e^{\gamma}$$

is called the *inner radius* of the domain B with respect to the point a, (see, e.g., [1, 10, 12–14, 17]).

Note that for simply connected domains the inner radius of the domain is equal to its conformal radius.

In 1934 Lavrentiev [16] solved the problem on maximum of the product of conformal radii of two non-overlapping simply connected domains. Namely, the following result is valid.

Theorem 1.1 ([16]). *Let a_1 and a_2 be some fixed points of the complex plane $\mathbb{C}$. Then for arbitrary non-overlapping simply connected domains B_1 and B_2 such that $a_k \in B_k$, $k \in \{1,2\}$ the following inequality holds:*

$$R(B_1,a_1)R(B_2,a_2) \leqslant |a_1 - a_2|^2. \tag{1.1}$$

Moreover, the equality in (1.1) *is achieved for the complementary half-planes B_1 and B_2 to some straight line $l \subset \mathbb{C}$, and any pair of points a_1, a_2 being symmetric with respect to l.*

In the case of three or more points, many authors considered estimates of a more general Möbius invariant of the form

$$T_n := \frac{\prod\limits_{k=1}^{n} r(B_k,a_k)}{\left(\prod\limits_{1\leqslant k<p\leqslant n} |a_k - a_p|\right)^{\frac{2}{n-1}}}$$

(for an infinitely distant point under the corresponding factor we understand the unit). In 1951 Goluzin [12] for $n=3$ obtained an exact estimation for T_3:

$$T_3 \leqslant \frac{64}{81\sqrt{3}},$$

where the equality is achieved when the points $a_k = \exp i\frac{2\pi}{3}(k-1)$ and the domains B_k, $k = \overline{1,3}$, are the poles and the circular domains, respectively, of the quadratic differential

$$Q(z)dz^2 = -\frac{z}{(z^3-1)^2}\,dz^2.$$

In 1980 Kuzmina [15] showed that the problem of the evaluation of T_4 reduces to the smallest capacity problem in a certain continuum family and obtained the exact inequality for T_4:

$$T_4 \leqslant 3^2\,4^{-8/3}.$$

One of the extreme systems is a system of circular domains B_k, $k=\overline{1,4}$, of the quadratic differential

$$Q(z)dz^2 = -\frac{z(z^3+8)}{(z^3-1)^2}\,dz^2.$$

This differential has second-order poles at the points 1, w, w^2, ∞, where $w^3=1$, and the boundaries of the domains B_k, $k=\overline{1,4}$, are formed by rectilinear segments and circular arcs. In this case, the extreme configurations correspond to the continuum $E(-1,1,i\sqrt{3})$, that joins three segments of the same equal length.

An exact estimate for the invariant T_5 was obtained by Kuzmina and Dubinin [10] under additional restriction on the set of points $\{a_k\}_{k=1}^5$, namely, the set must be symmetric with respect to a certain circle or line

$$T_5 \leqslant 4^{\frac{11}{3}}\,3^{-\frac{3}{4}}\,5^{-\frac{25}{6}}.$$

The equality is achieved for the points 1, $e^{-\frac{2\pi i}{3}}$, 0, $e^{\frac{2\pi i}{3}}$, ∞ and the domains B_k, $k=\overline{1,5}$, that are circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{z^6+7z^3+1}{z^2(z^3-1)^2}\,dz^2.$$

No other ultimate results in this problem for $n \geqslant 5$ are known at present. In 2021 [5] (see also [3]) an effective upper estimates was obtained for T_n, $n \geqslant 2$.

Theorem 1.2 ([5]). *Let $n \in \mathbb{N}$, $n \geqslant 2$, $a_k \in \mathbb{C}$, $k=\overline{1,n}$, be some set of fixed points of the complex plane. Then, for an arbitrary set of domains $B_k \subset \overline{\mathbb{C}}$, such that $a_k \in B_k$, $k=\overline{1,n}$, $B_i \cap B_j = \varnothing$, $i \neq j$, the following inequality holds:*

$$\prod_{k=1}^{n} r(B_k,a_k) \leqslant (n-1)^{-\frac{n}{4}}\Bigg(\prod_{1\leqslant p<k\leqslant n}|a_p-a_k|\Bigg)^{\frac{2}{n-1}} \tag{1.2}$$

Later, this topic was developed in different directions. In particular, problems where the conformal or inner radii of the domains were taken in some predetermined positive powers, problems where the points relative to which taken inner radii of the domains are not fixed, but have some freedom, as well as problems where the corresponding points are not arbitrary, but satisfy certain conditions were considered (see, e.g., [2, 4, 7, 8, 11]). In particular, in the works [6, 9] it was studied the case when the poles of the corresponding quadratic differentials belong to a certain ellipse.

The following question arises: *can the ideas proposed in these works be extended to the case when the points belong to some other figure, for example, some hyperbola?* The present paper is devoted to this question.

2. Estimate of the functional with fixed points on a hyperbola right branch

Let $M = \left\{ z = x + iy : \frac{x^2}{d^2} - \frac{y^2}{p^2} = 1, x > 0 \right\}$ be the right branch of some hyperbola, where $d^2 + p^2 = 1$ (see Fig. 2.1). It follows from the last condition that we can put $d = \cos\alpha$ and $p = \sin\alpha$ for some $0 < \alpha < \frac{\pi}{2}$, so the equation of the hyperbola can be rewritten as follows:

$$M = \left\{ z = x + iy : \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1, x > 0 \right\}. \tag{2.1}$$

Then, the following proposition is true.

Theorem 2.1. *Let M be the right branch of some hyperbola given by* (2.1)*, and $a_k = x_k + iy_k \in M$, $k = \overline{1,n}$, be $n \geqslant 3$ mutually distinct points, where $x_k = \cos\alpha\cosh t_k$ and $y_k = \sin\alpha\sinh t_k$ for some real values t_k.*

Then, for any set of mutually non-overlapping domains $\{B_k\}$, $k = \overline{1,n}$, such that $a_k \in B_k \subset \Omega$, $k = \overline{1,n}$, where

$$\Omega = \overline{\mathbb{C}} \setminus \left([-\infty; -1] \cup [1; +\infty] \right),$$

the following inequality holds:

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant 2^{\frac{n}{2n-1}} (2n-1)^{-\frac{n}{4}} \prod_{k=1}^{n} \left| \frac{\sqrt{a_k^2 - 1}}{a_k - \sqrt{a_k^2 - 1}} \right| \times \\ \times \prod_{1 \leqslant p < q \leqslant n} \left| e^{2t_q} - e^{2t_p} \right|^{\frac{2}{2n-1}} \prod_{k=1}^{n} e^{\frac{t_k}{2n-1}}. \tag{2.2}$$

Proof. Without loss of generality assume that

$$-\infty < t_1 < t_2 < \cdots < t_n < +\infty.$$

Consider the mapping $w(z) = z - \sqrt{z^2 - 1}$, where we take that branch of the multivalued function $z - \sqrt{z^2 - 1}$ which maps the domain Ω onto the upper half-plane $\mathrm{Im}(w) > 0$.

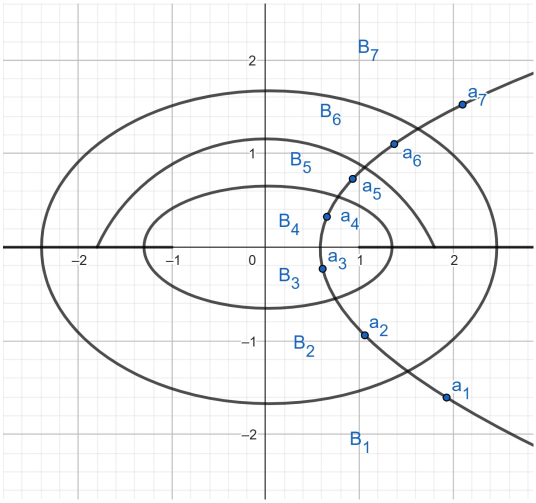

FIGURE 2.1. System example of n fixed points $a_k \in M$ (schematically)

The image of the given branch of the hyperbola M under this mapping will be the ray $w = \rho e^{i\alpha}$ for some $\rho > 0$ and the images of the points a_k under this mapping will be the points $a_k^* = e^{t_k + i\alpha}$ for $k = \overline{1, n}$.

The following lemma is holds.

Lemma 2.1.1. *Let $B_k^* = w(B_k)$ and $a_k^* = w(a_k)$ be the images of the domain B_k and the point a_k, $k = \overline{1, n}$, respectively, under the mapping $w(z)$, where we take that branch of the root, as in Theorem 2.1. Then, the following inequality holds:*

$$r(B_k, a_k) = \left| \frac{\sqrt{a_k^2 - 1}}{a_k - \sqrt{a_k^2 - 1}} \right| r(B_k^*, a_k^*). \tag{2.3}$$

Proof. Note that $\bigcup\limits_{k=0}^{n} B_k^* \subset D$, where $D = \{w : \mathrm{Im}(w) > 0\}$.

Let us determine how the inner radius of the domain changes under the mapping w. Since the Green function is invariant under conformal and univalent mapping, we have that

$$G_{B_k}(z, a_k) = G_{B_k^*}(w, a_k^*) = \ln \frac{1}{|w - a_k^*|} + \ln r(B_k^*, a_k^*) + o(1). \tag{2.4}$$

Now let us evaluate the expression $\ln\frac{1}{|w-a_k^*|}$. Using simple transformations, we get

$$\begin{aligned}
\ln\frac{1}{|w-a_k^*|} &= \ln\left|\frac{1}{z-\sqrt{z^2-1}-a_k+\sqrt{a_k^2-1}}\right| \\
&= \ln\frac{1}{|z-a_k|}+\ln\left|\frac{1}{1-\frac{\sqrt{z^2-1}-\sqrt{a_k^2-1}}{z-a_k}}\right| \\
&= \ln\frac{1}{|z-a_k|}+\ln\left|\frac{1}{1-\frac{(\sqrt{z^2-1}-\sqrt{a_k^2-1})(\sqrt{z^2-1}+\sqrt{a_k^2-1})}{(z-a_k)(\sqrt{z^2-1}+\sqrt{a_k^2-1})}}\right| \\
&= \ln\frac{1}{|z-a_k|}+\ln\left|\frac{1}{1-\frac{z+a_k}{\sqrt{z^2-1}+\sqrt{a_k^2-1}}}\right| \\
&= \ln\frac{1}{|z-a_k|}+\ln\left|\frac{\sqrt{z^2-1}+\sqrt{a_k^2-1}}{\sqrt{z^2-1}+\sqrt{a_k^2-1}-z-a_k}\right| \\
&= \ln\frac{1}{|z-a_k|}+\ln\left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right|+\ln\left|\frac{\sqrt{\frac{z^2-1}{a_k^2-1}}+1}{1+\frac{\sqrt{z^2-1}-z}{\sqrt{a_k^2-1}-a_k}}\right|.
\end{aligned}$$

Substituting this expression into (2.4) and also taking into account that

$$\ln\left|\frac{\sqrt{\frac{z^2-1}{a_k^2-1}}+1}{1+\frac{\sqrt{z^2-1}-z}{\sqrt{a_k^2-1}-a_k}}\right| \to 0 \quad \text{for} \quad z\to\ a_k,$$

we obtain that

$$G_{B_k}(z,a_k)=\ln\frac{1}{|z-a_k|}+\ln\left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right|r(B_k^*,a_k^*)+o(1).$$

Hence

$$r(B_k,a_k)=\left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right|r(B_k^*,a_k^*),$$

which proves the equality (2.3). □

Using the equalities (2.3) for each k, we obtain

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) = \prod_{k=1}^{n} \left| \frac{\sqrt{a_k^2 - 1}}{a_k - \sqrt{a_k^2 - 1}} \right| \prod_{k=1}^{n} r(B_k^*, a_k^*). \tag{2.5}$$

Now let us evaluate the expression $\prod\limits_{k=1}^{n} r(B_k^*, a_k^*)$. All points $a_k^* = e^{t_k + i\alpha}$ will be located on the ray $w = \rho e^{i\alpha}$ and all domains B_k^* will be located in the upper half-plane $\mathrm{Im}(w) > 0$.

For $k = \overline{1, n}$ denote

$$a_{-k}^* = -e^{t_k + i\alpha}, \qquad\qquad B_{-k}^* = \{w : -w \in B_k^*\}.$$

It is clear that for each k the point a_{-k}^* is symmetric to a_k^*, and the domains B_{-k}^* is symmetric to B_k^* with respect to the origin. Moreover, besides all points a_k^* and a_{-k}^* belong to the same straight line (see Fig. 2.2).

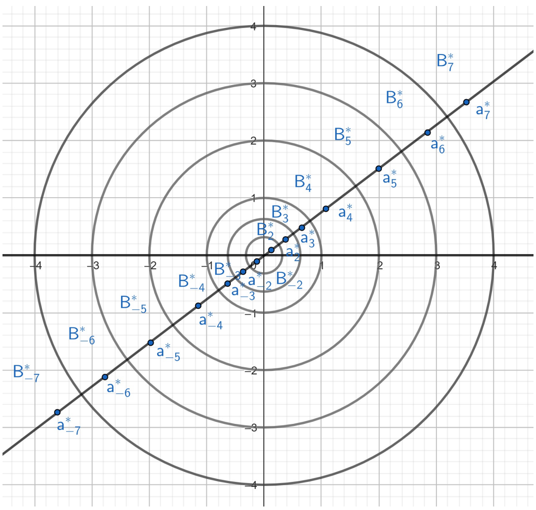

FIGURE 2.2. System example of $2n$ points and $2n$ non-overlapping domains (schematically)

It is obvious that

$$\prod_{k=1}^{n} r(B_k^*, a_k^*) = \prod_{k=1}^{n} r(B_{-k}^*, a_{-k}^*).$$

Therefore,

$$\prod_{k=1}^{n} r(B_k^*, a_k^*) = \sqrt{\prod_{k=1}^{n} r(B_k^*, a_k^*) \prod_{k=1}^{n} r(B_{-k}^*, a_{-k}^*)} = \sqrt{\prod_{\substack{k=-n \\ k\neq 0}}^{n} r(B_k^*, a_k^*)}. \tag{2.6}$$

Consider the expression

$$\prod_{\substack{k=-n \\ k\neq 0}}^{n} r(B_k^*, a_k^*).$$

We have a system of $2n$ points and $2n$ non-overlapping domains for which all conditions of [5, Theorem 2.1] are valid (see also Theorem 1.2). Therefore

$$\prod_{\substack{k=-n \\ k\neq 0}}^{n} r(B_k^*, a_k^*) \leqslant (2n-1)^{-\frac{n}{2}} \Bigg(\prod_{\substack{-n\leqslant p<q\leqslant n \\ p,q\neq 0}} |a_q^* - a_p^*| \Bigg)^{\frac{2}{2n-1}} \tag{2.7}$$

Note that for an arbitrary $0 < p < q \leqslant n$ the following equality is true:

$$|a_q^* - a_p^*| = |a_{-p}^* - a_{-q}^*| = e^{t_q} - e^{t_p}. \tag{2.8}$$

If $-n \leqslant p < 0 < q \leqslant n$, then

$$|a_q^* - a_p^*| = |a_{-p}^* - a_{-q}^*| = e^{t_q} + e^{t_{-p}}. \tag{2.9}$$

And finally, if $0 < -p = q \leqslant n$, then

$$|a_q^* - a_p^*| = 2e^{t_q}. \tag{2.10}$$

Taking into account the equalities (2.8)-(2.10), we obtain that

$$\prod_{\substack{-n\leqslant p<q\leqslant n \\ p,q\neq 0}} |a_q^* - a_p^*| = 2^n \prod_{1\leqslant p<q\leqslant n} \left(e^{2t_q} - e^{2t_p}\right)^2 \prod_{k=1}^{n} e^{t_k}.$$

Substituting the obtained expression first into the inequality (2.7) and then into the equality (2.6), we get

$$\prod_{k=1}^{n} r(B_k^*, a_k^*) \;\leqslant\; 2^{\frac{n}{2n-1}}\, (2n-1)^{-\frac{n}{4}} \prod_{1\leqslant p<q\leqslant n} \left(e^{2t_q} - e^{2t_p}\right)^{\frac{2}{2n-1}} \prod_{k=1}^{n} e^{\frac{t_k}{2n-1}}.$$

Substituting this inequality into the identity (2.5), we get the desired inequality (2.2). $\square$

Remark 2.2. Note that Theorem 2.1 can also be applied to hyperbolas whose semi-axes do not satisfy the identity $d^2 + p^2 = 1$, since this case can be reduced to the case that was considered by some linear transformation (taking into account some deformation of the set Ω to which the domains B_k belong).

3. Estimate of the functional with fixed points on hyperbola branches

Let $m, n \in \mathbb{N}$. A system of points

$$A_{m,n} \coloneqq \left\{a_{k,p} \in \mathbb{C} : \ k = \overline{1,m},\ p = \overline{1,n}\right\}$$

is called (m,n)*-ray*, if for all $k = \overline{1,m}$ and $p = \overline{1,n}$ the following relations are satisfied:

$$\begin{cases} 0 < |a_{k,1}| < \ldots < |a_{k,n}| < \infty; \\ \arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,n} =: \theta_k =: \theta_k(A_{m,n}); \\ 0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_m < 2\pi \leqslant \theta_{m+1} \coloneqq 2\pi + \theta_1. \end{cases} \tag{3.1}$$

Given a set of (m,n)-ray systems, consider the following quantities:

$$\begin{cases} \alpha_k \coloneqq \alpha_k(A_{m,n}) \coloneqq \frac{1}{\pi}\left[\theta_{k+1}(A_{m,n}) - \theta_k(A_{m,n})\right], \\ k = \overline{1,m}, \quad \alpha_{m+1} \coloneqq \alpha_1, \quad \alpha_0 \coloneqq \alpha_m. \end{cases} \tag{3.2}$$

The quantities $\alpha_k(A_{m,n})$, $k = \overline{1,m}$, will be called the *angular parameters* of the system $A_{m,n}$. It is obvious that

$$\sum_{k=1}^{m} \alpha_k(A_{m,n}) = \sum_{k=1}^{m} \alpha_k = 2.$$

For any (m,n)-ray system of points $A_{m,n} = \left\{a_{k,p} : \ k = \overline{1,m},\ p = \overline{1,n}\right\}$ we introduce the following “controlling” functional:

$$M(A_{m,n}) \coloneqq \prod_{k=1}^{m}\prod_{p=1}^{n}\left[\chi\left(|a_{k,p}|^{\frac{1}{\alpha_k}}\right)\chi\left(|a_{k,p}|^{\frac{1}{\alpha_{k-1}}}\right)\right]^{\frac{1}{2}}|a_{k,p}|, \tag{3.3}$$

where $\chi(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ be the Zhukovskii function.

Theorem 3.1 ([1, p.115]). *Let $m, n \in \mathbb{N}$, $m \geqslant 3$. Then, for any (m,n)-ray system of points $A_{m,n} = \{a_{k,p}\}$, $k = \overline{1,m}$, $p = \overline{1,n}$, and any collection of mutually non-overlapping domains $\{B_{k,p}\}$, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$, $k = \overline{1,m}$, $p = \overline{1,n}$, the following inequality holds:*

$$\prod_{k=1}^{m}\prod_{p=1}^{n} r(B_{k,p}, a_{k,p}) \leqslant \left(\frac{2}{n}\right)^{nm}\left(\prod_{k=1}^{m}\alpha_k\right)^{n} M(A_{m,n}). \tag{3.4}$$

The equality in (3.4) *is attained, if* $a_{k,p}$ *and* $B_{k,p}$, $k = \overline{1,m}$, $p = \overline{1,n}$, *are, respectively, poles and circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{w^{m-2}(1+w^m)^{2n-2}}{\left[(1-iw^{\frac{m}{2}})^{2n} + (1+iw^{\frac{m}{2}})^{2n}\right]^2}\, dw^2.$$

Let $N = \left\{z = x+iy : \frac{x^2}{d^2} - \frac{y^2}{p^2} = 1\right\}$ be some hyperbola with $d^2 + p^2 = 1$ and let

$$N_1 = \left\{z = x+iy : \frac{x^2}{d^2} - \frac{y^2}{p^2} = 1, x > 0\right\}$$
$$N_2 = \left\{z = x+iy : \frac{x^2}{d^2} - \frac{y^2}{p^2} = 1, x < 0\right\}$$

be its right and left branches, respectively. Then $d = \cos\alpha$ and $p = \sin\alpha$ for some $0 < \alpha < \frac{\pi}{2}$, and the equation of the hyperbola can be written as follows:

$$N = \left\{z = x+iy : \frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1\right\}. \tag{3.5}$$

Theorem 3.2. *Let* $n \in \mathbb{N}$, $n \geqslant 3$; N *be some hyperbola* (3.5), *where* $0 < \alpha < \frac{\pi}{2}$, N_1, N_2 *be, respectively, its right and left branches, and*

$$a_{k,p} = x_{k,p} + iy_{k,p} \in N_k,\ k \in \{1,2\},\ p = \overline{1,n},$$

be n *mutually disjoint points on each branch of* N, *where*

$$x_{1,p} = \cos\alpha\cosh(t_{1,p}), \qquad y_{1,p} = \sin\alpha\sinh(t_{1,p}),$$
$$x_{2,p} = -\cos\alpha\cosh(t_{2,p}), \qquad y_{2,p} = \sin\alpha\sinh(t_{2,p})$$

for some real values $t_{k,p}$ *such that*

$$-\infty < t_{1,1} < t_{1,2} < \ldots < t_{1,n} < +\infty,$$
$$-\infty < t_{2,1} < t_{2,2} < \ldots < t_{2,n} < +\infty.$$

Then, for any set of mutually non-overlapping domains $\{B_{k,p}\}$, $k \in \{1,2\}$, $p = \overline{1,n}$, *such that* $a_{k,p} \in B_{k,p} \subset \Omega$, *where* $\Omega = \overline{\mathbb{C}}\backslash([-\infty;-1] \cup [1;+\infty])$, *the following inequality holds:*

$$\prod_{k=1}^{2}\prod_{p=1}^{n} r\left(B_{k,p}, a_{k,p}\right) \leqslant \left(\frac{2}{n}\right)^{2n} \prod_{k=1}^{2}\prod_{p=1}^{n} \left|\frac{\sqrt{a_{k,p}^2 - 1}}{a_{k,p} - \sqrt{a_{k,p}^2 - 1}}\right| \times$$
$$\times \left(\frac{2\alpha}{\pi}\left(1 - \frac{2\alpha}{\pi}\right)\right)^n \prod_{k=1}^{2}\prod_{p=1}^{n} \left[\chi\left(e^{\frac{t_{k,p}\pi}{\pi-2\alpha}}\right)\chi\left(e^{\frac{t_{k,p}\pi}{2\alpha}}\right)\right]^{\frac{1}{2}} e^{t_{k,p}}. \tag{3.6}$$

Proof. As in Theorem 2.1 we consider the mapping $w(z) = z - \sqrt{z^2-1}$ where we take the branch of the multivalued function $z - \sqrt{z^2-1}$ which maps the domain Ω onto the upper half-plane $\mathrm{Im}(w) > 0$.

The image of the hyperbola branch N_1 under this mapping will be the ray $w = \rho e^{i\alpha}$ for some $\rho > 0$, and the image of the points $a_{1,p}$ under this mapping will be the points $a^*_{1,p} = e^{t_{1,p}+i\alpha}$ for $p = \overline{1,n}$. Hence, the image of the hyperbola branch N_2 under this mapping will be the ray $w = \rho e^{i(\pi-\alpha)}$ for some $\rho > 0$, and the image of the points $a_{2,p}$ under this mapping will be the points $a^*_{2,p} = e^{t_{2,p}+i(\pi-\alpha)}$ for $p = \overline{1,n}$. For $k \in \{1,2\}$, $p = \overline{1,n}$, denote by $B^*_{k,p}$ the domain that is the image of the domain $B_{k,p}$ under the mapping $w(z)$. Using the equalities (2.3) of Lemma 2.1.1 for the domains $B_{k,p}$, we obtain that

$$\prod_{k=1}^{2}\prod_{p=1}^{n} r\left(B_{k,p}, a_{k,p}\right) = \prod_{k=1}^{2}\prod_{p=1}^{n}\left|\frac{\sqrt{a_{k,p}^2-1}}{a_{k,p}-\sqrt{a_{k,p}^2-1}}\right| \prod_{k=1}^{2}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p}). \quad (3.7)$$

Let us now evaluate the expression $\prod_{k=1}^{2}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p})$. The points $a^*_{1,p}$ will be located on the ray $w = \rho e^{i\alpha}$, the points $a^*_{2,p}$ on the ray $w = \rho e^{i(\pi-\alpha)}$, and all domains $B^*_{k,p}$ will be located in the upper half-plane $\mathrm{Im}(w) > 0$. For $p = \overline{1,n}$ we denote

$$a^*_{3,p} = -e^{t_{1,p}+i\alpha}, \qquad B^*_{3,p} = \{w : -w \in B^*_{1,p}\},$$
$$a^*_{4,p} = -e^{t_{2,p}+i(\pi-\alpha)}, \qquad B^*_{4,p} = \{w : -w \in B^*_{2,p}\}.$$

Then it is obvious that

$$\prod_{k=1}^{2}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p}) = \prod_{k=3}^{4}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p}),$$

and therefore

$$\prod_{k=1}^{2}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p}) = \sqrt{\prod_{k=1}^{4}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p})}. \quad (3.8)$$

Consider the expression $\prod_{k=1}^{4}\prod_{p=1}^{n} r(B^*_{k,p}, a^*_{k,p})$. We have a $(4,n)$-ray system of points

$$A_{4,n} := \left\{a^*_{k,p} \in \mathbb{C} : \ k \in \{1,2,3,4\},\ p = \overline{1,n}\right\}$$

and mutually non-overlapping domains $B^*_{k,p}$ for which all conditions of Theorem 3.1 hold. From the equalities (3.2) we have

$$\alpha_1 = \alpha_3 = 1 - \frac{2\alpha}{\pi}, \quad \alpha_2 = \alpha_4 = \frac{2\alpha}{\pi}$$

and thus

$$\left(\prod_{k=1}^{4}\alpha_k\right)^n=\left(\frac{2\alpha}{\pi}\left(1-\frac{2\alpha}{\pi}\right)\right)^{2n}. \tag{3.9}$$

Taking into account that for $p=\overline{1,n}$ the equalities

$$|a^*_{1,p}|=|a^*_{3,p}|=e^{t_{1,p}}, \qquad |a^*_{2,p}|=|a^*_{4,p}|=e^{t_{2,p}}$$

hold, in the expression (3.3) we obtain

$$M(A_{4,n}) \coloneqq \prod_{k=1}^{2}\prod_{p=1}^{n}\chi\left(e^{\frac{t_{k,p}\pi}{\pi-2\alpha}}\right)\chi\left(e^{\frac{t_{k,p}\pi}{2\alpha}}\right)e^{2t_{k,p}}. \tag{3.10}$$

By substituting (3.9) and (3.10) in (3.4), we get

$$\prod_{k=1}^{2}\prod_{p=1}^{n} r\left(B^*_{k,p},a^*_{k,p}\right)\leqslant\left(\frac{2}{n}\right)^{4n}\left(\frac{2\alpha}{\pi}\left(1-\frac{2\alpha}{\pi}\right)\right)^{2n}\times$$
$$\times\prod_{k=1}^{2}\prod_{p=1}^{n}\chi\left(e^{\frac{t_{k,p}\pi}{\pi-2\alpha}}\right)\chi\left(e^{\frac{t_{k,p}\pi}{2\alpha}}\right)e^{2t_{k,p}}.$$

Then, substituting the resulting inequality successively into (3.8) and (3.7), we obtain the inequality (3.6). □

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