

Diagonals of strongly separately continuous functions

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Abstract. We study the diagonals $g(x) = f(x, \dots, x)$ of strongly separately continuous mappings $f: X^n \rightarrow Z$, that is, mappings which for a fixed value of one variable are jointly continuous with respect to the others variables. We prove that for any $n \geq 2$, any topological space X , any strongly σ -metrizable equiconnected space (Z, λ) with a perfect stratification assigned with a mapping λ and every Baire class one mapping $g: X \rightarrow \mathbb{R}$ there exists a strongly separately continuous mapping $f: X^n \rightarrow Z$ with the diagonal g . From this we obtain that for any PP -space space X and any strongly σ -metrizable equiconnected space (Z, λ) with a perfect stratification assigned with a mapping λ the diagonals of strongly separately continuous mappings $f: X^n \rightarrow Z$ are exactly Baire class one mappings. Moreover, we prove that for a countably compact space X the diagonals of strongly separately continuous functions $f: X^n \rightarrow \mathbb{R}$ coincide with the functions of Baire class one if and only if every system of functionally open pairwise disjoint sets in X is at most countable.

Анотація. Вивчаються діагоналі $g(x) = f(x, \dots, x)$ сильно нарізно неперервних відображень $f: X^n \rightarrow Z$, тобто відображень, які при фіксованому значенні однієї змінної є сукупно неперервними відносно решти $n-1$ змінних. Доведено, що для довільних $n \geq 2$, топологічного простору X , сильно σ -метризованого рівномірно зв'язного простору (Z, λ) з досконалим вичерпуванням, узгодженим з відображенням λ , і відображення першого класу Бера $g: X \rightarrow \mathbb{R}$ існує сильно нарізно неперервне відображення $f: X^n \rightarrow Z$ з діагоналлю g . Звідси випливає, що для довільного PP -простору простору X і довільного сильно σ -метризованого рівномірно зв'язного простору (Z, λ) з досконалим вичерпуванням, узгодженим з відображенням λ , діагоналі сильно нарізно неперервних відображень $f: X^n \rightarrow Z$ – це, в точності, відображення першого класу Бера. Крім того доведено, що для зліченно компактного простору X діагоналі сильно нарізно неперервних функцій $f: X^n \rightarrow \mathbb{R}$ збігаються з функціями першого класу Бера тоді і тільки тоді, коли кожна система функціонально відкритих попарно неперетинних множин в X є не більш, ніж зліченною.

2020 Mathematics Subject Classification: 54C05, 54C08, 54C30, 54D30

Keywords: separately continuous function, strongly separately continuous function, Baire one function, diagonal of mapping, countably compact space

DOI: <http://doi.org/10.15673/pigc.v18i3.3029>

1. INTRODUCTION

Let X be a topological space, $n \geq 2$ and $f: X^n \rightarrow \mathbb{R}$ be a mapping. We call a mapping $g: X \rightarrow \mathbb{R}$ defined by $g(x) = f(x, \dots, x)$ for every $x \in X$, the *diagonal* of f .

Investigations of diagonals of separately continuous functions $f: X^2 \rightarrow \mathbb{R}$ were initiated by R. Baire in [1]. It was shown in [1] that diagonals of separately continuous functions of two real variables are exactly pointwise limits of continuous functions. This result was generalized in [6, 8] by H. Lebesgue and H. Hahn for real-valued functions of n real variables.

The above results led to the study of the Baire classification of separately continuous mappings and their analogues. This topic has been intensively developed in the works of many mathematicians (see, for example, [2, 9, 11, 15, 21] and the literature given there). Notice that results on the Baire classification give necessary conditions on the diagonals of separately continuous mappings $f: X^n \rightarrow Z$ for corresponding classes of spaces X and Z . On the other hand, for an arbitrary topological space X the problem of constructing of a separately continuous function $f: X^n \rightarrow \mathbb{R}$ with a given diagonal of the $(n - 1)$ -th Baire class was solved in [16] (for $n = 2$) and [17] (for $n \geq 3$). Moreover, results on the construction of separately continuous mappings with values in abstract spaces Z with a given diagonal were obtained in [10, Corollary 3.2] and [11, Theorem 4].

Notice that the transition from separately continuous mappings of two variables to mappings of three or more variables can be made by considering strongly separately continuous mappings, that is, mappings which for a fixed value of one variable are jointly continuous with respect to the others variables. The history of studying the properties of strongly separately continuous mappings concerns only the investigation of the set of discontinuity points. J. C. Breckenridge and T. Nishiura in [3, Theorem 6.3] characterized the set of discontinuity points of strongly separately continuous functions defined on the product of n metric spaces. In doing so, they used the term *strongly $(n - 1)$ -continuous functions* for such functions. In [5, Theorem 2.1] the authors obtained a similar result for \mathcal{F}_k^+ -continuous functions, i.e. functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$, whose restriction on any k -dimensional flat parallel to vector subspace spanned by k coordinate vectors is continuous. The existence of strongly separately continuous functions on the product of abstract spaces with an one-point discontinuity was investigated in [12] and [13]. In this connection, the question on the diagonals of strongly separately continuous functions naturally arises.

In this paper we study the diagonals of strongly separately continuous mappings. First, using the standard approach, for an arbitrary $n \geq 2$, an arbitrary topological space X and a strongly σ -metrizable equiconnected

space (Z, λ) with a perfect stratification assigned with a mapping λ obtain a solution to the problem of constructing of strongly separately mapping $f: X^n \rightarrow Z$ with a given diagonal of the first Baire class. From this we obtain that for a PP -space X (in particular, metrizable space X), the diagonals of strongly separately continuous mappings $f: X^n \rightarrow Z$ are exactly Baire class one mappings. Furthermore, we prove that for a countably compact space X the diagonals of strongly separately continuous functions $f: X^n \rightarrow \mathbb{R}$ coincide with the functions of Baire class one if and only if every system of functionally open pairwise disjoint sets in X is at most countable.

2. SEPARATE \mathcal{S} -CONTINUITY AND STRONG SEPARATE CONTINUITY

Let $n \geq 2$, X_1, \dots, X_n be sets, $X = \prod_{k=1}^n X_k$, $S \subset \{1, \dots, n\}$ be a non-empty set and $T = \{1, \dots, n\} \setminus S$. We consider the following *projection mappings*

$$\text{pr}_S: X \rightarrow \prod_{k \in S} X_k \quad \text{and} \quad \text{pr}^S: X \rightarrow \prod_{k \in T} X_k$$

defined respectively by

$$\text{pr}_S(x) = (x_k)_{k \in S} \quad \text{and} \quad \text{pr}^S(x) = (x_k)_{k \in T}$$

for every $x = (x_k)_{k=1}^n \in X$. In the future, we will identify a point $x \in X$ and the pair $(\text{pr}_S(x), \text{pr}^S(x))$. If $S = \{k\}$ then we write pr_k and pr^k instead pr_S and pr^S respectively.

Moreover, let X_1, \dots, X_n be topological spaces, Y be a topological space and $f: X \rightarrow Y$ be a mapping. For every $v \in \prod_{k \in T} X_k$ we consider the mapping $f_{S,v}: \prod_{k \in S} X_k \rightarrow Y$ defined by

$$f_{S,v}(u) = f(u, v)$$

for every $u \in \prod_{k \in S} X_k$. If $S = \{1, \dots, n\} \setminus \{k\}$ and $x \in X_k$ then we write $f_{k,x}$ instead $f_{S,x}$.

The following notions were introduced in [14]. We say that f is *S-continuous at a point* $x \in X$ if the mapping $f_{S,\text{pr}^S(x)}$ is continuous at the point $\text{pr}_S(x)$, and f is *S-continuous* if f is *S-continuous* at every point $x \in X$.

Now let \mathcal{S} be a system of nonempty proper subset of the set $\{1, \dots, n\}$. We say that f is *separately S-continuous* if f is *S-continuous* for every $S \in \mathcal{S}$.

Proof of the following statement is obvious, and we omit it.

Proposition 2.1. *Let X_1, \dots, X_n be topological spaces, Y be a topological space and $f: \prod_{k=1}^n X_k \rightarrow Y$ be a mapping. Then*

- (1) f is separately continuous if and only if f is separately \mathcal{S} -continuous for the system $\mathcal{S} = \{\{1\}, \dots, \{n\}\}$;
- (2) if $S \subset \{1, \dots, n - 1\}$ for every $S \in \mathcal{S}$, then the \mathcal{S} -continuity of f is equivalent to the \mathcal{S} -continuity of every mapping from the family

$$(f_{n,x} : x \in X_n);$$

- (3) if $S_1 \subseteq S_2 \subset \{1, \dots, n\}$, then the S_2 -continuity of f implies the S_1 -continuity;
- (4) if f is separately \mathcal{S}_0 -continuous for the system

$$\mathcal{S}_0 = \{S \setminus \{k\} : 1 \leq k \leq n\},$$

then f is separately \mathcal{S} -continuous for any system \mathcal{S} of nonempty proper subset of the set $\{1, \dots, n\}$;

- (5) f is separately \mathcal{S}_0 -continuous for the system

$$\mathcal{S}_0 = \{S \setminus \{k\} : 1 \leq k \leq n\}$$

if and only if for every $k \leq n$ and every $x \in X_k$ the mapping $f_{k,x}$ is continuous.

Since the separate \mathcal{S}_0 -continuity from Proposition 2.1 is the strongest property among all separate \mathcal{S} -continuities, it is natural to call it a *strong separate continuity*.

Clearly, for every $n \geq 2$ the function $\text{sp}_n : \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$\text{sp}_n(x_1, \dots, x_n) = \begin{cases} \frac{x_1 x_2 \cdots x_n}{|x_1|^n + |x_2|^n + \cdots + |x_n|^n}, & x_1^2 + x_2^2 + \cdots + x_n^2 \neq 0, \\ 0, & x_1^2 + x_2^2 + \cdots + x_n^2 = 0, \end{cases}$$

for $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, is strongly separately continuous and discontinuous at $(0, 0, \dots, 0)$.

The following statement shows that for functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, where $n \geq 3$, the strong separate continuity is strictly the strongest property among all separate \mathcal{S} -continuities.

Proposition 2.2. *Let $n \geq 2$, \mathcal{S} be a system of nonempty proper subset of the set $\{1, \dots, n, n + 1\}$ and $\{1, \dots, n\} \notin \mathcal{S}$. Then there exists a separately \mathcal{S} -continuous function $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ which is not strongly separately continuous.*

Proof. It enough to consider the function

$$f(x_1, \dots, x_n, x_{n+1}) = \text{sp}_n(x_1, \dots, x_n). \quad \square$$

3. CONSTRUCTING OF STRONGLY SEPARATELY CONTINUOUS FUNCTIONS WITH GIVEN DIAGONAL

First, we will recall some definitions.

A mapping $f: X \rightarrow Y$ between topological spaces X and Y is called a *mapping of the first Baire class* if there exists a sequence $(f_i)_{i=1}^{\infty}$ of continuous mappings $f_i: X \rightarrow Y$ such that $f_i(x) \rightarrow f(x)$ for every $x \in X$.

A topological space X is called σ -*metrizable* if there exists an increasing sequence $(X_n)_{n=1}^{\infty}$ of closed metrizable subspaces X_n of X such that $X = \bigcup_{n=1}^{\infty} X_n$. The sequence $(X_n)_{n=1}^{\infty}$ is called a *stratification* of X . A σ -metrizable space X is called *strongly σ -metrizable* if there exists a stratification $(X_n)_{n=1}^{\infty}$ of X such that for any convergent sequence $(x_n)_{n=1}^{\infty}$ in X there exists $m \in \mathbb{N}$ with

$$\{x_n : n \in \mathbb{N}\} \subseteq X_m.$$

Moreover, we say that the stratification $(X_n)_{n=1}^{\infty}$ is *perfect* (see, [11]) if there exists a sequence $(\pi_n)_{n=1}^{\infty}$ of continuous projections $\pi_n: X \rightarrow X_n$.

An equiconnected space is a pair (X, λ) of a topological space X and a continuous function $\lambda: X \times X \times [0, 1] \rightarrow X$ such that

$$\lambda(x, y, 0) = x, \quad \lambda(x, y, 1) = y, \quad \lambda(x, x, t) = x,$$

for every $x, y \in X$ and $t \in [0, 1]$.

Let (X, λ) be an equiconnected strongly σ -metrizable space with a stratification $(X_n)_{n=1}^{\infty}$. Then we say that the stratification $(X_n)_{n=1}^{\infty}$ is *assigned with λ* if $\lambda(X_n \times X_n \times [0, 1]) \subseteq X_n$ for every $n \in \mathbb{N}$.

Following [22] we say that a topological space X is a *US-space* if every convergent sequence has exactly one limit to which it converges. Clearly, any T_2 -space is a *US-space*, and any *US-space* is a T_1 -space.

Proposition 3.1.

- (1) *Every strongly σ -metrizable space X is a US-space.*
- (2) *There exists a σ -metrizable space X which is not a US-space.*

Proof. (1). Let $(X_n)_{n=1}^{\infty}$ be a stratification of X such that for any convergent sequence $(x_n)_{n=1}^{\infty}$ in X there exists $m \in \mathbb{N}$ with $\{x_n : n \in \mathbb{N}\} \subseteq X_m$. Moreover, let $x_n \rightarrow a$ and $x_n \rightarrow b$ in X . Then there exists $m \in \mathbb{N}$ such that

$$\{x_n : n \in \mathbb{N}\} \cup \{a, b\} \subseteq X_m.$$

Then $x_n \rightarrow a$ and $x_n \rightarrow b$ in X_m . Therefore, $a = b$.

(2). Let $X = \{x_n : n \in \mathbb{N}\}$, where all x_n are distinct. Moreover, let τ be the topology on X such that $\{x_n : n \geq 2\}$ is a discrete subspace of (X, τ) and for every $i = 1, 2$ the sets

$$\{x_i\} \cup \{x_k : k \geq n\}$$

forms a τ -base at x_i . Since every $X_n = \{x_k : k \leq n\}$ is a discrete subspace of X , X is a σ -metrizable space. Moreover, $x_n \rightarrow x_1$ and $x_n \rightarrow x_2$ in X . \square

We say that a topological space X is a *countably compact space* if every countable open cover of X has a finite subcover.

Let X be a completely regular space and $C(X)$ be the collection of all continuous real-valued functions defined on X . By $C_p(X)$ we denote the space $C(X)$ endowed with the pointwise convergence topology on X .

We will need the following auxiliary statement.

Lemma 3.2. *Let X, Y be topological spaces, $f: X \rightarrow Y$, $x_0 \in X$ and $(U_i)_{i=1}^\infty$ be a sequence of neighborhoods U_i of x_0 in X such that $f(x_i) \rightarrow f(x_0)$ for every sequence $(x_i)_{i=1}^\infty$ of $x_i \in U_i$. Then f is continuous at x_0 .*

Proof. Let V be an arbitrary neighborhood of $f(x_0)$ in Y . Assume that $f(U_i) \not\subseteq V$ for every $i \in \mathbb{N}$. Then for every $i \in \mathbb{N}$ there exists $x_i \in U_i$ with $f(x_i) \notin V$. Now we have that $f(x_i)$ does not converge to $f(x_0)$, which gives a contradiction. \square

The following theorem can be proved analogously to [19, Theorem 1].

Theorem 3.3. *Let $n \geq 2$, X be a topological space, Y be a US-space, (Z, λ) be an equiconnected subspace of Y , $(G_i)_{i=0}^\infty$ and $(F_i)_{i=0}^\infty$ be sequences of open sets G_i and closed sets F_i in X^n , $(\varphi_i)_{i=1}^\infty$ be a sequence continuous functions $\varphi_i: X^n \rightarrow [0, 1]$, $(g_i)_{i=1}^\infty$ be a sequence of continuous mappings $g_i: X \rightarrow Z$ and $g: X \rightarrow Y$ be a mapping such that:*

- (1) $G_0 = F_0 = X^n$ and $\Delta_n = \{(x, \dots, x) \in X^n : x \in X\} \subseteq G_{i+1} \subseteq F_i \subseteq G_i$ for every $i \in \mathbb{N}$;
- (2) $X^n \setminus G_i \subseteq \varphi_i^{-1}(0)$ and $F_i \subseteq \varphi_i^{-1}(1)$ for every $i \in \mathbb{N}$;
- (3) for every $x \in X$, any sequence $(t_i)_{i=1}^\infty$ of $t_i \in [0, 1]$ and any sequence $(x_i)_{i=1}^\infty$ of $x_i \in X$ such that

$$(x_i, x) \in \bigcup_{k=2}^n \text{pr}_{\{1,k\}}(F_{i-1})$$

for all $i \in \mathbb{N}$ we have that

$$\lim_{i \rightarrow \infty} \lambda(g_i(x_i), g_{i+1}(x_i), t_i) = g(x).$$

Then the following function $f: X^n \rightarrow \mathbb{R}$, defined by

$$f(x) = \begin{cases} \lambda(g_i(x_1), g_{i+1}(x_1), \varphi_i(x)), & x \in F_{i-1} \setminus F_i; \\ g(x_1), & x \in E = \bigcap_{i=0}^\infty F_i. \end{cases}$$

for every $x = (x_1, \dots, x_n) \in X^n$, is strongly separately continuous on X^n and joint continuous on the open set $X^n \setminus E$.

Proof. The joint continuity of the function f on the open set $X^n \setminus E$ can be proved in a completely analogous way to the proof of [10, Theorem 3.1]. Notice that for every $x \in X$ we have $(x, x) \in \text{pr}_{\{1,2\}}(F_{i-1})$ for all $i \in \mathbb{N}$.

It follows from (3) that

$$\lim_{i \rightarrow \infty} g_i(x) = \lim_{i \rightarrow \infty} \lambda(g_i(x), g_{i+1}(x), 0) = g(x).$$

Moreover, for every $(x_1, \dots, x_n) \in E$ we have that

$$g(x_1) = g(x_2) = \dots = g(x_n).$$

Indeed, for every $k \geq 2$ we have that $(x_1, x_k) \in \text{pr}_{\{1,k\}}(F_{i-1})$ for all $i \in \mathbb{N}$ and, according to (3),

$$g(x_k) = \lim_{i \rightarrow \infty} \lambda(g_i(x_1), g_{i+1}(x_1), 0) = \lim_{i \rightarrow \infty} g_i(x_1) = g(x_1).$$

It remains to verify that f is strongly separately continuous at every point $x \in E$. Fix any $x = (x_1, \dots, x_n) \in E$ and $k \in \{1, 2, \dots, n\}$ and show that the mapping f_{k,x_k} is continuous at $\text{pr}^k(x)$. For every $i \in \mathbb{N}$ we set

$$U_i = F_{i-1} \cap \text{pr}_k^{-1}(x_k).$$

Let $i \in \mathbb{N}$ and $y = (y_1, \dots, y_n) \in U_i$. Notice that $y_k = x_k$. If $y \in F_i \cap E$, then

$$f(y) = g(y_1) = g(y_k) = g(x_k) = g(x_1) = f(x).$$

If $y \in F_i \setminus E$, then there is $j \geq i$ and $t \in [0, 1]$ such that

$$f(y) = \lambda(g_j(y_1), g_{j+1}(y_1), t).$$

Moreover,

$$(y_1, x_k) = (x_k, x_k) \in \bigcup_{l=2}^n \text{pr}_{\{1,l\}}(F_{j-1})$$

for $k = 1$ and

$$(y_1, x_k) \in \bigcup_{k=2}^n \text{pr}_{\{1,k\}}(F_{j-1})$$

for $k \geq 2$. The condition (3) implies that

$$\lim_{i \rightarrow \infty} f(y_i) = f(x)$$

for every sequence $(y_i)_{i=1}^\infty$ of $y_i \in U_i$. Therefore, f_{k,x_k} is continuous at $\text{pr}^k(x)$ by Lemma 3.2. □

We need the following statement [11, Proposition 3].

Proposition 3.4. *Let X be a metrizable space, (Z, λ) be a strongly σ -metrizable equiconnected space with a perfect stratification $(Z_i)_{i=1}^\infty$ assigned with a mapping λ and $g: X \rightarrow Z$ be a mapping of the first Baire class. Then there exist a sequence $(g_i)_{i=1}^\infty$ of continuous mappings $g_i: X \rightarrow Z_i$ and a sequence $(W_i)_{i=1}^\infty$ of open sets $W_i \subseteq X^2$ such that:*

- (1) $\Delta_2 \subseteq W_i$ for every $i \in \mathbb{N}$;
- (2) $\lim_{i \rightarrow \infty} g_i(x_i) = g(x)$ for every $x \in X$ and for any sequence $(x_i)_{i=1}^\infty$ of points $x_i \in X$ with $(x_i, x) \in W_i$ for all $i \in \mathbb{N}$.

Lemma 3.5. *Let (Z, λ) be an equiconnected space, $z \in Z$, $(z_i)_{i=1}^\infty$ and $(w_i)_{i=1}^\infty$ be sequences of $z_i, w_i \in Z$ such that $z_i \rightarrow z$ and $w_i \rightarrow z$. Then for every sequence $(t_i)_{i=1}^\infty$ of $t_i \in [0, 1]$ we have that $\lambda(z_i, w_i, t_i) \rightarrow z$.*

Proof. It follows from the continuity of λ and the compactness of $[0, 1]$ that for every neighborhood W of z there exists a neighborhood W_1 of z such that $\lambda(W_1 \times W_1 \times [0, 1]) \subseteq W$. □

Theorem 3.6. *Let $n \geq 2$, X be a topological space, (Z, λ) be a strongly σ -metrizable equiconnected space with a perfect stratification $(Z_i)_{i=1}^\infty$ assigned with a mapping λ and $g: X \rightarrow Z$ be a mapping of the first Baire class. Then there exists a strongly separately continuous mapping $f: X^n \rightarrow Z$ with the diagonal g .*

Proof. Since the stratification $(Z_i)_{i=1}^\infty$ is perfect, there is a sequence $(\pi_i)_{i=1}^\infty$ of continuous projections $\pi_i: Z \rightarrow Z_i$. Choose a sequence $(h_i)_{i=1}^\infty$ of continuous mappings $h_i: X \rightarrow Z$ which pointwise converges to g . Consider the sequence $(g_i)_{i=1}^\infty$ of continuous mappings $g_i: X \rightarrow Z_i$, $g_i = \pi_i \circ h_i$. Since $(Z_i)_{i=1}^\infty$ is a stratification of a strongly σ -metrizable space Z , we have that the sequence $(g_i)_{i=1}^\infty$ pointwise converges to g .

Consider the continuous mapping $\psi: X \rightarrow \prod_{i=1}^\infty Z_i$ defined by

$$\psi(x) = (g_i(x))_{i=1}^\infty$$

for every $x \in X$. Since all Z_i are metrizable, the space $Y = \psi(X)$ is metrizable. Moreover, all projection mappings $p_i: Y \rightarrow Z_i$, $p_i((z_j)_{j=1}^\infty) = z_i$, are continuous. Besides, for every $x \in X$ we have that

$$\lim_{i \rightarrow \infty} p_i(\psi(x)) = \lim_{i \rightarrow \infty} g_i(x) = g(x).$$

So, we can consider the mapping $\tilde{g}: Y \rightarrow Z$, defined by

$$\tilde{g}((z_i)_{i=1}^\infty) = \lim_{i \rightarrow \infty} z_i$$

for every $(z_i)_{i=1}^\infty \in Y$. Clearly, \tilde{g} is a Baire one mapping and $g(x) = \tilde{g}(\psi(x))$.

According to Proposition 3.4, there exist a sequence $(\tilde{g}_i)_{i=1}^\infty$ of continuous mappings $\tilde{g}_i: Y \rightarrow Z_i$ and a sequence $(W_i)_{i=1}^\infty$ of open sets $W_i \subseteq Y^2$ such that

$$\Omega = \{(y, y) : y \in Y\} \subseteq W_i, \quad \forall i \in \mathbb{N}, \quad \text{and} \quad \lim_{i \rightarrow \infty} \tilde{g}_i(y_i) = \tilde{g}(y)$$

for every $y \in Y$ and for any sequence $(y_i)_{i=1}^\infty$ of points $y_i \in Y$ such that $(y_i, y) \in W_i$ for all $i \in \mathbb{N}$.

Choose sequences $(\tilde{G}_i)_{i=1}^\infty$ and $(\tilde{F}_i)_{i=1}^\infty$ of open sets \tilde{G}_i and closed sets \tilde{F}_i in Y^2 such that

$$\Omega \subseteq \tilde{G}_{i+1} \subseteq \tilde{F}_i \subseteq \tilde{G}_i \subseteq W_{i+1} \cap W_{i+2}$$

for every $i \in \mathbb{N}$.

Consider also the continuous mapping $\phi: X^2 \rightarrow Y^2$ defined by

$$\phi(x_1, x_2) = (\psi(x_1), \psi(x_2))$$

for all $x_1, x_2 \in X$. Set $G_0 = F_0 = X^n$,

$$G_i = \bigcap_{k=2}^n \text{pr}_{\{1,k\}}^{-1}(\phi^{-1}(\tilde{G}_i)) \quad \text{and} \quad F_i = \bigcap_{k=2}^n \text{pr}_{\{1,k\}}^{-1}(\phi^{-1}(\tilde{F}_i))$$

for every $i \in \mathbb{N}$. Notice that the sequences $(G_i)_{i=0}^\infty$ and $(F_i)_{i=0}^\infty$ of functionally open in X^n sets G_i and functionally closed in X^n sets G_i satisfy the condition (1) from Theorem 3.3. Moreover, there is a sequence $(\varphi_i)_{i=1}^\infty$ of continuous functions $\varphi_i: X^n \rightarrow [0, 1]$ which satisfies (2) from Theorem 3.3.

Consider the sequence $(f_i)_{i=1}^\infty$ of continuous mappings $f_i: X \rightarrow Z$ defined by

$$f_i(x) = \tilde{g}(\psi(x))$$

for every $x \in X$. It remains to verify the condition (3) from Theorem 3.3 for the sequence $(f_i)_{i=1}^\infty$ and the mapping g . According to Lemma 3.5, it is enough to show that for any fixed $x \in X$ and any fixed sequence $(x_i)_{i=1}^\infty$ of $x_i \in X$ with $(x_i, x) \in \bigcup_{k=2}^n \text{pr}_{\{1,k\}}(F_{i-1})$ for all $i \in \mathbb{N}$ we have that

$$\lim_{i \rightarrow \infty} f_i(x_i) = \lim_{i \rightarrow \infty} f_{i+1}(x_i) = g(x).$$

Notice that

$$(\psi(x_i), \psi(x)) \in \tilde{F}_{i-1} \subseteq W_i \cap W_{i+1}.$$

Therefore,

$$\begin{aligned} \lim_{i \rightarrow \infty} f_i(x_i) &= \lim_{i \rightarrow \infty} \tilde{g}_i(\psi(x_i)) & \lim_{i \rightarrow \infty} f_{i+1}(x_i) &= \lim_{i \rightarrow \infty} \tilde{g}_{i+1}(\psi(x_i)) \\ &= \tilde{g}(\psi(x)) & \text{and} & & = \tilde{g}(\psi(x)) \\ &= g(x) & & & = g(x). \end{aligned}$$

Theorem is completed. □

4. CHARACTERIZATION OF DIAGONALS OF STRONGLY SEPARATELY CONTINUOUS MAPPINGS

The following simple observation reduces the study of necessary conditions on the diagonals of strongly separately continuous mappings to the case of mappings of two variables.

Remark 4.1. Let X and Z be topological spaces, $n \geq 2$ and $f: X^n \rightarrow Z$ be a strongly separately continuous mapping. Then the diagonal of f is the diagonal of the separately continuous mapping $h: X^2 \rightarrow Z$ defined by

$$h(x, y) = f(x, y, \dots, y)$$

for every $x, y \in X$.

Therefore, the theorems on the Baire classification of separately continuous mappings of two variables play a decisive role in obtaining the necessary conditions on the diagonal of strongly separately continuous functions. W. Rudin in [21], using locally finite partitions of the unit, showed that any separately continuous mapping defined on the product of a metrizable space and a topological space and with values in a locally convex space belongs to the first Baire class. Further development of Rudin's method led to the emergence of the following notion.

A topological space X is called a *PP-space* if there exist a sequence $((h_{n,i} : i \in I_n))_{n=1}^{\infty}$ of locally finite partitions of unity $(h_{n,i} : i \in I_n)$ on X and sequence $((x_{n,i} : i \in I_n))_{n=1}^{\infty}$ of families $(x_{n,i} : i \in I_n)$ of points $x_{n,i} \in X$ such that for any $x \in X$ and any neighborhood U of x there exists $n_0 \in \mathbb{N}$ such that $x_{n,i} \in U$ if $n \geq n_0$ and $h_{n,i}(x) \neq 0$. Notice that Hausdorff *PP-spaces* coincide with metrically quarter-stratifiable spaces (see [18]) introduced in [2].

Clearly, every metrizable space is a *PP-space*. A typical example of a non-metrizable *PP-space* is the Sorgenfrey line \mathbb{S} (see [7, p. 21]). In this case for every $n \in \mathbb{N}$ we can set $I_n = \mathbb{Z}$, $h_{n,i} = \chi_{[\frac{i}{n}, \frac{i+1}{n})}$ and $x_{n,i} = \frac{i+1}{n}$ for each $i \in \mathbb{Z}$, where χ_A means the characteristic function of the set A .

The following result gives one of the most general results on the Baire classification of separately continuous mappings with values in σ -metrizable spaces.

Theorem 4.2 ([11], Theorem 15). *Let X be a *PP-space*, Y be a topological space and Z be a strongly σ -metrizable equiconnected space with a perfect stratification $(Z_i)_{i=1}^{\infty}$, where every Z_i is an arcwise connected and locally arcwise connected subspace of Z . Then every separately continuous mapping $f: X \times Y \rightarrow Z$ is of the first Baire class.*

For a space Z which satisfies the conditions from Theorem 3.6 we obtain the following corollary.

Corollary 4.3. *Let X be a PP-space, Y be a topological space and (Z, λ) be a strongly σ -metrizable equiconnected space with a perfect stratification $(Z_i)_{i=1}^\infty$ assigned with a mapping λ . Then every separately continuous mapping $f: X \times Y \rightarrow Z$ is of the first Baire class.*

Proof. Since the stratification $(Z_i)_{i=1}^\infty$ assigned with a mapping λ , for every $i \in \mathbb{N}$ the pair (Z_i, λ_i) is an equiconnected space where λ_i is the restriction of λ on $X_i \times X_i \times [0, 1]$. Therefore, every Z_i is an arcwise connected and locally arcwise connected subspace of Z . \square

The following result is a main result of this section.

Theorem 4.4. *Let X be a topological space, (Z, λ) be a strongly σ -metrizable equiconnected space with a perfect stratification $(Z_i)_{i=1}^\infty$ assigned with a mapping λ . Suppose also that at least one of the following condition holds:*

- (1) *every separately continuous mapping $h: X^2 \rightarrow Z$ is of the first Baire class;*
- (2) *X is a PP-space;*
- (3) *X is a metrizable space.*

Then for any mapping $g: X \rightarrow Z$ the following conditions are equivalent:

- (i) *g is of the first Baire class;*
- (ii) *for some $n \geq 2$ there exists a strongly separately continuous mapping $f: X^n \rightarrow Z$ with the diagonal g ;*
- (iii) *for every $n \geq 2$ there exists a strongly separately continuous mapping $f: X^n \rightarrow Z$ with the diagonal g .*

Proof. Notice that (3) \Rightarrow (2). Moreover, (2) \Rightarrow (1) by Corollary 4.3. So, we can consider the case (1) only.

The implication (ii) \Rightarrow (i) follows immediately from Remark 4.1 and (1).

The implication (iii) \Rightarrow (ii) is obvious.

The implication (i) \Rightarrow (iii) follows immediately from Theorem 3.6. \square

5. CHARACTERIZATION OF DIAGONALS OF STRONGLY SEPARATELY CONTINUOUS FUNCTIONS ON THE PRODUCT OF COUNTABLY COMPACT SPACES

A separate direction of research on the Baire classification of separately continuous mappings concerns real-valued functions defined on the product of two spaces satisfying compactness-type conditions. These studies originate from the works [15] and [20]. It was proved in [15] and [20] that if X

and Y are compacts and X satisfies the countable chain condition (*CCC*) then every separately continuous function $f: X \times Y \rightarrow \mathbb{R}$ is of the first Baire class.

Using some result from [4], we will obtain a generalization of a Moran-Rosenthal's result.

Proposition 5.1. *Let X be a countably compact space such that every disjoint system of functionally open sets in X is at most countable, Y be a countably compact space, and $f: X \times Y \rightarrow \mathbb{R}$ be a separately continuous function. Then f is of the first Baire class.*

Proof. We use the following standard construction. Let \mathcal{F}_1 be the collection of all continuous functions $\varphi_1: X \rightarrow \mathbb{R}$, \mathcal{F}_2 be the collection of all continuous functions $\varphi_2: Y \rightarrow \mathbb{R}$, and $\Phi_1: X \rightarrow \mathbb{R}^{\mathcal{F}_1}$ and $\Phi_2: Y \rightarrow \mathbb{R}^{\mathcal{F}_2}$ be the diagonal mappings defined by

$$\Phi_1(x) = (\varphi_1(x))_{\varphi_1 \in \mathcal{F}_1} \quad \text{and} \quad \Phi_2(y) = (\varphi_2(y))_{\varphi_2 \in \mathcal{F}_2}$$

for every $x \in X$ and $y \in Y$. It is easy to check that Φ_1 and Φ_2 are continuous and $X_1 = \Phi_1(X)$ and $Y_1 = \Phi_2(Y)$ are completely regular countably compact spaces. Moreover, every disjoint system of open sets in X_1 is at most countable.

Consider the function $h: X_1 \times Y_1 \rightarrow \mathbb{R}$ defined by

$$h(\Phi_1(x), \Phi_2(y)) = f(x, y)$$

for every $x \in X$ and $y \in Y$. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ be such that $\Phi_1(x_1) = \Phi_1(x_2)$ and $\Phi_2(y_1) = \Phi_2(y_2)$. Then

$$f(x_1, y) = f(x_2, y) \quad \text{and} \quad f(x, y_1) = f(x, y_2)$$

for every $x \in X$ and $y \in Y$. Therefore,

$$f(x_1, y_1) = f(x_2, y_1) = f(x_2, y_2)$$

and the function h is well-defined. Moreover, since for every $x \in X$ the function $f^x(\cdot) = f(x, \cdot)$ belongs to \mathcal{F}_2 and for every $y \in Y$ defined the function $f_y(\cdot) = f(\cdot, y)$ belongs to \mathcal{F}_1 , h is separately continuous.

Consider the associated maps $\psi_1: X_1 \rightarrow C_p(Y_1)$ and $\psi_2: Y_1 \rightarrow C_p(X_1)$ defined by

$$\psi_1(u)(v) = \psi_2(v)(u) = h(u, v)$$

for every $u \in X_1$ and $v \in Y_1$. Since h is separately continuous, ψ_1 and ψ_2 are continuous. According to [4, Theorem 5.4], there exists a dense G_δ -set $A \subseteq X_1$ such that the mapping ψ_1 is continuous with respect to the sup-norm topology on $C(Y_1)$ at every point $u \in A$.

For every $i \in \mathbb{N}$ we choose a system \mathcal{G}_i of non-empty open pairwise disjoint sets G in X_1 such that the set $\bigcup \mathcal{G}_i = \bigcup_{G \in \mathcal{G}_i} G$ is dense in X_1 and

$$\sup_{v \in Y_1} |h(u_1, v) - h(u_2, v)| \leq \frac{1}{i}$$

for any $u_1, u_2 \in G \in \mathcal{G}_i$. Notice that every \mathcal{G}_i is at most countable. Therefore, the system $\mathcal{G} = \bigcup_{i=1}^\infty \mathcal{G}_i$ is at most countable. For any $G \in \mathcal{G}$ we choose $u_G \in G$ and consider the following at most countable set $B = \{u_G : G \in \mathcal{G}\}$.

Now consider the mapping $\phi: Y_1 \rightarrow \mathbb{R}^B$ defined by

$$\phi(v)(b) = h(b, v)$$

for every $v \in Y_1$ and $b \in B$. Since h is separately continuous, ϕ is continuous. Therefore, $\phi(Y_1)$ is a metrizable compact.

Show that for every $v_1, v_2 \in Y_1$ if $\phi(v_1) = \phi(v_2)$, then $h(u, v_1) = h(u, v_2)$ for every $u \in X_1$. Assume that $|h(u, v_1) - h(u, v_2)| \geq \frac{3}{i}$ for some $u \in X_1$ and $i \in \mathbb{N}$. Since the set $\bigcup \mathcal{G}_i$ is dense in Y , there exist $G \in \mathcal{G}_i$ and $u' \in G$ such that $|h(u, v_1) - h(u, v_2)| > \frac{2}{i}$. Now we have that

$$\begin{aligned} |h(u_G, v_1) - h(u_G, v_2)| &\geq |h(u', v_1) - h(u', v_2)| \\ &\quad - |h(u', v_1) \\ &\quad - h(u_G, v_1)| - |h(u', v_2) - h(u_G, v_2)| \\ &> \frac{2}{i} - \frac{1}{i} - \frac{1}{i} = 0. \end{aligned}$$

Let $Z = \psi_2(Y_1)$ then the restriction mapping $\pi: Z \rightarrow \phi(Y_1)$, $\pi(z) = z|_B$, is a bijection. Moreover, π is continuous, Z is a countably compact and $\phi(Y_1)$ is metrizable. Since for every closed subset C of Z the image $\pi(C)$ is closed in $\phi(Y_1)$, π is a homeomorphism and Z is a metrizable compact. According to the Rudin theorem [21], the evaluation mapping

$$e: X_1 \times Z \rightarrow \mathbb{R}, \quad (u, z) \mapsto z(u),$$

is a Baire one function. Choose a sequence $(e_i)_{i=1}^\infty$ of continuous function $e_i: X_1 \times Z \rightarrow \mathbb{R}$ which pointwise converges to e . Then the sequence of continuous functions $f_i: X \times Y \rightarrow \mathbb{R}$, defined by

$$f_i(x, y) = e_i(\Phi_1(x), \psi_2(\Phi_2(y))),$$

pointwise converges to f . □

The following statement shows that the additional condition on the countably compact space X from Proposition 5.1 is necessary for obtaining a characterization of the diagonals of separately continuous functions on the product of two countably compact spaces.

Proposition 5.2. *Let $n \geq 2$ and X be a T_1 -space such that*

- (1) every closed discrete subspace of X is at most countable;
 (2) the diagonal of any strongly separately continuous function $f: X^n \rightarrow \mathbb{R}$ is of the first Baire class.

Then every disjoint system of functionally open sets in X is at most countable.

Proof. Assume that there exists a uncountable system \mathcal{U} of pairwise disjoint non-empty functionally open sets in X . For every $U \in \mathcal{U}$ choose $x_U \in U$ and continuous function $\varphi_U: X \rightarrow [0, 1]$ such that $\varphi_U(x_U) = 1$ and $\varphi_U(x) = 0$ for every $x \in X \setminus U$.

Consider the function $f: X^n \rightarrow \mathbb{R}$, defined by

$$f(x_1, \dots, x_n) = \sum_{U \in \mathcal{U}} \prod_{k=1}^n \varphi_U(x_k)$$

for every $(x_1, \dots, x_n) \in X^n$. Show that f is strongly separately continuous. Fix any $k \leq n$, $x \in X$ and consider the system $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$. If $\mathcal{U}_x = \emptyset$, then $f_{k,x} = 0$ and therefore, it is continuous. If $\mathcal{U}_x \neq \emptyset$, then $\mathcal{U}_x = \{U\}$ for some $U \in \mathcal{U}$. Therefore, $f_{k,x} = h_{k,x}$ where

$$h(x_1, \dots, x_n) = \prod_{k=1}^n \varphi_U(x_k)$$

for every $(x_1, \dots, x_n) \in X^n$. Thus, the function $f_{k,x}$ is continuous.

Now consider the diagonal g of f . According to (2), g is of the first Baire class. Therefore, there exists a sequence $(g_i)_{i=1}^\infty$ of continuous functions $g_i: X \rightarrow \mathbb{R}$ such that $g(x) = \lim_{i \rightarrow \infty} g_i(x)$ for every $x \in X$. Clearly,

$$g(x) = \sum_{U \in \mathcal{U}} (\varphi_U(x))^n \quad \forall x \in X.$$

In particular, $g(x_U) = 1$ for every $U \in \mathcal{U}$ and $g(x) = 0$ for every $x \in X \setminus G$, where $G = \bigcup_{U \in \mathcal{U}} U$.

For every $i \in \mathbb{N}$ set

$$\mathcal{U}_i = \bigcap_{j=1}^\infty \{U \in \mathcal{U} : g_j(x_U) \geq \frac{1}{2}\}.$$

Since $\mathcal{U} = \bigcup_{i=1}^\infty \mathcal{U}_i$ and \mathcal{U} is uncountable, there exists $i \in \mathbb{N}$ such that \mathcal{U}_i is uncountable. It remains to show that the uncountable discrete subspace $A = \{x_U : U \in \mathcal{U}_i\}$ is closed in X .

Let $x \in \bar{A}$. Then for every $j \geq i$ we have that

$$x \in \bar{A} \subseteq g_j^{-1}([\frac{1}{2}, +\infty)).$$

Therefore, $g(x) = \lim_{j \rightarrow \infty} g_j(x) \geq \frac{1}{2}$ and so, $x \in G$. Then there exists an element $U \in \mathcal{U}$ such that $x \in U$. Since $U \cap A \neq \emptyset$ and \mathcal{U} is pairwise disjoint, $U \in \mathcal{U}_i$. Now taking into account that X is a T_1 -space we obtain that

$$x \in \overline{U \cap A} = \overline{\{x_U\}} = \{x_U\} \subseteq A.$$

Thus, A is closed. □

The following theorem is a main result of this section.

Theorem 5.3. *Let X be a countably compact space. Then the following conditions are equivalent:*

- (i) *for every $n \geq 2$ the diagonals of all strongly separately continuous functions $f: X^n \rightarrow \mathbb{R}$ are exactly the functions of the first Baire class;*
- (ii) *for some $n \geq 2$ the diagonals of all strongly separately continuous functions $f: X^n \rightarrow \mathbb{R}$ are exactly the functions of the first Baire class;*
- (iii) *every disjoint system of functionally open sets in X is at most countable.*

Proof. Notice that according to Theorem 3.6, the condition (i) (the condition (ii)) means for every $n \geq 2$ (for some $n \geq 2$) the diagonal of any strongly separately continuous function $f: X^n \rightarrow \mathbb{R}$ is of the first Baire class.

The implication (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). Let \mathcal{F} be the collection of all continuous functions $\varphi: X \rightarrow \mathbb{R}$, $\Phi: X \rightarrow \mathbb{R}^{\mathcal{F}}$ be the continuous diagonal mapping defined by

$$\Phi(x) = (\varphi(x))_{\varphi \in \mathcal{F}}$$

for every $x \in X$ and $Y = \Phi(X)$. Clearly, Y is a completely regular countably compact space.

Firstly, we show that diagonal of any strongly separately continuous function $f: Y^n \rightarrow \mathbb{R}$ is of the first Baire class. Let $f: Y^n \rightarrow \mathbb{R}$ be a strongly separately continuous function. Since Φ is continuous, the function $h: X^n \rightarrow \mathbb{R}$ defined by

$$h(x_1, \dots, x_n) = f(\Phi(x_1), \dots, \Phi(x_n))$$

is strongly separately continuous. According to (ii), there exists a sequence $(\varphi_i)_{i=1}^{\infty}$ of functions $\varphi_i \in \mathcal{F}$ such that $\lim_{i \rightarrow \infty} \varphi_i(x) = h(x, \dots, x)$ for every $x \in X$. Notice that for any $i \in \mathbb{N}$ the function $g_i: Y \rightarrow \mathbb{R}$, which for every $y = (y_{\varphi})_{\varphi \in \mathcal{F}}$ defined by $g_i(y) = y_{\varphi_i}$, is continuous. Moreover, for every

$y \in Y$ and $x \in X$ with $y = \Phi(x)$ we have that

$$\lim_{i \rightarrow \infty} g_i(y) = \lim_{i \rightarrow \infty} y_{\varphi_i} = \lim_{i \rightarrow \infty} \varphi_i(x) = h(x, \dots, x) = f(y, \dots, y).$$

Thus, the diagonal of f is of the first Baire class.

It follows from Proposition 5.2 that every disjoint system of functionally open sets in Y is at most countable. It is easy to see that every disjoint system of functionally open sets in X is at most countable. Really, let \mathcal{U} be a disjoint system of nonempty functionally open sets in X . Then for every $U \in \mathcal{U}$ there exists $\varphi_U \in \mathcal{F}$ such that

$$U = \{x \in X : \varphi(x) \neq 0\}.$$

For every $U \in \mathcal{U}$ we consider the continuous function $\psi_U: Y \rightarrow \mathbb{R}$ defined by

$$\psi_U(y) = y_{\varphi_U}$$

for every $y = (y_\varphi)_{\varphi \in \mathcal{F}} \in Y$ and set

$$V_U = \{y \in Y : \psi_U(y) \neq 0\}.$$

Since the family $(V_U : U \in \mathcal{U})$ is disjoint and consists of nonempty functionally open in Y sets, \mathcal{U} is at most countable.

(iii) \Rightarrow (i). This immediately follows from Remark 4.1, Proposition 5.1 and Theorem 3.6. \square

Notice that Theorem 5.3 is non-comparable with the part of Theorem 4.4 which concerns the case of a PP -space X and $Z = \mathbb{R}$. For example, a discrete space X of infinite cardinality is metrizable (and, therefore, a PP -space), but it is not countably compact. On the other hand, the following example shows that Theorem 5.3 does not follow immediately from Theorem 4.4 for PP -spaces.

Example 5.4. Let X be a subspace of $[0, 1]^{[0, 1]}$ which consists of all functions $x: [0, 1] \rightarrow [0, 1]$ with at most countable support $\{t \in [0, 1] : x(t) \neq 0\}$. According to [7, Example 3.10.17], X is a countably compact space. Since X is dense in the separable space $[0, 1]^{[0, 1]}$, X satisfies the countable chain condition. Moreover, any countably compact PP -space is separable (because all set I_n from the definition of PP -space must be finite). Therefore, X is not a PP -space.

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Received: January 21, 2025, accepted: September 16, 2025.

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