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On weakly 1-convex and weakly 1-semiconvex sets in real Euclidean spaces

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Abstract. The present work concerns generalized convex sets in the real multi-dimensional Euclidean space, known as weakly 1-convex and weakly 1-semiconvex sets. An open set is called *weakly* 1-*convex* (resp. *weakly* 1-*semiconvex*) if, through every boundary point of the set, there passes a straight line (resp. a closed ray) not intersecting the set. A closed set is called *weakly* 1-*convex* (resp. *weakly* 1-*semiconvex*) if it is approximated from the outside by a family of open weakly 1-convex (resp. weakly 1-semiconvex) sets. A point of the complement of a set to the whole space is a 1-*nonconvexity* (resp. 1-*nonsemiconvexity*) *point* of the set if every straight line passing through the point (resp. every ray emanating from the point) intersects the set. It is proved that if the collection of all 1-nonconvexity (resp. 1-nonsemiconvexity) points corresponding to an open weakly 1-convex (resp. weakly 1-semiconvex) set is non-empty, then it is open. It is also proved that the non-empty interior of a closed weakly 1-convex (resp. weakly 1-semiconvex) set in the space is weakly 1-convex (resp. weakly 1-semiconvex).

Анотація. У цій роботі розглядаються узагальнено опуклі множини у дійсному багатовимірному евклідовому просторі, відомі як слабко 1-опуклі та слабко 1-напівопуклі множини. Відкрита множина називається *слабко* 1-*опуклою* (*слабко* 1-*напівопуклою*), якщо через кожну точку її межі проходить пряма (замкнений промінь), що не перетинає множину. Замкнена множина називається *слабко* 1-*опуклою* (*слабко* 1-*напівопуклою*), якщо вона апроксимується ззовні сім'єю відкритих слабко 1-опуклих (слабко 1-напівопуклих) множин. Точка доповнення множини до всього простору є *точкою* 1-*неопуклості* (1-*ненапівопуклості*) множини, якщо кожна пряма, що проходить через точку (кожний промінь, що виходить з точки) перетинає множину. Доведено, що якщо сукупність усіх точок 1-неопуклості (1-ненапівопуклості), які відповідають відкритій слабко 1-опуклій (слабко 1-напівопуклій) множині, непорожня, то вона відкрита. Також доведено, що непорожня внутрішність замкненої

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слабко 1-опуклої (слабко 1-напівопуклої) множини у просторі є слабко 1-опуклою (слабко 1-напівопуклою).

1. Introduction

Weakly m-convex and weakly m-semiconvex sets, $m = 1, 2, \ldots, n-1$, in the real space $\mathbb{R}^n$, $n \geqslant 2$, with the Euclidean norm, can be seen as a generalization of convex sets. These notions were coined by Yurii Zelinskii [11,12]. First, recall the following definitions.

Any m-dimensional affine subspace of $\mathbb{R}^n$, $m = 0, 1, 2, \ldots, n-1$, $n \geqslant 1$, is called an *m-dimensional plane*. A 1-dimensional plane is also known as a *straight line*.

One of two parts of an m-dimensional plane, $m = 1, 2, \ldots, n-1$, of the space $\mathbb{R}^n$, $n \geqslant 2$, into which it is divided by any of its $(m-1)$-dimensional planes (herewith, the points of the $(m-1)$-dimensional plane are included) is said to be an *m-dimensional half-plane*. A 1-dimensional half-plane is also known as a *ray*.

Let us use the following standard notations. For a subset $A \subset \mathbb{R}^n$, let $\overline{A}$ be its closure, $\operatorname{Int} A$ be its interior, and $\partial A = \overline{A} \setminus \operatorname{Int} A$ be its boundary.

Definition 1.1 (Zelinskii [11,12])**.** An open subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, is called *weakly m-convex* (resp. *weakly m-semiconvex*), $m = 1, 2, \ldots, n-1$, if for any point $x \in \partial E$, there exists an m-dimensional plane L (resp. m-dimensional half-plane L) such that $x \in L$ and $L \cap E = \varnothing$.

Say that a set A *is approximated from the outside* by a family of open sets A_k, $k = 1, 2, \ldots$, if $\overline{A}_{k+1}$ is contained in A_k, and $A = \bigcap\limits_k A_k$, ([1]).

It can be proved that any set approximated from the outside by a family of open sets is closed.

Definition 1.2 (Zelinskii [11,12])**.** A closed subset $E \subset \mathbb{R}^n$, $n \geqslant 2$, is called *weakly m-convex* (resp. *weakly m-semiconvex*), $m = 1, 2, \ldots, n-1$, if it can be approximated from the outside by a family of open weakly m-convex (resp. weakly m-semiconvex) sets.

The class of weakly m-convex sets in $\mathbb{R}^n$ is denoted by $\mathbf{WC^n_m}$ and the class of weakly m-semiconvex sets in $\mathbb{R}^n$ is denoted by $\mathbf{WS^n_m}$.

The properties of the class of generalized convex sets on Grassmannian manifolds which are closely related to the properties of the conjugate sets (see [12, Definition 2]) are investigated in [12]. This class includes $\mathbf{WC^n_m}$. Geometric and topological properties of weakly m-convex sets are also investigated in [2,3].

The theory of weakly m-semiconvex sets is newish and is based on the research of some subclass as well as further investigation of weakly m-convex sets also focuses on the similar subclass. In order to determine these subclasses, we need the following definition.

Definition 1.3. A point $x \in \mathbb{R}^n \backslash E$ is called an *m-nonconvexity* (resp. *m-nonsemiconvexity*) *point* of a subset $E \subset \mathbb{R}^n$ if every m-dimensional plane (resp. m-dimensional half-plane) passing through x intersects E.

The set of all m-nonconvexity (resp. m-nonsemiconvexity) points of a subset $E \subset \mathbb{R}^n$ with respect to a fixed $m \in \{1, 2, \ldots, n-1\}$ is called the *m-nonconvexity-point* (resp. *m-nonsemiconvexity-point*) *set* corresponding to E and is denoted by $E_m^{\triangle}$ ($E_m^{\diamondsuit}$). Moreover, $E^{\triangle} := E_1^{\triangle}$, $E^{\diamondsuit} := E_1^{\diamondsuit}$.

The class of weakly m-convex sets in $\mathbb{R}^n$ with non-empty m-nonconvexity-point set is denoted by $\mathbf{WC^n_m} \backslash \mathbf{C^n_m}$ and the class of weakly m-semiconvex sets with non-empty m-nonsemiconvexity-point set in $\mathbb{R}^n$ is denoted by $\mathbf{WS^n_m} \backslash \mathbf{S^n_m}$.

The disconnectedness of any open weakly 1-semiconvex set with non-empty 1-nonsemiconvexity-point set in the plane was established by Zelinskii [11, Theorem 7]. Moreover, the following result is true.

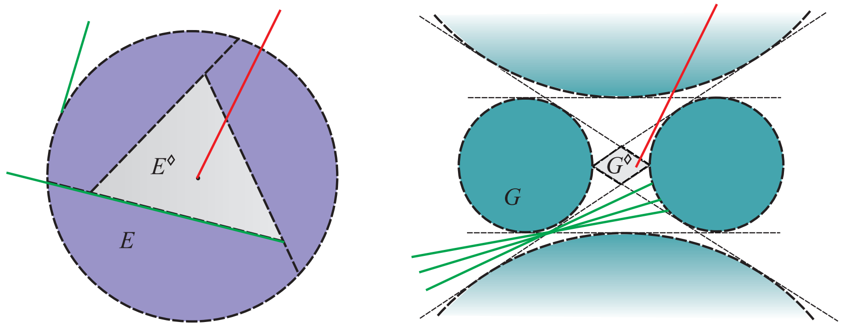
Lemma 1.4 (Dakhil [2], Osipchuk [8]). *An open set or a closed set belonging to the class $\mathbf{WS^2_1} \backslash \mathbf{S^2_1}$ consists of not less than three connected components.*

Interestingly, the number of components of a set belonging to the class $\mathbf{WS^2_1} \backslash \mathbf{S^2_1}$ is also affected by the smoothness of its boundary.

Lemma 1.5 (Osipchuk [4]). *Suppose that an open bounded subset $E \subset \mathbb{R}^2$ with smooth boundary belongs to the class $\mathbf{WS^2_1} \backslash \mathbf{S^2_1}$. Then E consists of not less than four connected components.*

Lemma 1.6 (Osipchuk [8]). *Suppose that a closed bounded subset $E \subset \mathbb{R}^2$ with smooth boundary and such that* $\operatorname{Int} E$ *is not* 1*-semiconvex belongs to the class $\mathbf{WS^2_1} \backslash \mathbf{S^2_1}$. Then E consists of not less than four connected components.*

The example of an open set $E \in \mathbf{WS^2_1} \backslash \mathbf{S^2_1}$ consisting of three components is in Figure 1.1 a), and an open set $G \in \mathbf{WS^2_1} \backslash \mathbf{S^2_1}$ with smooth boundary and four components is in Figure 1.1 b). Moreover, if we want to construct an open set belonging to the class $\mathbf{WS^2_1} \backslash \mathbf{S^2_1}$ with countably infinite number of components, then, instead of a triangle inside a convex set, we should throw away a closed convex generalized polygon (the convex hull of a bounded countably infinite set of points in the plane with boundary containing countably infinite number of vertices). The example of a closed



a) *b)*

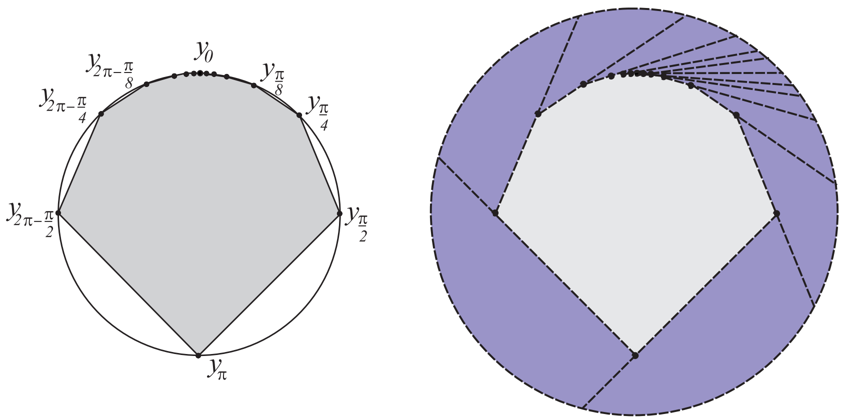
FIGURE 1.1.

convex generalized polygon is the convex hull of the points

$$y_0, \quad y_\pi, \quad y_{\frac{\pi}{2}}, \quad y_{2\pi-\frac{\pi}{2}}, \quad y_{\frac{\pi}{4}}, \quad y_{2\pi-\frac{\pi}{4}}, \quad \ldots, \quad y_{\frac{\pi}{2^k}}, \quad y_{2\pi-\frac{\pi}{2^k}}, \quad \ldots$$

in Figure 1.2 a). And also cut the obtained set along rays containing the polygon sides and the accumulation points of the polygon vertices as it is shown in Figure 1.2 b).

Examples of closed sets belonging to $\mathbf{WS_1^2} \backslash \mathbf{S_1^2}$ with non-smooth or smooth boundary see in [8].



a) *b)*

FIGURE 1.2.

Notice that the above properties of weakly m-semiconvex sets could so far be established only in the plane, in contrast to weakly m-convex sets.

Lemma 1.7 (Dakhil [2], Osipchuk [6])**.** *An open set or a closed set that belongs to the class* $\mathbf{WC^n_{n-1}} \backslash \mathbf{C^n_{n-1}}$ *consists of not less than three connected components.*

But unlike weakly 1-semiconvex sets with smooth boundary, *any open weakly* $(n-1)$*-convex set in* $\mathbb{R}^n$ *with smooth boundary does not have* $(n-1)$*-nonconvexity points* [2, Proposition 2.3.7].

An example of sets belonging to the class $\mathbf{WC^2_1} \backslash \mathbf{C^2_1}$ can be constructed by cutting an open convex set without closed convex polygon or generalized polygon in Figures 1.1 a) and 1.2 b) along the straight lines containing the sides and the accumulation points of vertices of the polygons instead of rays. Examples of open and closed sets belonging to $\mathbf{WC^n_{n-1}} \backslash \mathbf{C^n_{n-1}}$ are presented in [5].

For $n \geqslant 3$ and $m = 1, 2, \ldots, n-2$, the disconnectedness property is violated for both weakly m-convex and weakly m-semiconvex sets.

Lemma 1.8 (Osipchuk [6, 8])**.** *There exist domains and closed connected sets in the space* $\mathbb{R}^n$*,* $n \geqslant 3$*, belonging to the class* $\mathbf{WC^n_m} \backslash \mathbf{C^n_m}$ *(resp.* $\mathbf{WS^n_m} \backslash \mathbf{S^n_m}$*),* $1 \leqslant m < n-1$*.*

Of special interest are the properties of m-nonconvexity-point sets corresponding to weakly m-convex sets and m-nonsemiconvexity-point sets corresponding to weakly m-semiconvex sets. The following results were obtained.

Lemma 1.9 (Osipchuk [7], Osipchuk and Tkachuk[10])**.** *Suppose that an open subset* $E \subset \mathbb{R}^2$ *belongs to the class* $\mathbf{WC^2_1} \backslash \mathbf{C^2_1}$*. Let* $E^{\triangle}_j$*,* $j \in N \subseteq \mathbb{N}$*, be the components of* $E^{\triangle}$*. Then*

(a) $E^{\triangle}$ *is open and weakly* 1*-convex;*

(b) $E^{\triangle}_j$*,* $j \in N$*, are convex (bounded or unbounded);*

Lemma 1.10 (Osipchuk [9])**.** *Suppose that an open subset* $E \subset \mathbb{R}^2$ *belongs to the class* $\mathbf{WS^2_1} \backslash \mathbf{S^2_1}$*. Let* $E^{\diamondsuit}_j$*,* $j \in N \subseteq \mathbb{N}$*, be the components of* $E^{\diamondsuit}$*. Then*

(a) $E^{\diamondsuit}$ *is open and weakly* 1*-semiconvex;*

(b) $E^{\diamondsuit}_j$*,* $j \in N$*, are convex and bounded;*

(c) *any connected subset of* $\partial E^{\diamondsuit}_j$*,* $j \in N$*, consisting of only smooth points is a line segment or a point;*

(d) *there exists a collection of rays* $\{\eta^k\}_{k \in M}$*,* $M \subseteq \mathbb{N}$*, such that*
  - $\bigcup\limits_k \eta^k \supset \partial E^{\diamondsuit}$*,*

- *the set* $\bigcup_k \eta^k \bigcup E^{\diamondsuit}$ *does not contain rays emanating from* $E^{\diamondsuit}$,
- $\bigcup_k \eta^k \bigcap E = \varnothing$.

In other words, Lemma 1.10 shows that the 1-nonsemiconvexity-point set corresponding to a flat weakly 1-semiconvex set is the union of open convex polygons and open convex generalized polygons. But they cannot be arbitrarily placed in the plane. Their arrangement is constrained by property (d).

The methods developed to prove item (a) in Lemmas 1.9 and 1.10 allow us to obtain the following result for the closed weakly 1-convex (weakly 1-semiconvex) sets in the plane.

Lemma 1.11 (Osipchuk [9], Osipchuk and Tkachuk [10])**.** *Let* $E \subset \mathbb{R}^2$ *be a closed subset such that* $\operatorname{Int} E \neq \varnothing$. *If* E *is weakly* 1*-convex* (*resp. weakly* 1*-semiconvex*), *then* $\operatorname{Int} E$ *is weakly* 1*-convex* (*resp. weakly* 1*-semiconvex*).

In this study, we focus on establishing the general topological properties of the 1-nonconvexity-point set corresponding to an open weakly 1-convex set and the 1-nonsemiconvexity-point set corresponding to an open weakly 1-semiconvex set in $\mathbb{R}^n$, $n \geqslant 2$.

First, we prove that *the* 1*-nonsemiconvexity-point set* $E^{\diamondsuit}$ *corresponding to an open set* $E \in \mathbf{WS_1^n} \backslash \mathbf{S_1^n}$, $n \geqslant 2$, *is open.* Therefore, we generalize Lemma 1.10 (a) to the real Euclidean space of any dimension $n \geqslant 2$. The proof is similar to the proof of Lemma 1.10 (a). Its essence is to find, for every fixed point $y \in E^{\diamondsuit}$ and each ray emanating from y, points $x_\alpha(y)$, $\alpha \in S^{n-1}$, on these rays, and the number $d(y) > 0$ such that the points $x_\alpha(y)$ are contained in E together with open balls of the same radii $d(y)$ and centered at $x_\alpha(y)$. This allows us to assert that any ray emanating from an open ball with center at y and radius $\varepsilon \leqslant d(y)$ intersects the union of the balls contained in E. Thus, we show that y is an inner point of $E^{\diamondsuit}$.

To find $x_\alpha(y) \in E \subset \mathbb{R}^2$, $\alpha \in [0, 2\pi]$, it was used the connectedness of the components of E. Namely, there were constructed a finite number of curves contained in E and such that every ray emanating from y intersects the union of the curves. Moreover, it was shown that points $x_\alpha(y)$ are actually placed on those curves and $d(y)$ is the minimum value of the restrictions of the distance functions defined on the components of E to the respective curves. This trick fails for the set E in the spaces of higher dimensions, obviously. But we are lucky to find not one-dimensional compacts that meet our requirements.

Using the same algorithm, we also prove that *the* 1*-nonconvexity-point set* $G^{\triangle}$ *corresponding to an open set* $G \in \mathbf{WC_1^n} \backslash \mathbf{C_1^n}$, $n \geqslant 2$, *is open.* But in this case, we show that, for every fixed point $y \in G^{\triangle}$ and each straight line passing through y, there exist points $x_\alpha(y)$, $\alpha \in S^{n-1}$, on these lines, and the number $d(y) > 0$ such that the points $x_\alpha(y)$ are contained in G together with open balls of the same radii $d(y)$.

The methods developed to prove the first two results allow us to generalize Lemma 1.11 on closed weakly 1-semiconvex and closed weakly 1-convex sets in $\mathbb{R}^n$, $n \geqslant 2$.

Property (d) of Lemma 1.10 easily extends to all spaces with dimensions $n \geqslant 2$, and is also inherent to weakly 1-convex sets of those spaces.

Our final result refutes the validity of Lemma 1.9 (b) and Lemma 1.10 (b) for the spaces $\mathbb{R}^n$, $n \geqslant 3$, as we construct examples of simultaneously weakly 1-convex and weakly 1-semiconvex open sets in $\mathbb{R}^n$, $n \geqslant 3$, which non-empty 1-nonconvexity-point sets are connected, non-convex, bounded (or unbounded), and coincide with their 1-nonsemiconvexity-point sets.

We give here briefly some directions for further research. Probably the most natural next question to study would be to investigate the general topological properties of $E^{\diamondsuit}_m$, $E \in \mathbf{WS^n_m} \backslash \mathbf{S^n_m}$, and $G^{\triangle}_m$, $G \in \mathbf{WC^n_m} \backslash \mathbf{C^n_m}$, for $n \geqslant 2$, $m \geqslant 1$. We expect $E^{\diamondsuit}_m$ and $G^{\triangle}_m$ to be open (resp. closed) if E and G are open (resp. closed). In addition, the question of estimating the number of components of the sets belonging to the class $\mathbf{WS^n_{n-1}} \backslash \mathbf{S^n_{n-1}}$ remains open for $n > 2$.

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2. Main results

Given two points $x, y \in \mathbb{R}^n$, we will denote by xy the open line segment between those points and by $\|x - y\|$ its length. Let also

$$\begin{aligned} U(y,\varepsilon) &:= \{x \in \mathbb{R}^n : \|x-y\| < \varepsilon\}, \quad y \in \mathbb{R}^n, \quad \varepsilon > 0; \\ S^{n-1} &:= \{z \in \mathbb{R}^n : \|z\| = 1\}; \\ \eta_\alpha(y) &:= \{t\alpha + y : t \in [0,+\infty)\}, \quad \alpha \in S^{n-1}, \quad y \in \mathbb{R}^n; \\ \gamma_\alpha(y) &:= \{t\alpha + y : t \in (-\infty,+\infty)\}, \quad \alpha \in S^{n-1}, \quad y \in \mathbb{R}^n. \end{aligned}$$

The following lemma combines two statements:

(i) for a set with non-empty 1-nonsemiconvexity-point set;
(ii) for a set with non-empty 1-nonconvexity-point set.

For convenience, blue text corresponds to case (i), red text corresponds to case (ii), and black text corresponds to both cases.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be an open subset and $E^{\diamondsuit} \neq \varnothing$ (resp. $E^{\triangle} \neq \varnothing$). Let also $y \in E^{\diamondsuit}$ (resp. $y \in E^{\triangle}$). Then for any ray $\eta_\alpha(y)$ (resp. straight line $\gamma_\alpha(y)$), $\alpha \in S^{n-1}$, there exists a point*

$$x_\alpha(y) \in \eta_\alpha(y) \cap E \quad (\text{resp. } x_\alpha(y) \in \gamma_\alpha(y) \cap E)$$

such that $U(x_\alpha(y), d(y)) \subset E$, where $d(y) > 0$ depends on only y and does not depend on α.

Proof. Let $O \in \mathbb{R}^n$ be the origin. Consider the homeomorphism

$$\phi\colon \mathbb{R}^n \backslash \{O\} \to S^{n-1} \times (0, +\infty)$$

defined by the formula

$$\phi(z) := \Big(\frac{z}{\|z\|}, \|z\|\Big).$$

Let also $\sigma : S^{n-1} \times (0, +\infty) \to S^{n-1}$ be the central projection on the sphere, i.e.,

$$\sigma(z) := \frac{z}{\|z\|}.$$

Then σ is open.

Fix an arbitrary point $y \in E^{\diamondsuit}$ (resp. $y \in E^{\triangle}$). Without loss of generality, suppose that $y = O$.

Let z_α be an arbitrary fixed point of $\eta_\alpha(y) \cap E$ (resp. $\gamma_\alpha(y) \cap E$), $\alpha \in S^{n-1}$. Since E is open, there exist open balls $U_\alpha := U(z_\alpha, \varepsilon_\alpha)$, $\alpha \in S^{n-1}$, such that $\overline{U_\alpha} \subset E$. Then the images $\sigma(U_\alpha)$, $\alpha \in S^{n-1}$, are open subsets of the sphere S^{n-1} (resp. the projective space $\mathbb{R}\mathbf{P}^{n-1}$). Moreover,

$$\bigcup_{\alpha \in S^{n-1}} \sigma(U_\alpha)$$

is a cover of S^{n-1} (resp. $\mathbb{R}\mathbf{P}^{n-1}$). By the Heine-Borel theorem, there exists a subcover

$$\bigcup_{j \in M} \sigma(U_{\alpha_j}), \quad \alpha_j \in S^{n-1}, \quad j \in M, \quad M \text{ is finite},$$

of S^{n-1} (resp. $\mathbb{R}\mathbf{P}^{n-1}$).

Let E_i, $i \in N \subseteq \mathbb{N}$, be the components of E. Then for any $j \in M$ there exists $i(j) \in N$ such that $\overline{U_{\alpha_j}} \subset E_{i(j)}$.

Consider the distance functions

$$d_i(x) := \inf_{x^0 \in \partial E_i} \|x - x^0\|, \quad x \in E_i, \quad i \in N.$$

They are continuous in the domains E_i, $i \in N$. Then their restrictions to the compacts $\overline{U_{\alpha_j}}$ attain their minimum values $d_j > 0$ on that compacts,

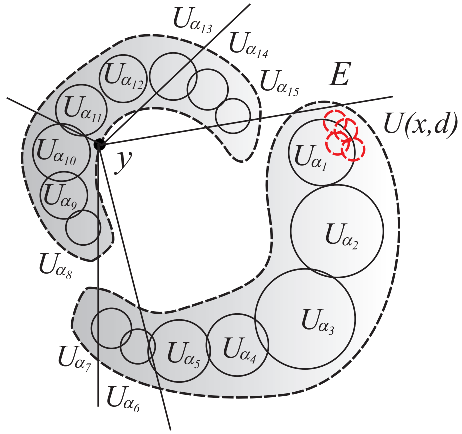


FIGURE 2.1.

i.e.,

$$d_j \coloneqq \min_{x \in \overline{U_{\alpha_j}}} d_{i(j)}(x), \quad j \in M.$$

Since M is finite, there exists

$$d \coloneqq \min_{j \in M} d_j > 0.$$

Then $U(x, d) \subset E$ for any point $x \in \overline{U_{\alpha_j}}$, $j \in M$, see Figure 2.1. And for any $\alpha \in S^{n-1}$, there exists $j \in M$ such that $\eta_\alpha(y) \cap \overline{U_{\alpha_j}} \neq \varnothing$ (resp. $\gamma_\alpha(y) \cap \overline{U_{\alpha_j}} \neq \varnothing$) by the construction. $\square$

Theorem 2.2. *Suppose that an open subset $E \subset \mathbb{R}^n$ belongs to the class $\mathbf{WS_1^n} \backslash \mathbf{S_1^n}$. Then $E^\diamondsuit$ is open.*

Proof. Fix an arbitrary point $y \in E^\diamondsuit$ and show that it is an inner point of $E^\diamondsuit$. Since E is weakly 1-semiconvex, it follows that $y \notin \partial E$. Then there exists a number $\varepsilon_1 > 0$ such that $U(y, \varepsilon_1) \subset (\mathbb{R}^n \backslash \overline{E})$.

By Lemma 2.1, for the fixed y there exist points

$$x_\alpha \in \eta_\alpha(y) \cap E, \quad \alpha \in S^{n-1},$$

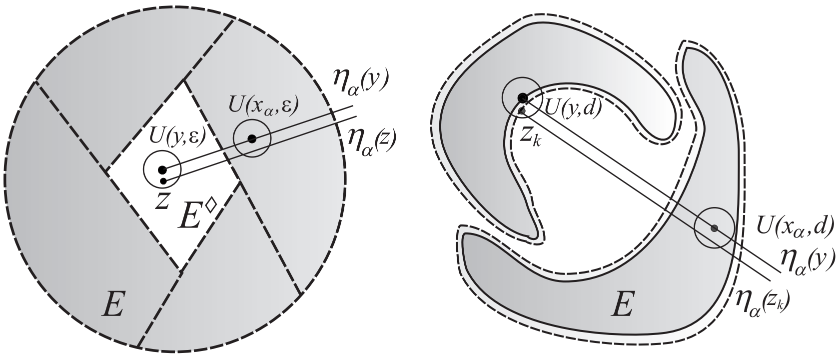
and a constant $d > 0$ such that $U(x_\alpha, d) \subset E$.

Let $\varepsilon \coloneqq \min\{\varepsilon_1, d\}$. Consider the neighborhood $U(y, \varepsilon)$ of the point y. Let $z \in U(y, \varepsilon)$ and let $\eta_\alpha(z)$, $\alpha \in S^{n-1}$, be an arbitrary ray with initial point at z. Draw the ray $\eta_\alpha(y)$ parallel to the ray $\eta_\alpha(z)$.

Since $U(x_\alpha, \varepsilon) \subseteq U(x_\alpha, d) \subset E$ for the point x_α corresponding to $\eta_\alpha(y)$, it follows that

$$\eta_\alpha(z) \cap U(x_\alpha, \varepsilon) \neq \varnothing$$

and, therefore, $\eta_\alpha(z) \cap E \neq \varnothing$ for any $\alpha \in S^{n-1}$, see Figure 2.2 a). Thus, z is a 1-nonsemiconvexity point of E. Since z is arbitrary, it implies that all points of $U(y, \varepsilon)$ are 1-nonsemiconvexity points of E. Hence, y is an inner point of $E^\diamondsuit$. ☐



a) b)

FIGURE 2.2.

Corollary 2.3. *Suppose that an open subset $E \subset \mathbb{R}^n$ belongs to the class $\mathbf{WS_1^n} \backslash \mathbf{S_1^n}$. Then $E^\diamondsuit$ is weakly 1-semiconvex.*

Proof. Since $E^\diamondsuit$ is open, for any point $y \in \partial E^\diamondsuit$ there exists a ray $\eta_{\alpha'}(y)$, $\alpha' \in S^{n-1}$, not intersecting E. Then $\eta_{\alpha'}(y) \cap E^\diamondsuit = \varnothing$ by Definition 1.3. Thus, $E^\diamondsuit$ is weakly 1-semiconvex. ☐

Theorem 2.4. *Suppose that an open subset $E \subset \mathbb{R}^n$ belongs to the class $\mathbf{WC_1^n} \backslash \mathbf{C_1^n}$. Then $E^\triangle$ is open.*

Proof. The scheme of proving this theorem is exactly the same as for Theorem 2.2. We fix an arbitrary point of $E^\triangle$ and show that there exists a neighborhood of this point which belongs to $E^\triangle$.

To do so, we use Lemma 2.1 with respect to the points $y \in E^\triangle$ and the straight lines $\gamma_\alpha(y)$, $\alpha \in S^{n-1}$, and we also replace the rays with the straight lines everywhere in the proof of Theorem 2.2. ☐

Corollary 2.5. *Suppose that an open subset $E \subset \mathbb{R}^n$ belongs to the class $\mathbf{WC_1^n} \backslash \mathbf{C_1^n}$. Then $E^\triangle$ is weakly 1-convex.*

Theorem 2.6. *Let $E \subset \mathbb{R}^n$ be a closed subset such that* $\operatorname{Int} E \neq \varnothing$*. If E is weakly 1-semiconvex, then* $\operatorname{Int} E$ *is weakly 1-semiconvex.*

Proof. Suppose that $\operatorname{Int} E$ is not weakly 1-semiconvex. Then there exists a 1-nonsemiconvexity point $y \in \partial E$ of the set $\operatorname{Int} E$.

By Lemma 2.1, for the point y, there exist points $x_\alpha \in \eta_\alpha(y) \cap \operatorname{Int} E$, $\alpha \in S^{n-1}$, and a constant $d > 0$ such that $U(x_\alpha, d) \subset \operatorname{Int} E$.

Consider the neighborhood $U(y, d)$ of the point y; see Figure 2.2 b). Since E is weakly 1-semiconvex, there exists a family of open weakly 1-semiconvex sets G_k, $k = 1, 2, \ldots$, approximating E from the outside. Then there exists an index k_0 such that

$$\partial G_k \cap U(y, d) \neq \varnothing \quad \forall k \geqslant k_0.$$

For each $k \geqslant k_0$, choose a point $z_k \in \partial G_k \cap U(y, d)$ and draw an arbitrary ray $\eta_\alpha(z_k)$, $\alpha \in S^{n-1}$, with initial point at z_k. Consider the ray $\eta_\alpha(y)$ parallel to $\eta_\alpha(z_k)$. Since $U(x_\alpha, d) \subset \operatorname{Int} E \subset E$ for the point x_α corresponding to $\eta_\alpha(y)$, it follows that

$$\eta_\alpha(z_k) \cap U(x_\alpha, d) \neq \varnothing$$

and, therefore, $\eta_\alpha(z_k) \cap E \neq \varnothing$. Since $G_k \supset E$, $k = 1, 2, \ldots$, we obtain that

$$\eta_\alpha(z_k) \cap G_k \neq \varnothing, \quad \forall k \geqslant k_0.$$

Since the ray $\eta_\alpha(z_k)$ is arbitrary, the point $z_k \in \partial G_k$ is a 1-nonsemiconvexity point of G_k for all $k \geqslant k_0$, which gives a contradiction. □

Theorem 2.7. *Let $E \subset \mathbb{R}^n$ be a closed subset such that $\operatorname{Int} E \neq \varnothing$. If E is weakly 1-convex, then $\operatorname{Int} E$ is weakly 1-convex.*

Proof. The proof of this theorem is the same as the proof of Theorem 2.6. We only consider weakly 1-convex sets instead of weakly 1-semiconvex and replace the rays with the straight lines everywhere in the proof of Theorem 2.6. □

Proposition 2.8. *Suppose that an open subset $E \subset \mathbb{R}^n$ belongs to the class $\mathbf{WS^n_1} \backslash \mathbf{S^n_1}$. Then there exists a collection of rays $\{\eta(x)\}_{x \in \partial E^\diamondsuit}$ such that*

- *the set $\bigcup\limits_{x \in \partial E^\diamondsuit} \eta(x) \bigcup E^\diamondsuit$ does not contain rays emanating from $E^\diamondsuit$,*
- $\bigcup\limits_{x \in \partial E^\diamondsuit} \eta(x) \bigcap E = \varnothing$.

Proof. Since $E^\diamondsuit$ is open, for any point $x \in \partial E^\diamondsuit$, there exists a ray $\eta(x)$ such that $\eta(x) \cap E = \varnothing$. Moreover, $\bigcup\limits_{x \in \partial E^\diamondsuit} \eta(x) \cup E^\diamondsuit$ does not contain any ray emanating from $E^\diamondsuit$, otherwise, each ray

$$\eta(y) \subset \bigcup_{x \in \partial E^\diamondsuit} \eta(x) \cup E^\diamondsuit, \quad y \in E^\diamondsuit,$$

does not intersect E, which contradicts the definition of 1-nonsemiconvexity point. □

Proposition 2.9. *Suppose that an open subset $E \subset \mathbb{R}^n$ belongs to the class $\mathbf{WC_1^n} \backslash \mathbf{C_1^n}$. Then there exists a collection of straight lines $\{\gamma(x)\}_{x \in \partial E^{\triangle}}$ such that*

- *the set $\bigcup\limits_{x \in \partial E^{\triangle}} \gamma(x) \bigcup E^{\triangle}$ does not contain straight lines passing through $E^{\triangle}$,*
- $\bigcup\limits_{x \in \partial E^{\triangle}} \gamma(x) \bigcap E = \varnothing$.

Proof. The statements are similar to the proof of Proposition 2.8. We only consider straight lines instead of rays. □

Lemma 2.10. *There exists an open set $E^3 \in \left(\mathbf{WS_1^3} \backslash \mathbf{S_1^3}\right) \cap \left(\mathbf{WC_1^3} \backslash \mathbf{C_1^3}\right)$ such that the set $(E^3)^{\diamond} = (E^3)^{\triangle}$ is bounded (or unbounded), connected, and non-convex.*

Proof. Let

$$P^3 := P^2 \times (-1, 1) \subset \mathbb{R}^3,$$

where $P^2 \subset \mathbb{R}^2$ is an open connected non-convex subset such that $\overline{P^2}$ does not contain rays; see Figure 2.3 a)-c).

Let L_- and L_+ be the planes parallel to the coordinate plane x_1Ox_2 at the distance 1 below and above x_1Ox_2, respectively.

Consider the set

$$E^3 := \mathbb{R}^3 \backslash \left(P^3 \cup L_- \cup L_+ \cup \left(\partial P^2 \times (-\infty, +\infty)\right)\right).$$

see Figure 2.3 c).

It is not difficult to see that $E^3 \in \mathbf{WC_1^3}$ and, thus, $E^3 \in \mathbf{WS_1^3}$, since

$$\partial E^3 \subset L := L_- \cup L_+ \cup \left(\partial P^2 \times (-\infty, +\infty)\right),$$

and the set L consists of straight lines not intersecting E^3.

Now prove that

$$(E^3)^{\triangle} = (E^3)^{\diamond} = P^3.$$

First, show that

$$(E^3)^{\triangle} \supset (E^3)^{\diamond} \supset P^3.$$

Consider an arbitrary point $x \in P^3$. Let $\eta(x)$ be an arbitrary ray emanating from x. Show that $\eta(x) \cap E^3 \neq \varnothing$. Notice that

$$E^3 = \Big(\left[(\mathbb{R}^2 \backslash \overline{P^2}) \times (-\infty, +\infty)\right] \cup \left[P^2 \times ((-\infty, -1) \cup (1, +\infty))\right]\Big) \backslash (L_- \cup L_+).$$

If $\eta(x)$ is parallel to the coordinate axis O_{x_3}, then

$$\eta(x) \cap P^2 \times ((-\infty, -1) \cup (1, +\infty)) \neq \varnothing.$$

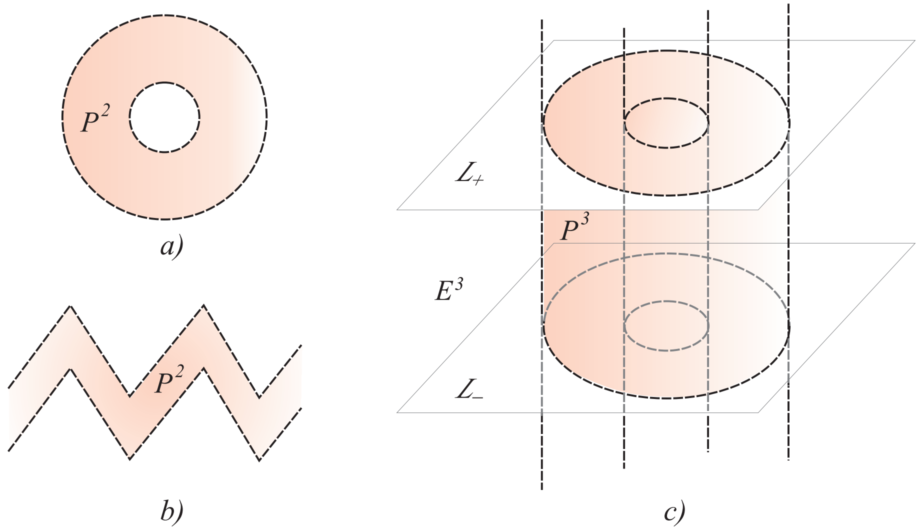


FIGURE 2.3.

Therefore, $\eta(x) \cap E^3 \neq \varnothing$, since

$$\big(P^2 \times ((-\infty,-1) \cup (1,+\infty))\big) \cap (L_- \cup L_+) = \varnothing.$$

If $\eta(x)$ is not parallel to the coordinate axis O_{x_3}, then its orthogonal projection onto the plane x_1Ox_2 is a ray emanating from a point of P^2 and intersecting $\mathbb{R}^2\backslash\overline{P^2}$, since $\overline{P^2}$ does not contain any ray by the condition. Thus,

$$\eta(x) \cap \big((\mathbb{R}^2\backslash\overline{P^2}) \times (-\infty,+\infty)\big) \neq \varnothing$$

and so

$$\eta(x) \cap \big((\mathbb{R}^2\backslash\overline{P^2}) \times (-\infty,+\infty)\big)\backslash(L_- \cup L_+) \neq \varnothing.$$

Therefore, $\eta(x) \cap E^3 \neq \varnothing$. Moreover, if $x \in (E^3)^{\diamondsuit}$, then $x \in (E^3)^{\triangle}$.

Now show that

$$(E^3)^{\triangle} \subset (E^3)^{\diamondsuit} \subset P^3.$$

By Definition 1.3,

$$(E^3)^{\triangle} \subset \mathbb{R}^3\backslash E^3 = P^3 \cup L.$$

Since L consists of straight lines not intersecting E^3, it follows that $(E^3)^{\triangle} \cap L = \varnothing$. Thus, $(E^3)^{\triangle} \subset P^3$.

Moreover, the set P^3 is bounded, connected, and non-convex if P^2 is bounded; see Figure 2.3 a), and P^3 is unbounded, connected, and non-convex if P^2 is unbounded; see Figure 2.3 b). $\square$

Theorem 2.11. *There exists an open set $E^n \in (\mathbf{WS^n_1}\backslash\mathbf{S^n_1}) \cap (\mathbf{C^n_1}\backslash\mathbf{C^n_1})$, $n \geqslant 3$, such that the set $(E^n)^\diamondsuit = (E^n)^\triangle$ is bounded (or unbounded), connected, and non-convex.*

Proof. Prove theorem by the induction. For $n = 3$, the theorem holds by Lemmas 2.10. Suppose that, for $n > 3$, an open set

$$E^{n-1} \in \left(\mathbf{WS^{n-1}_1}\backslash\mathbf{S^{n-1}_1}\right) \cap \left(\mathbf{WC^{n-1}_1}\backslash\mathbf{C^{n-1}_1}\right),$$

and $P^{n-1} := (E^{n-1})^\diamondsuit = (E^{n-1})^\triangle$ is bounded (or unbounded), connected, and non-convex.

Consider the following sets:

$$\begin{aligned}\widetilde{E}^n &:= E^{n-1} \times (-1, 1),\\ D^n_- &:= D^{n-1} \times \left(-1\tfrac{1}{2}, -1\right),\\ D^n_+ &:= D^{n-1} \times \left(1, 1\tfrac{1}{2}\right),\end{aligned}$$

where $D^{n-1} \subset \mathbb{R}^{n-1}$ is the convex hull of E^{n-1}. Denote also

$$E^n := D^n_- \cup \widetilde{E}^n \cup D^n_+.$$

First, show that

$$(E^n)^\triangle \supset (E^n)^\diamondsuit \supset P^{n-1} \times (-1, 1).$$

Consider an arbitrary point $x \in P^{n-1} \times (-1, 1)$, and an arbitrary ray $\eta(x)$ emanating from x.

If $\eta(x) \cap (D^n_- \cup D^n_+) \neq \varnothing$, then $\eta(x) \cap E^n \neq \varnothing$.

If $\eta(x) \cap (D^n_- \cup D^n_+) = \varnothing$, then consider the orthogonal projection of $\eta(x)$ onto the coordinate subspace $\mathbb{R}^{n-1}$. It is a ray $\eta(x_0)$ emanating from the point $x_0 \in P^{n-1}$ which is the orthogonal projection of x onto $\mathbb{R}^{n-1}$.

Therefore, $R := \eta(x_0) \cap E^{n-1} \neq \varnothing$, which gives that

$$\eta(x) \cap (R \times (-1, 1)) \neq \varnothing,$$

so $\eta(x) \cap E^n \neq \varnothing$. Moreover, if $x \in (E^n)^\diamondsuit$, then $x \in (E^n)^\triangle$.

Now, prove that $E^n \in \mathbf{WC^n_1}$, therefore, $E^n \in \mathbf{WS^n_1}$, and

$$(E^n)^\triangle \subset (E^n)^\diamondsuit \subset P^{n-1} \times (-1, 1).$$

It is enough to show that if

$$z \notin E^n \cup \left(P^{n-1} \times (-1, 1)\right),$$

then $z \notin (E^n)^\triangle$. Let L be the $(n-1)$-dimensional plane passing through z parallel to the coordinate subspace $\mathbb{R}^{n-1}$. Then the intersection $L \cap E^n$ is either 1) empty or 2) congruent to D^{n-1}, or 3) congruent to E^{n-1}.

1) Any straight line passing through z in L does not intersect E^n.

2) Since $L \cap E^n$ is convex in L, there exists a straight line passing through z in L and not intersecting $L \cap E^n$, therefore, not intersecting E^n.
3) $L \cap E^n \in \mathbf{WC_1^{n-1}} \backslash \mathbf{C_1^{n-1}}$ and

$$L \cap \big(P^{n-1} \times (-1,1)\big) = (L \cap E^n)^{\triangle}$$

with respect to L. Since $z \notin L \cap \big(P^{n-1} \times (-1,1)\big)$, there exists a straight line passing through z in L and not intersecting $L \cap E^n$, therefore, not intersecting E^n.

The set $P^{n-1} \times (-1,1)$ is bounded (or unbounded), connected, and non-convex, obviously. $\square$

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