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The generalized homotopy axiom and Alexandroff–Čech cohomology theory

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Dedicated to Professor Aleksei Viktorovich Chernavsky

Abstract. Let T be a connected compact metric space, $r, s \in T$ be two points, and X be a locally compact paracompact space. We prove that the mappings $\phi_r, \phi_s \colon X \to X \times T$, defined by $\phi_t(x) = (x, t)$ for $t = r, s \in T$, induce the same homomorphisms of Alexandroff–Čech cohomology if and only if the following natural homomorphism $\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T)$ splits.

Анотація. Нехай T – зв'язний компактний метричний простір, $r, s \in T$ – дві точки, а X – локально компактний паракомпактний простір. В роботі доведено, що відображення $\phi_r, \phi_s \colon X \to X \times T$, визначені за формулами $\phi_r(x) = (x, r)$ та $\phi_s(x) = (x, s)$, індукують однакові гомоморфізми когомологій Александрова-Чеха тоді і лише тоді, коли природній гомоморфізм $\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T)$ розщеплюється.

1. Introduction

Axiomatic characterization of the theory of Alexandroff–Čech cohomology, based on the axioms of Eilenberg and Steenrod [4] on the category of paracompact spaces, was given by Bacon [1]. One of the axioms is the Homotopy Axiom for cohomology, which is equivalent to the following statement:

Axiom. *Let $i_0, i_1 \colon X \to X \times [0;1]$ be embeddings of a X intto $X \times [0,1]$, defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$. Then they induce the same homomorphisms of cohomologies:*

$$i_0^* = i_1^* \colon \check{H}^*(X \times [0,1]) \to \check{H}^*(X).$$

It is known that on the category of compact spaces the unit interval $[0,1]$ in the Axiom can be replaced by any connected space T, cf. [3, Corollary 3]:

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Theorem 1.1. *Let T and X be any connected and compact spaces respectively, and $i_t\colon X \to X \times T$ the family of maps, defined by $i_t(x) = (x, t)$. Then $i_t^*\colon \check{H}^*(X \times T) \to \check{H}^*(X)$ is independent of t.*

It will be convenient to formulate the following

Axiom (Generalized Homotopy Axiom for $(T, \{r, s\})$)**.** *Let X and T be topological spaces, $r, s \in T$ be two points, and $i_r, i_s\colon X \to X \times T$ be embeddings defined by $i_r(x) = (x, r)$ and $i_s(x) = (x, s)$. Then*

$$i_r^* = i_s^*\colon \check{H}^*(X \times T) \to \check{H}^*(X).$$

Then Theorem 1.1 can be formulated so that the Generalized Axiom holds for any connected compact spaces X and T.

On the other hand, in the paper [7] it is constructed a connected compact space T and a locally compact polyhedron X such that $i_{t_1}^* \neq i_{t_2}^*$ for some points $t_1, t_2 \in T$.

The following theorem is the main result of this paper:

Theorem 1.2. *Let T be a connected compact metric space, $r, s \in T$ be two points, and X be any locally compact paracompact space. Then the mappings $\phi_r, \phi_s\colon X \to X \times T$, defined by $\phi_t(x) = (x, t)$ at $t = r, s$, induce the same homomorphisms of Alexandroff–Čech cohomology if and only if the following natural homomorphism $\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T)$ splits.*

Thus, for such X and T, the Generalized Homotopy Axiom is consistent with Alexandroff-Čech cohomology if and only if the natural homomorphism $\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T)$ splits.

2. Preliminaries

We start by fixing some terminology and notations, and recall several statements which will be used in the proof of the Main Theorem 1.2. All undefined terms can be found in [2,4,5,8]. By $\check{H}$ we mean Alexandroff–Čech cohomology with integer coefficients. It is well known, that on the category of paracompact spaces the cohomology of Alexandroff–Čech, Alexander–Kolmogorov–Spinier cohomology, sheaf cohomology with integer coefficients are isomorphic, see, e.g. [2].

By a *group* we always mean an abelian group. A homomorphism of abelian groups $A \xrightarrow{f} B$ *splits*, if there exists a homomorphism $B \xrightarrow{g} A$ such that the composition $B \xrightarrow{g} A \xrightarrow{f} B$ is the identity mapping of the group B.

Lemma 2.1. *For each finitely generated free (abelian) group F there is a natural isomorphism $\psi\colon$ $\mathrm{Hom}(F, \mathbb{Z}) \otimes A \to \mathrm{Hom}(F, A)$.*

Proof. Let $\{v_i\}_{i=1}^n$ be some basis of F and $\{\phi_i\colon F \to \mathbb{Z}\}_{i=1}^n$ be the dual basis of $\mathrm{Hom}(F,\mathbb{Z})$, i.e. $\phi_i(v_i) = 1$ and $\phi_i(v_j) = 0$ for $i \neq j$. Then for each $f \in F$ and $h \in \mathrm{Hom}(F,\mathbb{Z})$ we have that

$$f = \sum_{i=1}^n \phi_i(f) v_i, \qquad h = \sum_{i=1}^n h(v_i)\phi_i, \qquad h(f) = \sum_{i=1}^n h(v_i)\phi_i(f).$$

Notice that the following map $e\colon \mathrm{Hom}(F,\mathbb{Z}) \times A \to \mathrm{Hom}(F,A)$

$$e(h,a)(f) := h(f)a = \Big(\sum_{i=1}^n h(v_i)\phi_i(f)\Big)a.$$

is bilinear and therefore it induces a certain homomorphism

$$\psi\colon \mathrm{Hom}(F,\mathbb{Z}) \otimes A \to \mathrm{Hom}(F,A).$$

To prove that ψ is an isomorphism, we need to show that for each bilinear map $g\colon \mathrm{Hom}(F,\mathbb{Z}) \times A \to G$ into some group G, there exists a homomorphism $\overline{g}\colon \mathrm{Hom}(F,A) \to G$ such that $g = \overline{g} \circ e$, [5, Theorem 50.1].

Indeed, since g is bilinear, we have that

$$g(h,a) = g\big(\sum_{i=1}^n h(v_i)\phi_i, a\big) = \sum_{i=1}^n h(v_i) g(\phi_i, a).$$

Define the homomorphism $\overline{g}\colon \mathrm{Hom}(F,A) \to G$ as follows

$$\overline{g}(q) = \sum_{i=1}^n g(\phi_i, q(v_i)).$$

Then

$$\begin{aligned}\overline{g} \circ e(h,a) &= \sum_{i=1}^n g\big(\phi_i, e(h,a)(v_i)\big) = \sum_{i=1}^n g\big(\phi_i, h(v_i)a\big) \\ &= \sum_{i=1}^n h(v_i) g\big(\phi_i, a\big) = g(h,a).\end{aligned}$$ □

Lemma 2.2. *A homomorphism $A \xrightarrow{f} B$ of countable torsion free groups splits if and only if for any inverse sequence $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$ of finitely generated free groups the natural homomorphism*

$$\varprojlim(G_i \otimes A) \to \varprojlim(G_i \otimes B)$$

is surjective.

Proof. Suppose $A \xrightarrow{f} B$ splits. Since $\varprojlim$ is a covariant functor, it follows that the homomorphism $\varprojlim(G_i \otimes A) \to \varprojlim(G_i \otimes B)$ is surjective.

Conversely, suppose that for each inverse sequence

$$G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$$

the homomorphism $\varprojlim(G_i \otimes A) \to \varprojlim(G_i \otimes B)$ is surjective. Since the group B is countable and torsion free, it follows that B is a union of increasing system of finitely generated free groups $F_1 \hookrightarrow F_2 \hookrightarrow F_3 \hookrightarrow \cdots$, that is $B = \varinjlim F_i$. Denote $G_i = \mathrm{Hom}(F_i, \mathbb{Z})$. Then we get the corresponding inverse spectrum $G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$.

By assumption, the homomorphism

$$\varprojlim(G_i \otimes A) \to \varprojlim(G_i \otimes B) \tag{2.1}$$

is surjective. Hence, by Lemma 2.1, we get canonical isomorphisms:

$$G_i \otimes A = \mathrm{Hom}(F_i, \mathbb{Z}) \otimes A = \mathrm{Hom}(F_i, A),$$
$$\varprojlim(G_i \otimes A) = \varprojlim(\mathrm{Hom}(F_i, A)) = \mathrm{Hom}(\varinjlim F_i, A) = \mathrm{Hom}(B, A).$$

Similarly

$$\varprojlim(G_i \otimes B) = \varprojlim(\mathrm{Hom}(F_i, B)) = \mathrm{Hom}(\varinjlim F_i, B) = \mathrm{Hom}(B, B).$$

Since all these isomorphisms are functorial, the epimorphism (2.1) yields an epimorphism $\mathrm{Hom}(B, A) \to \mathrm{Hom}(B, B)$. In particular, for homomorphism $\phi\colon B \to B$ there exists a homomorphism $\psi\colon B \to A$ such that $\phi = f \circ \psi$. In particular, for the identity map $\mathrm{id}_B\colon B \to B$ there exists a homomorphism $g\colon B \to A$ satisfying $f \circ g = \mathrm{id}_B$, i.e. f splits. □

Lemma 2.3. *Let* $G_1 \xleftarrow{\pi^1} G_2 \xleftarrow{\pi^2} G_3 \xleftarrow{\pi^3} \cdots$ *be an inverse sequence of finitely generated free groups. Then there exist finite CW-complexes* P_n, $n \in \mathbb{N}$, *such that* $P_n \subset \mathrm{int}(P_{n+1})$, *the groups* $\check{H}^1(P_n)$ *are isomorphic to* G_n, *and the natural homomorphisms* $\check{H}^1(P_n) \leftarrow \check{H}^1(P_{n+1})$ *can be identified with* π^n.

Proof. Let S_i be the bouquet of the same number of circles, as is the number of generators in the group G_i. Let C_i be the cylinder of the mapping $S_i \to S_{i+1}$, which induces on 1-dimensional cohomology the homomorphism π^i. The space S_i is the lower base and S_{i+1} is the upper base of the cylinder C_i. Let us define the space P_n as quotient space of topological sum $\sum_{i=1}^{n-1} C_i$ by the identification of upper base of C_i with lover base of C_{i-1}.

After that it is easy to check that $P_n \subset \mathrm{int}(P_{n+1})$, $\check{H}^1(P_n) = G_n$ and homomorphisms, generated by the inclusion $P_n \subset P_{n+1}$, are π^n. □

Lemma 2.4. *One-dimensional Čech cohomology of any space* X *does not have nontrivial element of finite order.*

Proof. Recall that for each polyhedron P the following universal coefficients formula in dimension 1 holds:

$$0 \to \mathrm{Ext}(H_0(P), \mathbb{Z}) \to H^1(P) \to \mathrm{Hom}(H_1(P), \mathbb{Z}) \to 0.$$

Note also that the group $\mathrm{Hom}(H_1(P), \mathbb{Z})$ does not have nontrivial elements of finite order, since if a homomorphism $\varphi \in \mathrm{Hom}(H_1(P), \mathbb{Z})$ is nontrivial, then $n\varphi$ is also nontrivial for any natural number n.

On the other hand, the group $\mathrm{Ext}(H_0(P), \mathbb{Z})$ is trivial because $H_0(P)$ is free group, see [5, Chapter 9, §52, p. 222, Property (A)]. Therefore and by exactness of the row it follows, that the group $H^1(P)$ does not have torsion.

By definition, the Čech cohomology group $\check{H}^1(X)$ is a direct limit of cohomology groups of the nerves of open coverings of X. Since the nerves are polyhedra and direct limit of the groups without torsion also does not have nontrivial torsion, we obtain that $\check{H}^1(X)$ must be torsion free. □

Lemma 2.5 ([6, Theorem 1 (1)])**.** *Let $\mathcal{A}$ be a sheaf on a topological space X and $X = \bigcup_{n=1}^{\infty} U_n$, where U_n are open sets such that $U_n \subset U_{n+1}$. Then there exists a natural exact sequence*

$$0 \to \varprojlim^{(1)} \check{H}^{p-1}(U_n; \mathcal{A}) \to \check{H}^p(X; \mathcal{A}) \to \varprojlim \check{H}^p(U_n; \mathcal{A}) \to 0.$$

Lemma 2.6 ([2, Proposition 12.2], [8, Chapter 10, §7, p. 328])**.** *Cohomology of a pair (X, F) are isomorphic to the cohomology of a single space with coefficients in some special sheaf $\mathcal{A}_{X \setminus F}$:*

$$\check{H}^p(X, F; \mathcal{A}) \cong \check{H}^p(X; \mathcal{A}_{X \setminus F}).$$

3. Proof of the main results

Let T be a connected compact metric space and $r, s \in T$ be two points. Define the following mappings $\phi_r, \phi_s \colon X \to X \times T$, $\phi_t(x) = (x, t)$ for $t = r, s$.

For a subset $Y \subset T$ it will be convenient to denote $X_Y := X \times Y$. For instance, $X_T = X \times T$, $X_{r,s} = X \times \{r, s\}$, $X_r = X \times \{r\}$ etc.

Lemma 3.1. *The homomorphisms $\phi_r^n, \phi_s^n \colon \check{H}^n(X \times T) \to \check{H}^n(X)$ coincide if and only if the homomorphism*

$$\check{H}^n(X \times T, X \times \{r, s\}) \xrightarrow{i^n} \check{H}^n(X \times T, X \times \{r\}),$$

induced by the embedding $i \colon (X_T, X_r) \subset (X_T, X_{r,s})$, is surjective.

Proof. It follows from the exact sequence of the triple $(X_T, X_{r,s}, X_r)$:

$$\check{H}^n(X_T, X_{r,s}) \xrightarrow{i^n} \check{H}^n(X_T, X_r) \xrightarrow{j^n} \check{H}^n(X_{r,s}, X_r)$$

that it is enough to prove that $\phi_r^n = \phi_s^n$ if and only if the homomorphisms j^n are trivial.

Necessity. Suppose, that $\phi_r^n = \phi_s^n$. Then we have commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\check{H}^{n-1}(X_T) & \xrightarrow{j_1^{n-1}} & \check{H}^{n-1}(X_r) & \xrightarrow{\delta_1^{n-1}} & \check{H}^n(X_T, X_r) & \xrightarrow{i_1^n} & \check{H}^n(X_T) & \xrightarrow{j_1^n} & \\
\downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle g^{n-1}} & & \downarrow{\scriptstyle j^n} & & \downarrow{\scriptstyle f^n} & & \\
\check{H}^{n-1}(X_{r,s}) & \xrightarrow{j_2^{n-1}} & \check{H}^{n-1}(X_r) & \xrightarrow{\delta_2^{n-1}} & \check{H}^n(X_{r,s}, X_r) & \xrightarrow{i_2^n} & \check{H}^n(X_{r,s}) & \xrightarrow{j_2^n} &
\end{array} \tag{3.1}$$

Let us show that j^n is trivial. For this end, consider the following diagram

$$\begin{array}{ccccc}
\xrightarrow{i_1^n} & \check{H}^n(X_T) & \xrightarrow{j_1^n} & \check{H}^n(X_r) \\
 & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle g^n} \\
\xrightarrow{i_2^n} & \check{H}^n(X_{r,s}) & \xrightarrow{j_2^n} & \check{H}^n(X_r) \\
 & \downarrow{\scriptstyle h^n} & & \downarrow{\scriptstyle p^n} \\
 & \check{H}^n(X_s) & \xrightarrow{q^n} & \check{H}^n(X)
\end{array} \tag{3.2}$$

Since X_r is a retract of both X_T and $X_{r,s}$, the mappings j_1^{n-1} and j_2^{n-1} in Diagram (3.1) are epimorphisms, whence i_1^n and i_2^n are injective.

Further, it follows from Diagram (3.2) that

$$\phi_r^n = p^n\, g^n\, j_1^n, \qquad \phi_s^n = q^n\, h^n\, f^n = p^n\, j_2^n\, f^n.$$

Moreover, the homomorphisms g^n, p^n, q^n are obviously isomorphisms. Hence by exactness of rows in Diagram (3.1) it follows that $j_1^n\, i_1^n = 0$.

Therefore, for each element $a \in \check{H}^n(X_T, X_r)$,

$$\phi_r^n\, i_1^n(a) = p^n\, g^n\, j_1^n\, i_1^n(a) = p^n\, j_2^n\, f^n\, i_1^n(a) = 0,$$

whence $j_2^n\, f^n\, i_1^n(a) = 0$, since p^n is isomorphism. By assumption $\phi_r^* = \phi_s^*$, whence $\phi_s^*\, i_1^n(a) = 0$ and $q^n\, h^n\, f^n\, i_1^n(a) = 0$. Since q^n is isomorphism, we get that $h^n\, f^n\, i_1^n(a) = 0$.

Note that the group $\check{H}^n(X_{r,s})$ is naturally isomorphic to the direct sum of the groups $\check{H}^n(X_r)$ and $\check{H}^n(X_s)$. Hence $f^n\, i_1^n(a) = 0$. Now, from the commutativity of Diagram (3.1), it follows that $i_2^n\, j^n(a) = 0$. Since i_2^n is injective, we obtain that $j^n(a) = 0$ for all $a \in \check{H}^n(X_T, X_r)$.

Sufficiency. Assuming that j^n is trivial we should prove that $\phi_r^n = \phi_s^n$. Take any element $b \in \check{H}^n(X_T)$. We are going to show that $\phi_r^n(b) = \phi_s^n(b)$.

Let $\pi\colon X_T \to X$ be the continuous mapping given by $\pi(x,t) = x$. Consider the following diagram

$$\begin{array}{ccccc} & & \check{H}^1(X) & & \\ & & \downarrow{\scriptstyle \pi^n} & & \\ \check{H}^n(X) & \xleftarrow{\phi_r^n} & \check{H}^n(X_T) & \xrightarrow{\phi_s^n} & \check{H}^n(X) \end{array}$$

Let $a = b - \pi^n\, \phi_r^n(b)$. Obviously, $\phi_r^n(a) = \phi_r^n(b) - \phi_r^n\, \pi^n\, \phi_r^n(b) = 0$, since $\phi_r^n\, \pi^n$ is the identity mapping. As was mentioned above, $\phi_r^n = p^n\, g^n\, j_1^n$ and $p^n\, g^n$ is isomorphism. Therefore $j_1^n(a) = 0$ and, due to the exactness of the row in Diagram (3.2), there exists some $c \in \check{H}^n(X_T, X_r)$ such that $i_1^n(c) = a$. By assumption $j^n = 0$, whence

$$i_2^n\, j^n(c) = f^n\, i_1(c) = f^n(a) = 0$$

and $j_2^n\, f^n(a) = 0$. However, since $\phi_s^n(a) = p^n\, j_2^n\, f^n(a)$, it follows that

$$\phi_s^n(a) = \phi_s^n(b) - \phi_s\, \pi^n\, \phi_r^n(b) = 0$$

and $\phi_s^n(b) = \phi_s^n\, \pi^n\, \phi_r^n(b)$ or $\phi_s^n(b) = \phi_r^n(b)$, since $\phi_s^n\, \pi^n$ is the identity mapping. □

Suppose a polyhedron P is a union $P = \bigcup_{i=1}^{\infty} P^i$ of finite polyhedra P^i such that $P^i \subset \operatorname{int}(P^{i+1})$. Again, for a subset $Y \subset T$, it will be conveinent to denote $P_Y = P \times Y$ and $P_Y^i = P^i \times Y$.

Lemma 3.2. *Homomorphism* $\check{H}^n(P_T, P_{r,s}) \xrightarrow{i^n} \check{H}^n(P_T, P_r)$ *is surjective for all* n*, if and only if the homomorphism*

$$\varprojlim \check{H}^n(P_T^i, P_{r,s}^i) \xrightarrow{j^n} \varprojlim \check{H}^n(P_T^i, P_r^i)$$

is surjective for all n.

Proof. Necessity. According to Lemmas 2.5 and 2.6 and since $P^i \subset \operatorname{int}(P^{i+1})$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & \varprojlim^{(1)} \check{H}^{n-1}(P_T^i, P_{r,s}^i) & \xrightarrow{\psi_1'} & \check{H}^n(P_T, P_{r,s}) & \xrightarrow{\psi_2'} & \varprojlim \check{H}^{n-1}(P_T^i, P_{r,s}^i) & \to & 0 \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle i^n} & & \downarrow{\scriptstyle j^n} & & \\ 0 & \to & \varprojlim^{(1)} \check{H}^{n-1}(P_T^i, P_{r,s}^i) & \xrightarrow{\psi_1} & \check{H}^n(P_T, P_{r,s}) & \xrightarrow{\psi_2} & \varprojlim \check{H}^{n-1}(P_T^i, P_{r,s}^i) & \to & 0 \end{array}$$

If i^n is surjective and since ψ_2' is surjective it follows, that j^n is also surjective.

Sufficiency. Suppose now, that j^n is surjective. Since the Axiom of Homotopy is valid on the category of compact spaces, see Theorem 1.1, and polyhedra P^i are compact, it follows from Lemma 3.1 that the homomorphisms $\check{H}^{n-1}(P^i_T, P^i_{r,s}) \xrightarrow{j^n} \check{H}^{n-1}(P^i_T, P^i_r)$ are surjective. And since the functor $\varprojlim^{(1)}$ is exact from the right, it follows that f^{n-1} is surjective. Therefore i^n is surjective as well. □

Lemma 3.3. *Homomorphism*

$$\varprojlim \check{H}^n(P^i_T, P^i_{r,s}) \to \varprojlim \check{H}^n(P^i_T, P^i_r)$$

is surjective for every n, if and only if the homomorphism

$$\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T, \{r\})$$

splits.

Proof. Necessity. Suppose, $\check{H}^n(P^i_T, P^i_r)$ is surjective. Let us prove, that the homomorphism $\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T, \{r\})$ splits. Consider commutative diagram with exact rows, which is obtained from the Künneth formulas by passing to the inverse limit

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \varprojlim A_i & \xrightarrow{i_1} & \varprojlim B_i & \xrightarrow{i_2} & \varprojlim C_i & \xrightarrow{i_3} & \varprojlim^{(1)} A_i & \xrightarrow{i_4} & \varprojlim^{(1)} B_i \\ & & \downarrow{\scriptstyle i_5} & & \downarrow{\scriptstyle i_6} & & \downarrow{\scriptstyle i_7} & & \downarrow{\scriptstyle i_8} & & \downarrow{\scriptstyle i_9} \\ 0 & \longrightarrow & \varprojlim A'_i & \xrightarrow{i_{10}} & \varprojlim B'_i & \xrightarrow{i_{11}} & \varprojlim C'_i & \xrightarrow{i_{12}} & \varprojlim^{(1)} A'_i & \xrightarrow{i_{13}} & \varprojlim^{(1)} B'_i \end{array}$$

where

$$A_i = \sum_{p+q=n} \check{H}^p(P^i) \otimes \check{H}^q(T, \{r, s\}), \qquad A'_i = \sum_{p+q=n} \check{H}^p(P^i) \otimes \check{H}^q(T, \{r\}),$$

$$B_i = \check{H}^n(P^i_T, P^i_{r,s}), \qquad B'_i = \check{H}^n(P^i_T, P^r_r),$$

$$C_i = \sum_{p+q=n+1} \check{H}^p(P^i) \otimes \check{H}^q(T, \{r, s\}), \qquad C'_i = \sum_{p+q=n+1} \check{H}^p(P^i) \otimes \check{H}^q(T, \{r\}).$$

By Lemma 2.4, the 1-dimensional cohomology with coefficients in $\mathbb{Z}$ does not have the nontrivial elements of finite order. Therefore, the following torsion products vanish:

$$\mathrm{Tor}(\check{H}^n(P^i), \check{H}^1(T, \{r, s\})) = 0, \qquad \mathrm{Tor}(\check{H}^n(P^i), \check{H}^1(T, \{r\})) = 0,$$

see [5, Chapter 10, §62, Property B], and since $\check{H}^q(T, \{r, s\}) \approx \check{H}^q(T, \{r\})$ at $q > 1$ it follows, that i_7 is isomorphism.

Hence, the homomorphism

$$\sum_{p+q=n} \varprojlim \check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\}) \xrightarrow{i_5} \sum_{p+q=n} \varprojlim \check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r\})$$

is surjective, whence the homomorphism

$$\varprojlim \check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\}) \to \varprojlim \check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r\})$$

is surjective as well. By Lemmas 2.2 and 2.3 the homomorphism

$$\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T, \{r\})$$

splits.

Sufficiency. Suppose now that the homomorphism

$$\check{H}^1(T, \{r, s\}) \xrightarrow{\pi} \check{H}^1(T, \{r\})$$

splits. Let us prove that then the mapping

$$\varprojlim \check{H}^n(P^i_T, P^i_{r,s}) \to \varprojlim \check{H}^n(P^i_T, P^i_r)$$

or, equivalently, the mapping $\varprojlim B_i \xrightarrow{i_6} \varprojlim B'_i$ is an epimorphism.

For this it is enough to show that

$$\ker i_4 \cap \ker i_8 = 0. \tag{3.3}$$

Indeed, suppose (3.3) holds. Consider any element $\alpha \in \varprojlim B'_i$. Since i_7 is an isomorphism, there exists $\beta \in \varprojlim C_i$ such that $i_7(\beta) = i_{11}(\alpha)$ and $i_3(\beta) \in \ker i_4 \cap \ker i_8$. It then follows that $i_3(\beta) = 0$.

By exactness of row it follows, that there exists $\alpha' \in \varprojlim B_i$ for which $i_3(\alpha') = \beta$ and by the commutativity of diagram $i_{11} - i_6(\alpha') = 0$. Since by assumption the homomorphism π splits, i_5 must be surjective and in this case there should exists $\gamma \in \varprojlim A_i$ such that $i_{10}\, i_5(\gamma) = \alpha - i_6(\alpha')$ or $i_6\, i_1(\gamma) = \alpha - i_6(\alpha')$. Thus $\alpha = i_6(i_1(\gamma) + \alpha')$ and therefore i_6 is also being surjective.

It remains to prove (3.3). Consider following commutative diagram with natural homomorphisms:

$$\begin{array}{ccccc}
\overset{(1)}{\varprojlim} T(\check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\})) & & \overset{(1)}{\varprojlim} \check{H}^{n-1}(P^i) \otimes \check{H}^0(\{r, s\}, \{r\}) & & \\
\downarrow j_1 & \swarrow j_2 & \downarrow j_3 & & \\
\overset{(1)}{\varprojlim} \check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\}) & \xrightarrow{j_4} & \overset{(1)}{\varprojlim} A_i & \xrightarrow{j_5} & \overset{(1)}{\varprojlim} B_i \\
\downarrow j_6 & & \downarrow j_7 & & \downarrow j_8 \\
\overset{(1)}{\varprojlim} F(\check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\})) & \xrightarrow{j_9} & \overset{(1)}{\varprojlim} F(A_i) & \xrightarrow{j_{10}} & \overset{(1)}{\varprojlim} F(B_i)
\end{array}$$

Here $T(G)$ is a torsion of the group G, $F(G) = G/T(G)$ is a quotient group of the group G by the subgroup $T(G)$. Note that the group

$$\varprojlim^{(1)} T(\check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\}))$$

is trivial, because P^i are finite polyhedra and, therefore, the Mittag-Leffler condition is valid. From exactness of first column it follows that j_6 is isomorphism. Furthermore, the homomorphism j_2 is injective, since the following exact triple splits:

$$0 \to \check{H}^0(\{r, s\}, \{r\}) \to \check{H}^1(T, \{r, s\}) \to \check{H}^1(T, \{r\}) \to 0.$$

Also note that the homomorphism j_9 is injective, because the group

$$\check{H}^{n-1}(P^i) \otimes \check{H}^1(T, \{r, s\})$$

is a direct summand of $\sum\limits_{p+q=n} \check{H}^p(P^i) \otimes \check{H}^q(T, \{r, s\})$. Finally, the homomorphism j_{10} is isomorphism, since by the Künneth formula

$$\check{H}^n(P^i_T, P^i_{r,s}) = A_i \oplus C_i$$

and, therefore,

$$F\big(\check{H}^n(P^i_T, P^i_{r,s})\big) = F(A_i) \oplus F(C_i).$$

But $F(C_i) = 0$, since C_i is a torsion group, see [5, Chapter X, §62, Property A]. This implies that $j_{10}\, j_9\, j_6\, j_2$ is injective and therefore $j_5\, j_3$ is injective as well.

Consider the following commutative diagram with an exact column:

$$\begin{array}{ccc}
\varprojlim^{(1)} \check{H}^{n-1}(P^i) \otimes \check{H}^0(\{r, s\}, \{r\}) & & \\
\downarrow{\scriptstyle j_3} & & \\
\varprojlim^{(1)} A_i & \xrightarrow{j_5 = i_4} & \varprojlim^{(1)} B_i \\
\downarrow{\scriptstyle j_8} & & \\
\varprojlim^{(1)} A'_i & &
\end{array}$$

Let $a \in \ker i_4 \cap \ker i_8$. Due to the exactness of that column there exists b such that $j_3(b) = a$. Then $i_4\, j_3(b) = i_4(a) = 0$. As was mentioned above $j_5\, j_3 = i_4\, j_3$ is injective, whence $b = 0$, $a = 0$, and $\ker i_4 \cap \ker i_8 = 0$. This completes the proof of the lemma. □

Lemma 3.4. *For every locally compact paracompact space X and every element $\alpha \in \check{H}^n(X \times T)$ there exist a locally finite polyhedron P and a continuous mapping $\Psi\colon X \to P$ such that a is contained in the image of the homomorphism*

$$(\Psi \times \mathrm{id})^n\colon \check{H}^n(P \times T) \to \check{H}^n(X \times T).$$

Proof. For a locally finite open cover ω of $X \times T$ we will denote by $N(\omega)$ the nerve of ω and by $f_\omega\colon X \times T \to N(\omega)$ the natural map.

Since the product $X \times T$ is paracompact, it follows from definition of Čech cohomology that for the element $a \in \check{H}^n(X \times T)$, there exists an open cover ω of $X \times T$ such that a is contained in the image of the induced homomorphism $\check{H}^n(N(\omega)) \to f^n\check{H}^n(X \times T)$, see e.g. [8, Chapter 3, page 152, Exercise G].

Let ω_1 be some refinement of the cover ω whose elements are of the folowing form:

$$\{U_\alpha \times V_\beta : \alpha \in A, \beta \in B_\alpha\},$$

where $\{U_\alpha\}_{\alpha\in A}$ is some open cover of X, A is an index set, for each $\alpha \in A$ the collection $\{V_\beta\}_{\beta\in B_\alpha}$ is an open cover of T, and the nerve $N(\omega_1)$ is a locally finite polyhedron.

It is easy to see that *such a cover always exists*. Indeed, for every point $x \in X$ the space $x \times T$ is compact and therefore for the point x there exists a neighborhood U_x and a finite cover $\{V_\beta : \beta \in B_x\}$ of T such that $\{U_x \times V_\beta\}$ refines the cover ω. Since the space X is locally compact and paracompact, there exists a locally finite refinement $\{U_\alpha : \alpha \in A\}$ of $\{U_x : x \in X\}$, whose elements have compact closures. Note also that for every U_α there exists a finite covering each element of which intersects only finitely many elements of the system $\{U_\alpha : \alpha \in A\}$. This implies that U_α can intersect only finitely many elements of $\{U_\alpha : \alpha \in A\}$, i.e. the nerve $N(\omega_1)$ is a locally finite polyhedron.

Let P be the nerve of $\{U_\alpha : \alpha \in A\}$. By definition, the points of P are functions $p\colon \{U_\alpha : \alpha \in A\} \to [0;1]$, which are non-zero only for finitely many indexes $\alpha \in A$ and for thos indexes $\sum\limits_{\alpha\in A} p(U_\alpha) = 1$, see [4].

The following function $p_\alpha\colon \{U_\alpha : \alpha \in A\} \to [0;1]$ given by $p_\alpha(U_\alpha) = 1$ and $p_\alpha(U_{\alpha'}) = 0$ for $\alpha \neq \alpha'$, is called the *vertex* of the nerve P.

Let $\{\Psi_\alpha : \alpha \in A\}$ be some partition of unity subordinate to $\{U_\alpha : \alpha \in A\}$ and $\{\Psi^\alpha_\beta : \beta \in B_\alpha\}$ be some partition of unity subordinate to $\{V_\beta : \beta \in B_\alpha\}$. Then $\{\Psi_\alpha \cdot \Psi^\alpha_\beta : \alpha \in A, \beta \in B_\alpha\}$ is a partition of unity subordinate to ω_1. We have the following maps:

$$\Psi\colon X \to P, \qquad x \mapsto \{\Psi_\alpha(x) : \alpha \in A\},$$
$$\lambda\colon X \times T \to N(\omega_1), \qquad (x,t) \mapsto \{\Psi_\alpha(x) \cdot \Psi^\alpha_\beta(t) : \alpha \in A, \beta \in B_\alpha\},$$

$$\gamma\colon P \times T \to N(\omega_1), \qquad (p,t) \mapsto \{p_\alpha \cdot \Psi^\alpha_\beta(t) : p_\alpha = p(U_\alpha)\},$$

and the corresponding commutative diagram:

$$\begin{array}{ccc} X \times T & \xrightarrow{\Psi\times\mathrm{id}} & P \times T \\ {\scriptstyle\lambda}\downarrow & & \downarrow{\scriptstyle\gamma} \\ N(\omega_1) & \xrightarrow{\mathrm{id}} & N(\omega_1) \end{array}$$

which induces the respective commutative diagram on cohomologies:

$$\begin{array}{ccc} \check{H}^n(X \times T) & \xleftarrow{(\Psi\times\mathrm{id})^n} & \check{H}^n(P \times T) \\ {\scriptstyle\gamma^n}\uparrow & & \uparrow{\scriptstyle\lambda^n} \\ \check{H}^n(N(\omega_1)) & \xleftarrow{\mathrm{id}^n} & \check{H}^n(N(\omega_1)) \end{array}$$

Since ω_1 is a refinement of ω, the element a is contained in the image of

$$\lambda^n\colon \check{H}^n(N(\omega_1)) \to \check{H}^n(X \times T)$$

and, therefore, a is contained in the image of

$$(\Psi \times \mathrm{id})^n\colon \check{H}^n(P \times T) \to \check{H}^n(X \times T). \qquad \square$$

Now we are ready to prove the main Theorem 1.2:

Proof. Necessity. Suppose that $\phi^*_r = \phi^*_s\colon \check{H}^*(X \times T) \to \check{H}^*(X)$ for any locally compact paracompact space X. Then it is true for any locally compact connected polyhedron P. Such polyhedra can be represented as a union $P = \bigcup_{i=1}^\infty P^i$ of finite polyhedra P^i such that $P^i \subset \mathrm{int}(P^{i+1})$. According to Lemmas 3.2 and 3.3 the homomorphism $\check{H}^1(T, \{r,s\}) \to \check{H}^1(T)$ splits.

Sufficiency. Conversely, suppose that the homomorphism

$$\check{H}^1(T, \{r,s\}) \to \check{H}^1(T)$$

splits. Let $\alpha \in \check{H}^*(X \times T)$ be any element. By Lemma 3.4 there exists a continuous map $\Psi\colon X \to P$ such that α is contained in the image of

$$(\Psi \times \mathrm{id})^n\colon \check{H}^n(P \times T) \to \check{H}^n(X \times T).$$

Consider the following commutative diagram

$$\begin{array}{ccc} \check{H}^n(P \times T) & \xrightarrow{(\Psi\times\mathrm{id})^n} & \check{H}^n(X \times T) \\ {\scriptstyle\phi^n_{P_r}}\downarrow\downarrow{\scriptstyle\phi^n_{P_s}} & & {\scriptstyle\phi^n_r}\downarrow\downarrow{\scriptstyle\phi^n_s} \\ \check{H}^n(P) & \xrightarrow{\Psi^n} & \check{H}^n(X) \end{array}$$

Then it follows from Lemmas 3.3, 3.2, and 3.1 that $\phi^n_{P_r} = \phi^n_{P_s}$, and therefore $\phi^n_r(\alpha) = \phi^n_s(\alpha)$. □

The following is open:

Problem 3.5. Does the main Theorem 1.2 hold for any paracompact space?

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