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Semi-reducible pseudo-Riemannian spaces with additional conditions

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Abstract. We consider semi-reducible pseudo-Riemannian spaces with algebraic conditions on the Ricci tensor and the Riemann tensor. For almost Einstein and weakly recurrent spaces we find the type of tensor characteristic of semi-reducibility. Semi-reducible almost Einstein spaces and weakly recurrent spaces are divided into types depending on the properties of the vector fields that exist in them by necessity. The study is carried out locally in the tensor form.

Анотація. В роботі розглядаються напівзвідні псевдоріманові простори з алгебраїчними умовами на тензор Річчі та тензор Рімана. Для майже ейнштейнових та слаборекурентних просторів знайдено вид тензорної ознаки напівзвідності. Напівзвідні майже ейнштейнові простори та слаборекурентні простори розділені на типи в залежності від властивостей векторних полів, які в них існують за необхідністю. Дослідження ведуться локально в тензорній формі.

1. Introduction

The semi-reducible expansion of a metric of a pseudo-Riemannian space V_n $(n > 2)$ with a metric tensor g_{ij} is its representation in the form:

$$ds^2 = ds_1^2(x^1, x^2, \dots, x^r) + \sigma(x^1, x^2, \dots, x^r)\, ds_2^2(x^{r+1}, x^{r+2}, \dots, x^n),$$

where ds_1^2 and ds_2^2 are independent metrics, depending on different coordinates, and the function σ depends only on the coordinates ds_1^2. A space V_n admitting at least one semi-reducible expansion is called *semi-reducible*. Semi-reducible spaces are a generalization of reducible spaces [17].

A pseudo-Riemannian space V_n is semi-reducible if and only if there exists a symmetric idempotent tensor, not proportional to the metric tensor, such that

$$b_{\alpha i} b^{\alpha}_{j} = b_{ij}, \tag{1.1}$$

$$b_{ij,k} = u_i b_{jk} + u_j b_{ik}, \tag{1.2}$$

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where $u_i = u_{,i} = \partial_i u$ is some gradient vector, comma is a sign of covariant derivative, and $b^i_j = b_{\alpha j} g^{\alpha i}$, g^{ij} are elements of a matrix inverse to g_{ij}.

Equations (1.1) and (1.2) are called *tensor characteristic of semi-reducibility of pseudo-Riemannian spaces* [3].

The vector u_i in the tensor characteristic of semi-reducible pseudo-Riemannian spaces satisfies the following conditions:

$$u_\alpha b^\alpha_j = 0. \tag{1.3}$$

$$u_{\alpha,i} b^\alpha_j = -u_\alpha u^\alpha b_{ij}, \tag{1.4}$$

where $u^i = u_\alpha g^{\alpha i}$. Moreover,

$$(u_{l,i} - u_l u_i) b = (u^\alpha_{\alpha,} - u_\alpha u^\alpha) b_{il} + b_{\alpha i} R^\alpha_l - b_{\alpha\beta} R^{\alpha\beta}_{.il.}. \tag{1.5}$$

In this identity

$$b = b_{\alpha\beta} g^{\alpha\beta}, \qquad u^\alpha_{\alpha,} = u_{\alpha,\beta} g^{\alpha\beta}, \qquad R^{hj}_{il} = R^h_{il\beta} g^{\alpha j},$$

where R^h_{ijk} is the Riemann tensor, and $R_{ij} = R^\alpha_{ij\alpha}$ is the Ricci tensor [3].

Let us recall some of earlier results.

A pseudo-Riemannian space V_n is called *spaces of constant curvature* if the Riemannian tensor V_n satisfies the following condition

$$R^h_{ijk} = \frac{R}{n(n-1)} \big(\delta^h_k g_{ij} - \delta^h_j g_{ik}\big),$$

where δ^h_k is Kronecker symbols, and $R \coloneqq R_{\alpha\beta} g^{\alpha\beta}$ is a scalar curvature.

In semi-reducible spaces of constant curvature, the following identity holds

$$b_{ij} = \gamma \rho_i \rho_j,$$

where γ is some invariant. Moreover $\gamma \rho_\alpha \rho^\alpha = 1$, see [3, 4].

A pseudo-Riemannian space V_n is called *Ricci symmetric*, whenever the following conditions hold for the Ricci tensor V_n, [6, 10]:

$$R_{ij,k} = 0.$$

In semi-reducible Ricci symmetric spaces we get the following relation:

$$u_\alpha R^\alpha_i = \tfrac{1}{\tau} u_i, \qquad b_{\alpha i} R^\alpha_j = \tfrac{1}{\tau} b_{ij},$$

where $\frac{1}{\tau}$ is some constant.

In semi-reducible Ricci-symmetric spaces, the vector u_i satisfies the following conditions:

$$b\big(u_{i,j} - u_i u_j\big) = \big(u^\alpha{}_{,\alpha} - u_\alpha u^\alpha + \tfrac{1}{\tau}\big) b_{ij} - b_{\alpha\beta} R^\alpha{}_{ij}{}^\beta,$$

$$u^\alpha{}_{,\alpha k} - u^\alpha{}_{,k\alpha} = \tfrac{1}{\tau} u_i.$$

Spaces in which the following conditions hold for the Ricci tensor

$$R_{ij} = \frac{R}{n} g_{ij},$$

are called *Einstein spaces.*

For Einstein spaces, the scalar curvature R is always constant. Einstein spaces belong to the Ricci symmetric spaces.

In semi-reducible Einstein spaces, $\overset{1}{\tau} = \dfrac{R}{n}$, [2].

A pseudo-Riemannian space is *conformally flat* if and only if following conditions hold:

$$P_{ij,k} - P_{ik,j} = 0.$$
$$R_{hijk} = P_{hk}g_{ij} - P_{hj}g_{ik} + P_{ij}g_{hk} - P_{ik}g_{hj},$$

where

$$P_{ij} = \frac{1}{n-2}\Big(R_{ij} - \frac{1}{2(n-1)} R g_{ij}\Big).$$

If a conformally flat space admits a semi-reducible expansion, then the vector u_i satisfies equations

$$u_{i,j} - u_i u_j + P_{ij} + \mu g_{ij} - e^{2u} b_{ij} = 0,$$

where μ is some invariant such that

$$\mu_k = u_\alpha P^\alpha_k - \mu u_k.$$

In this paper, we will continue the study of special types of semi-reducible spaces started in [3, 19, 20]. The following three lemmas will be used. We will present their proofs in Section 4.

Lemma 1.1. *If a vector w^l_i is an eigenvector of a tensor b_{ij} with a tensor characteristic of semi-reducibility, then its eigenvalue is either zero or one.*

Lemma 1.2. *If the tensor b_{ij} from the tensor characteristic of semi-reducibility satisfies the condition*

$$b_{ij} = \tau_i \tau_j,$$

then for the Ricci tensor of the semi-reducible pseudo-Riemannian space V_n the followign identity holds true:

$$b_{\alpha i} R^\alpha_j = b^{\alpha\beta} R_{\alpha\beta} b_{ij},$$

where τ_i is some vector, and $b^{ij} = b_{\alpha\beta} g^{\alpha i} g^{\beta j}$.

Lemma 1.3. *Suppose that the following conditions are met for the vector u_i from equation* (1.2)

$$u_{i,j} - u_i u_j + u_\alpha u^\alpha b_{ij} = \tau_i \tau_j. \tag{1.6}$$

Then the vector τ_i is either zero or an eigenvector of the tensor matrix b_{ij} from the tensor characteristic of semi-reducibility with zero eigenvalue.

2. Semi-reducible spaces with conditions on the Ricci tensor

Let us consider semi-reducible pseudo-Riemannian spaces with algebraic conditions imposed on the Ricci tensor.

The following tensor

$$E_{ij} := R_{ij} - \frac{R}{n} g_{ij}, \tag{2.1}$$

is called an *Einstein's tensor.* Note that Einstein spaces are characterized by the identity:

$$E_{ij} = 0. \tag{2.2}$$

Spaces in which the Einstein tensor has the form

$$E_{ij} = \lambda_i \lambda_j, \tag{2.3}$$

for some nonzero gradient vector λ_i are called *almost Einstein spaces*, [5,15].

Taking into account equations (2.1), (2.2) and (2.3), we obtain that

$$R_{ij} - \frac{R}{n} g_{ij} = \lambda_i \lambda_j. \tag{2.4}$$

Note that the vector λ_i is isotropic, i.e. $\lambda_\alpha \lambda^\alpha = 0$, whence

$$\lambda_{\alpha,i} \lambda^\alpha = 0. \tag{2.5}$$

Differentiating (2.4) covariantly we obtain that

$$R_{ij,k} - \frac{R_{,k}}{n} g_{ij} = \lambda_{i,k} \lambda_j + \lambda_i \lambda_{j,k}.$$

Multiplying the latter identity by g^{ik} and wrapping it up by the indices i and k we get that

$$\frac{n-2}{2n} R_{,j} = \lambda_{\alpha,}{}^{\alpha} \lambda_j,$$

where $\lambda_{\alpha,}{}^{\alpha} = \lambda_{\alpha,\beta} g^{\alpha\beta}$.

Thus, a pseudo-Riemannian almost Einstein space V_n $(n > 2)$ has constant scalar curvature only if $\lambda_{\alpha,}{}^{\alpha} = 0$ [13].

The integrability conditions for equation (1.2) can be written down as follows, see [4, 11]:

$$\begin{aligned} b_{\alpha i} R^{\alpha}_{jkl} + b_{\alpha j} R^{\alpha}_{ikl} = (u_{l,i} - u_l u_i) b_{jk} + (u_{l,j} - u_l u_j) b_{ik} \\ - (u_{k,i} - u_k u_i) b_{jl} - (u_{k,j} - u_k u_j) b_{il}. \end{aligned} \tag{2.6}$$

Cycling the latter by the indices i, k, l, we obtain that

$$b_{\alpha i} R^{\alpha}_{jkl} + b_{\alpha k} R^{\alpha}_{jli} + b_{\alpha l} R^{\alpha}_{jik} = 0. \tag{2.7}$$

Multiplying (2.7) by the tensor g^{jk} and wrapping it by the indices j and k:

$$b_{\alpha i} R^{\alpha}_{l} - b_{\alpha l} R^{\alpha}_{i} = 0. \tag{2.8}$$

Substituting the equations (2.2) and (2.1) into (2.8) we finally obtain that

$$b_{\alpha i} E^{\alpha}_{l} - b_{\alpha l} E^{\alpha}_{i} = 0. \tag{2.9}$$

Thus the following theorem holds

Theorem 2.1. *The Einstein tensor of a semi-reducible pseudo-Riemannian space V_n, $(n > 2)$, satisfies the condition* (2.9).

Consider now semi-reducible Einstein spaces. Taking into account (2.3), the equation (2.9) will be written as follows:

$$b_{\alpha i} \lambda^{\alpha} \lambda_l - b_{\alpha l} \lambda^{\alpha} \lambda_i = 0, \tag{2.10}$$

where $\lambda^i = \lambda_\alpha g^{\alpha i}$.

Since $\lambda_i \neq 0$, we can choose a vector ξ^i satisfying $\xi^\alpha \lambda_\alpha = 1$. Multiplying the equation (2.10) by ξ^l and wrapping it by the index l we get that

$$b_{\alpha i} \lambda^{\alpha} = \overset{2}{\tau} \lambda_i,$$

where $\overset{2}{\tau} = b_{\alpha\beta} \lambda^{\alpha} \xi^{\beta}$.

Now we get from Lemma 1.1 the following

Theorem 2.2. *In a semi-reducible pseudo-Riemannian almost Einstein space, the vector λ_i from equation* (2.4) *is an eigenvector of the tensor matrix b_{ij} corresponding to the eigenvalue* 0 *or* 1.

The above theorem allows us to split semi-reducible almost Einstein pseudo-Riemannian spaces into two types:

(A) when $\overset{2}{\tau} = 0$, and accordingly $\lambda^{\alpha} b_{\alpha i} = 0$;

(B) when $\overset{2}{\tau} = 1$ and then $\lambda^{\alpha} b_{\alpha i} = \lambda_i$.

Multiplying equation (2.6) by the tensor g^{jk} and wrapping it up by the indices j and k, we get the equation (1.5).

Substituting equation (2.4) into (1.5) we obtain that

$$b(u_{l,i} - u_l u_i) = \Big(u_{\alpha,}{}^{\alpha} - u_\alpha u^\alpha + \frac{R}{n}\Big) b_{il} + \lambda_l b_{\alpha i} \lambda^{\alpha} - b_{\alpha\beta} R^{\alpha}{}_{il}{}^{\beta}.$$

For spaces of type (A) the latter equation reduces to the following one:

$$b(u_{l,i} - u_l u_i) = \Big(u_{\alpha,}{}^{\alpha} - u_\alpha u^\alpha + \frac{R}{n}\Big) b_{ij} - b_{\alpha\beta} R^{\alpha}{}_{il}{}^{\beta}, \tag{2.11}$$

while for space of type (B) it will have the following form:

$$b(u_{l,i} - u_l u_i) = \Big(u_{\alpha,}{}^{\alpha} - u_\alpha u^\alpha + \frac{R}{n}\Big) b_{li} + \lambda_l \lambda_i - b_{\alpha\beta} R^{\alpha}{}_{ij}{}^{\beta}. \tag{2.12}$$

Hence the following statement holds:

Theorem 2.3. *If a space is semi-reducible, almost Einstein and pseudo-Riemannian, then the vector u_i in the equation* (1.2) *satisfies the condition* (2.11) *for spaces of type* (A) *and the condition* (2.12) *for spaces of type* (B).

In next section we will consider semi-reducible spaces with certain restrictions on the Riemannian tensor.

3. Semi-reducible pseudo-Riemannian spaces with conditions on the Riemannian tensor

A pseudo-Riemannian space is called *recurrent* if its Riemannian tensor satisfies the following condition:

$$R^h_{ijk,l} = \rho_l R^h_{ijk}, \tag{3.1}$$

for some non-zero vector ρ_l.

Cycling (3.1) through the indices j, k, l, we obtain that

$$\rho_l R^h_{ijk} + \rho_j R^h_{ikl} + \rho_k R^h_{ilj} = 0. \tag{3.2}$$

Note that the condition (3.2) is satisfied for recurrent spaces, however the converse statement is not true in general. That is, the class of pseudo-Riemannian spaces in which the condition (3.2) holds is wider than the class of all recurrent spaces.

A pseudo-Riemannian space in which the condition (3.2) holds are called *weakly recurrent spaces.*

Let us wrap the equation (3.2) by the indices l and h

$$\rho_\alpha R^\alpha_{ijk} = \rho_k R_{ij} - \rho_j R_{ik}. \tag{3.3}$$

Using this identity, we consider further semi-reducible weakly recurrent pseudo-Riemannian spaces.

Let us multiply the equation (2.6) by the vector ρ_m and cycle through the indices k, l, m. Taking into account (3.2), we then obtain that

$$\begin{aligned}\rho_m\big((u_{l,i} - u_i u_l) b_{jk} + (u_{l,j} - u_j u_l) b_{ik} \\ - (u_{k,i} - u_i u_k) b_{jl} - (u_{k,j} - u_j u_k) b_{il}\big) \\ + \rho_k\big((u_{m,i} - u_i u_m) b_{jl} + (u_{m,j} - u_j u_m) b_{il} \\ - (u_{l,i} - u_i u_l) b_{jm} - (u_{l,j} - u_j u_l) b_{im}\big)\end{aligned}$$

$$
\begin{aligned}
&+\rho_l\big((u_{k,i}-u_iu_k)b_{jm}+(u_{k,j}-u_ju_k)b_{im}\\
&\qquad-(u_{m,i}-u_iu_m)b_{jk}-(u_{m,j}-u_ju_m)b_{ik}\big)=0.
\end{aligned}
$$

Let us group the summands:

$$
\begin{aligned}
&b_{jk}\left(\rho_m u_{l,i}-\rho_m u_i u_l-\rho_l u_{m,i}+\rho_l u_i u_m\right)\\
&\quad+b_{ik}\left(\rho_m u_{l,j}-\rho_m u_j u_l-\rho_l u_{m,j}+\rho_l u_j u_m\right)\\
&\quad+b_{jl}\left(\rho_k u_{m,i}-\rho_k u_i u_m-\rho_m u_{k,i}+\rho_m u_k u_i\right)\\
&\quad+b_{il}\left(\rho_k u_{m,j}-\rho_k u_j u_m-\rho_m u_{k,j}+\rho_m u_k u_j\right)\\
&\quad+b_{im}\left(\rho_l u_{k,j}-\rho_l u_j u_k-\rho_k u_{l,j}+\rho_k u_l u_j\right)\\
&\quad+b_{jm}\left(\rho_l u_{k,i}-\rho_l u_i u_k-\rho_k u_{l,i}+\rho_k u_l u_i\right)=0. \qquad (3.4)
\end{aligned}
$$

Multiplying (3.4) by g^{jk}, and wrapping by indices j and k we get

$$
\begin{aligned}
&u_{l,i}\left(b\rho_m-\rho_\alpha b^\alpha_m\right)-u_{m,i}\left(b\rho_l-\rho_\alpha b^\alpha_l\right)\\
&\quad-u_iu_l\left(b\rho_m-\rho_\alpha b^\alpha_m\right)+u_iu_m\left(b\rho_l-\rho_\alpha b^\alpha_l\right) \qquad (3.5)\\
&\quad+b_{il}\left(\rho_\alpha u^\alpha_{,m}-\rho_\alpha u^\alpha u_m-u^\alpha_{\alpha,}\rho_m+u_\alpha u^\alpha\rho_m\right)\\
&\quad-b_{im}\left(\rho_\alpha u^\alpha_{,l}-\rho_\alpha u^\alpha u_l-u^\alpha_{\alpha,}\rho_l+u_\alpha u^\alpha\rho_l\right)=0 \qquad (3.6)
\end{aligned}
$$

Let us multiply further (3.6) by tensor b^{ij}, wrap it by index i, and swap indices i and j:

$$
\begin{aligned}
&b_{il}\left(\rho_\alpha u^\alpha_{,m}-\rho_\alpha u^\alpha u_m-u^\alpha_{\alpha,}\rho_m+u_\alpha u^\alpha\rho_m-u_\alpha u^\alpha b\rho_m+u_\alpha u^\alpha\rho_\alpha b^\alpha_m\right)\\
&\quad-b_{im}\left(\rho_\alpha u^\alpha_{,l}-\rho_\alpha u^\alpha u_l-u^\alpha_{\alpha,}\rho_l+u_\alpha u^\alpha\rho_l-u_\alpha u^\alpha b\rho_l+u_\alpha u^\alpha\rho_\beta b^\beta_l\right)=0.
\end{aligned}
$$

This means that

$$\overset{3}{\tau}_m b_{il}-\overset{3}{\tau}_l b_{im}=0, \qquad (3.7)$$

where

$$\overset{3}{\tau}_m=\rho_\alpha u^\alpha_{,m}-\rho_\alpha u^\alpha u_m-u^\alpha_{\alpha,}\rho_m+u_\alpha u^\alpha\rho_m-u_\alpha u^\alpha b\rho_m+u_\alpha u^\alpha b^\beta_l\rho_\beta. \qquad (3.8)$$

If the vector $\overset{3}{\tau}_m\neq 0$, then we can choose a vector ξ^m so that $\xi^\alpha\,\overset{3}{\tau}_\alpha=1$ and multiply equation (3.7) by the vector ξ^m. After wrapping by the index m, we see that

$$b_{il}=\overset{3}{\tau}_l b_{i\alpha}\xi^\alpha. \qquad (3.9)$$

Let us re-alternate equation (3.9) and take into account the symmetry of the tensor b_{il}

$$\overset{3}{\tau}_l b_{i\alpha}\xi^\alpha-\overset{3}{\tau}_i b_{l\alpha}\xi^\alpha=0. \qquad (3.10)$$

Multiply further the equation (3.10) by ξ^l and wrap by the index lL

$$b_{i\alpha}\xi^\alpha=b_{\alpha\beta}\xi^\alpha\xi^\beta\,\overset{3}{\tau}_i.$$

Then (3.9) will be written down as follows:

$$b_{il} = b_{\alpha\beta}\xi^{\alpha}\xi^{\beta} \overset{3}{\tau}_i \overset{3}{\tau}_l. \tag{3.11}$$

It follows from (1.1) that

$$\left(b_{\alpha\beta}\xi^{\alpha}\xi^{\beta}\right)^2 \overset{3}{\tau}_j \overset{3}{\tau}{}^{\gamma} \overset{3}{\tau}_i \overset{3}{\tau}_{\gamma} = b_{\alpha\beta}\xi^{\alpha}\xi^{\beta} \overset{3}{\tau}_i \overset{3}{\tau}_j,$$

where $\overset{3}{\tau}{}^{i} = \overset{3}{\tau}_{\alpha} g^{\alpha i}$. In other words,

$$b_{\alpha\beta}\xi^{\alpha}\xi^{\beta}\left(b_{\alpha\beta}\xi^{\alpha}\xi^{\beta} \overset{3}{\tau}{}^{\gamma} \overset{3}{\tau}_{\gamma} - 1\right) = 0,$$

whence (3.11) can finally we written as follows:

$$b_{ij} = \frac{1}{\overset{3}{\tau}_{\alpha} \overset{3}{\tau}{}^{\alpha}} \overset{3}{\tau}_i \overset{3}{\tau}_j. \tag{3.12}$$

Note that the vectors u_i and $\overset{3}{\tau}_i$ are orthogonal, i.e.

$$u_{\alpha} \overset{3}{\tau}{}^{\alpha} = 0.$$

Weakly recurrent semi-reducible spaces in which the tensor b_{ij} from the tensor characteristic of semi-reducibility satisfies the condition (3.12) are called *spaces of the first type*. That is, the tensor b_{ij} satisfies the identity

$$b_{ij} = \tau_i \tau_j.$$

In this case, τ_i is a vector such that

$$\tau_i = \frac{1}{\sqrt{\overset{3}{\tau}_{\alpha} \overset{3}{\tau}{}^{\alpha}}} \overset{3}{\tau}_i.$$

Hence, the tensor b_{ij} satisfies the assumption of the semi-reducibility condition of Lemma 1.3 and for the Ricci tensor the following equation holds

$$b_{\alpha i} R^{\alpha}_{j} = b^{\alpha\beta} R_{\alpha\beta} b_{ij}.$$

Thus, (1.5) reduces to the following identity:

$$b(u_{i,j} - u_i u_j) = (u_{\alpha,}{}^{\alpha} - u_{\alpha}u^{\alpha} + b^{\alpha\beta}R_{\alpha\beta})b_{ij} - b_{\alpha\beta}R^{\alpha}{}_{ij}{}^{\beta}. \tag{3.13}$$

Thus we have proved the following

Theorem 3.1. *In semi-reducible weakly recurrent spaces of the first type, a vector from tensor characteristic of semi-reducibility conforms to the condition* (3.13).

Let us consider the case when the vector $\overset{3}{\tau}_i$ is zero. Then, taking into account (3.8), the equation (3.6) reduces to

$$(b\rho_m - \rho_\alpha b^\alpha_m)\,(u_{l,i} - u_i u_l + u_\alpha u^\alpha b_{il}) \\ - (b\rho_l - \rho_\alpha b^\alpha_l)\,(u_{m,i} - u_i u_m + u_\alpha u^\alpha b_{im}) = 0. \qquad (3.14)$$

Denote

$$\overset{4}{\tau}_m := b\rho_m - \rho_\alpha b^\alpha_m.$$

A pseudo-Riemannian semi-reducible weakly recurrent space will be called a *space of the second type* if $\overset{4}{\tau}_m \neq 0$ and *of the third type* if $\overset{4}{\tau}_m = 0$.

For spaces of the second type, one can choose a vector ξ^i such that $\overset{4}{\tau}_\alpha \xi^\alpha = 1$. Then the equation (3.14) will be written as follows:

$$u_{l,i} - u_i u_l + u_\alpha u^\alpha b_{il} = \overset{4}{\tau}_l (u_{\alpha,i} - u_\alpha u_i + u_\beta u^\beta b_{\alpha i})\xi^\alpha.$$

Since the vector u_i is gradient and the tensor b_{il} is symmetric, we obtain that

$$u_{l,i} - u_i u_l + u_\alpha u^\alpha b_{il} = c\,\overset{4}{\tau}_l\,\overset{4}{\tau}_i, \qquad (3.15)$$

where c is some invariant such that

$$c = (u_{\alpha,\beta} - u_\alpha u_\beta + u_\gamma u^\gamma b_{\alpha\beta})\xi^\alpha \xi^\beta.$$

Theorem 3.2. *In semi-reducible weakly recurrent spaces of the second type, a vector from tensor characteristic of semi-reducibility satisfies the condition* (3.15).

It follows from Lemma 1.3, that either $c = 0$ or $\overset{4}{\tau}_i$ is an eigenvector of the tensor b_{ij} with zero eigenvalue.

That is, either

$$u_{l,i} - u_i u_l + u_\alpha u^\alpha b_{il} = 0,$$

or

$$(b\rho_\beta - \rho_\alpha b^\alpha_\beta) b^\beta_i = 0. \qquad (3.16)$$

Taking into account (1.1), the equation (3.16) implies that:

$$(b-1)\rho_\alpha b^\alpha_i = 0.$$

Thus, semi-reducible weakly recurrent spaces of the second type can be split into spaces in which either $c = 0$ or $b = 1$ or $\rho_\alpha b^\alpha_i = 0$.

Consider now semi-reducible weakly recurrent spaces of the third type, in which

$$\rho_\alpha b^\alpha_m = b\rho_m.$$

According to Lemma 1.1, the invariant b can take the value 0 or 1.

Theorem 3.3. *In semi-reducible weakly recurrent spaces of the third type with $b = 1$ the following identity holds:*

$$u_{i,j} - u_i u_j + u^\alpha u_\alpha b_{ij} = b_i^\alpha R_{\alpha j} - R_{ij}.$$

Proof. Multiply the equation (2.6) by the vector $\rho^l = \rho_\alpha g^{\alpha l}$ and wrap it by the index l:

$$\begin{aligned} b_{\alpha i} R^\alpha_{jk\beta}\rho^\beta + b_{\alpha j} R^\alpha_{ik\beta}\rho^\beta = \rho^\alpha(u_{\alpha,i} - u_\alpha u_i) b_{jk} + \rho^\alpha(u_{\alpha,j} - u_\alpha u_j) b_{ik} \\ - (u_{k,i} - u_k u_i) b_{\alpha j}\rho^\alpha - (u_{k,j} - u_k u_j) b_{\alpha i}\rho^\alpha. \end{aligned}$$

Taking into account equations (2.8) and (3.3), the latter identity reduces to the following one:

$$\rho_i A_{kj} + \rho_j A_{ki} = \overset{5}{\tau}_i b_{jk} + \overset{5}{\tau}_j b_{ik}, \tag{3.17}$$

where

$$A_{kj} = R_{kj} - b_j^\alpha R_{\alpha k} + b(u_{k,j} - u_k u_j).$$

On the other hand,

$$\overset{5}{\tau}_i = \rho^\alpha(u_{\alpha,i} - u_\alpha u_i).$$

Let us now re-alternate (3.17) with respect to the indices i and k. Since the tensor A_{ij} is symmetric, i.e. $A_{ij} = A_{ji}$, we obtain that

$$\rho_i A_{kj} - \rho_k A_{ij} = \overset{5}{\tau}_i b_{jk} - \overset{5}{\tau}_k b_{ij}.$$

Swapping the indices k and j we get:

$$\rho_i A_{kj} - \rho_j A_{ik} = \overset{5}{\tau}_i b_{jk} - \overset{5}{\tau}_j b_{ik}. \tag{3.18}$$

Add up equations (3.18) and (3.17) we obtain that

$$\rho_i A_{kj} = \overset{5}{\tau}_i b_{jk}. \tag{3.19}$$

Finally, multiplying (3.19) by tensor b_l^i and wrapping by index i we get that

$$b\rho_l A_{kj} = -u_\alpha u^\alpha b \rho_l b_{kj}.$$

Since $\rho_l \neq 0$ and $b = 1$, the theorem is proved. □

Consider the case when $b = 0$. Then the equation (3.19) reduces to the following one:

$$\rho_i (R_{kj} - b_j^\alpha R_{\alpha k}) = \overset{5}{\tau}_i b_{jk}. \tag{3.20}$$

Multiply equation (3.20) by the tensor b_l^j and wrap it up by the index j. Taking further into account equation (1.1), we will obtain that $\overset{5}{\tau}_i$. Hence we get the following

Theorem 3.4. *In semi-reducible weakly recurrent pseudo-Riemannian spaces of the third type at $b = 0$ for the Ricci tensor the following conditions hold*

$$R_{ij} = b_i^\alpha R_{\alpha j}. \tag{3.21}$$

Multiply the equation (3.21) by the vector u^i and wrap by the index i. Then we obtain that

$$u^\alpha R_{\alpha j} = 0.$$

Let us differentiate covariantly (3.21):

$$R_{ij,k} = u_i R_{jk} + b_i^\alpha R_{\alpha j,k}. \tag{3.22}$$

Recall that a *Ricci flat space* is a pseudo-Riemannian space with vanishing Ricci tensor. Then equation (3.22) leads to the following:

Corollary 3.5. *Suppose that in a semi-reducible weakly recurrent pseudo-Riemannian space of the third type with $b = 0$, its Riemannian tensor satisfies the recurrence condition* (3.1)*. Then the space is Ricci flat.*

Let us multiply equation (3.21) by the vector ρ^i and wrap up by the index i. From equation (3.3) and the former expression, we can see that

$$2\rho^\alpha R_{\alpha i} = R\rho_i = 0.$$

This gives the following

Corollary 3.6. *A semi-reducible weakly recurrent space of type with $b = 0$ is a space of constant scalar curvature.*

4. Proof of Lemmas 1.1, 1.2 and 1.3

Proof of Lemma 1.1. Let's multiply equation (2.8) by the tensor b_j^i and wrap up the index i. Taking into account the equation (1.1), we get

$$\big(\overset{2}{\tau} - 1\big)\lambda^\alpha b_{\alpha i} = 0.$$

Thus, $\overset{2}{\tau}$ is a constant, that equals either 1 or 0. □

Proof of Lemma 1.2. Consider the case when the tensor b_{ij} is decomposed into a bivector, i.e.

$$b_{ij} = \tau_i \tau_j, \tag{4.1}$$

for some vector τ_i.

Condition (1.1) then implies that

$$\tau_i \tau_\alpha \tau^\alpha \tau_j = \tau_i \tau_j.$$

Indeed, otherwise

$$\tau_\alpha \tau^\alpha = 1,$$

and the equation (1.2) reduces to the following one:

$$b_{ij,k} = \tau_k(u_i\tau_j + u_j\tau_i).$$

Multiplying (1.3) by g^{ij} and wrapping by indices i and j we obtain

$$b_{,k} = 2u_\alpha\tau^\alpha\tau_k.$$

Hence

$$u_\alpha\tau^\alpha = 0.$$

Substituting condition (4.1) in equation (2.5) we get

$$\tau_i\tau_\alpha R^\alpha_{jkl} + \tau_j\tau_\alpha R^\alpha_{ikl} = (u_{l,i} - u_l u_i)\tau_j\tau_k + (u_{l,j} - u_l u_j)\tau_i\tau_k \\ - (u_{k,i} - u_k u_i)\tau_j\tau_l - (u_{k,j} - u_k u_j)\tau_i\tau_l.$$

Grouping the terms in the latter equation we obtain

$$\tau_i\left(\tau_\alpha R^\alpha_{jkl} - (u_{l,j} - u_l u_j)\tau_k + (u_{k,j} - u_k u_j)\tau_l\right) \\ + \tau_j\left(\tau_\alpha R^\alpha_{ikl} - (u_{l,i} - u_l u_i)\tau_k - (u_{k,i} - u_k u_i)\tau_l\right) = 0. \tag{4.2}$$

Multiplying (4.2) by τ^i and wrapping by index i

$$\tau_\alpha R^\alpha_{jkl} - (u_{l,j} - u_l u_j)\tau_k + (u_{k,j} - u_k u_j)\tau_l \\ + \tau_j\left(-\tau_\alpha u_{l,\alpha}\tau_k + \tau_l\tau^\alpha u_{k,\alpha}\right) = 0. \tag{4.3}$$

Substituting (4.3) into (4.2) we obtain

$$\tau_i\tau_j\left(\tau_l\tau^\alpha u_{k,\alpha} - \tau_k u_{l,\alpha}\tau^\alpha\right) = 0. \tag{4.4}$$

Hence (4.3) reduces to

$$\tau_\alpha R^\alpha_{jkl} = \tau_k(u_{l,j} - u_l u_j) - \tau_l(u_{k,j} - u_k u_j). \tag{4.5}$$

It also follows from (4.4) that

$$\tau^\alpha u_{k,\alpha} = u_{\alpha,\beta}\tau^\alpha\tau^\beta\tau_k.$$

Multiplying (4.5) by vector τ^k and wrapping by index k, we get

$$\tau_\alpha\tau^\beta R^\alpha_{j\beta l} = u_{l,j} - u_l u_j - u_{\alpha,\beta}\tau^\alpha\tau^\beta\tau_l\tau_j. \tag{4.6}$$

Taking into account equation (4.6) in (4.5)

$$\tau_\alpha R^\alpha_{jkl} = \tau_l\tau_\alpha\tau^\beta R_{\alpha jk\beta} - \tau_k\tau^\alpha\tau^\beta R_{\alpha jl\beta}.$$

In other words,

$$\tau_\alpha R^\alpha_{jkl} = \tau_l b^{\alpha\beta} R_{\alpha jk\beta} - \tau_k\tau^\alpha\tau^\beta R_{\alpha jl\beta}. \tag{4.7}$$

Multiplying (4.7) by g^{jk} and wrapping by indices j and k we get

$$\tau_\alpha R^\alpha_l = b^{\alpha\beta} R_{\alpha\beta}\tau_l. \tag{4.8}$$

Multiplying furhter (4.8) by vector τ_k we finally get

$$b_{k\alpha} R^{\alpha}_{l} = b^{\alpha\beta} R_{\alpha\beta} b_{kl},$$

which proves the lemma. □

Proof of Lemma 1.3. Multiply the equation (1.6) by the tensor b^i_k and wrap it by the index i:

$$b^{\alpha}_{k} u_{\alpha,j} - b^{\alpha} u_{\alpha} u_j + u_{\alpha} u^{\alpha} b^{\alpha}_{k} b_{\alpha j} = b^{\alpha}_{k} \tau_{\alpha} \tau_j.$$

Then taking into account equations (1.4), (1.3) and (1.1), respectively, we obtain that $b^{\alpha}_{k} \tau_{\alpha} \tau_j = 0$. □

Conclusions

The specialization of pseudo-Riemannian spaces is one of the most effective methods of research in differential geometry. In our case, it is the possibility of reducing the metric tensor of a pseudo-Riemannian space to a special form in a certain coordinate system. Such pseudo-Riemannian spaces are called *almost reduced spaces.*

Almost reduced spaces are wide classes of pseudo-Riemannian spaces from general relativity [1], geodesic mapping theory [8], quasi-mapping theory [16, 18] and conformal mapping theory [12, 14], as well as other branches of differential geometry and topology [21].

The need of a choice of a coordinate system complicates the study, so the tensor characteristic of such spaces is used. The tensor characteristic of semi-reducible pseudo-Riemannian spaces is a system of two equations, algebraic and differential, with respect to some tensor b_{ij} and gradient vector u_i. The study of this system leads to significant technical difficulties and, therefore, the method of additional specialization by the type of internal objects [7, 9].

We also described certain properties of the vector from the tensor semi-reducibility characteristic of almost Einstein spaces and weakly recurrent pseudo-Riemannian spaces.

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