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On boundary distortion estimates of homeomorphisms with a fixed point

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Abstract. This article is devoted to the study of homeomorphisms that distort the modulus of families of paths according to a Poletsky-type inequality. We consider the case when the majorant corresponding to the distortion of the modulus has finite mean oscillation at any point or satisfies a Lehto-type integral divergence condition. We obtain boundary Hölder continuity for such mappings, provided that the image of at least one inner point is located at a fixed distance from the boundary of the mapped domain. In particular, this condition holds for fixed-point mappings. The manuscript deals with cases of both good boundaries and domains with prime ends.

Анотація. Стаття присвячена вивченню гомеоморфізмів, які спотворюють модуль сімей кривих по типу нерівності Полецького. Розглядається випадок, коли мажоранта, що відповідає спотворенню модуля, має скінченне середнє коливання в будь-якій точці, або задовольняє умову інтегральної розбіжності типу Лехто. Нами отримана межова неперервність за Гельдером таких відображень за умови, що образ принаймні однієї внутрішньої точки знаходиться на фіксованій відстані від межі відображеної області. Зокрема, вказана умова виконується для відображень з фіксованою точкою. Ми окремо розглядаємо гарні межі та області з простими кінцями.

1. Introduction

This article is devoted to the boundary behavior of mappings with finite distortion. In particular, we deal with mappings associated with the distortion of the modulus of families of paths (see, e.g., [14]). It should be noted that boundary behavior occupies a special place in the theory of quasiregular mappings and mappings with finite distortion. In turn, the qualitative

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behavior of mappings on the boundary is associated with Lipschitz and Hölder inequalities (see, for example, [4, 13, 18]).

Several recent publications by the first author are also devoted to the above problems. In particular, we have obtained boundary estimates for distortion under mappings satisfying a weighted Poletsky inequality (see, e.g., [1, 21, 2], cf. [20]). We have considered domains with different geometries, including domains that are locally connected at their boundary, domains satisfying a Poincaré inequality, domains with prime ends, etc. Besides this, we are interested in the case when the majorant Q, which is responsible for the distortion of the modulus of families of paths, has finite means over infinitesimal balls.

In this paper, we continue our research on this topic by considering more general conditions on Q. Namely, we investigate the condition of finite mean oscillation or the divergence of a Lehto-type integral. Another important necessary condition is the requirement that the image of some point under the mapping does not approach the boundary of the image domain (cf. [2]).

In what follows, D is a domain in $\mathbb{R}^n$ $(n \geqslant 2)$, and we use the following notation:

$$\begin{aligned} &B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, && \mathbb{B}^n = B(0,1), \\ &S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, && \mathbb{S}^{n-1} = S(0,1), \\ &\Omega_n = m(\mathbb{B}^n), && \omega_{n-1} = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}), \end{aligned}$$

where m is the Lebesgue measure in $\mathbb{R}^n$, $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure,

$$A(x_0, r_1, r_2) := \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\},$$

and $M(\Gamma)$ denotes the *conformal modulus of the family of paths* Γ (see [22]).

Let $Q\colon \mathbb{R}^n \to [0, \infty]$ be a Lebesgue-measurable function equal to zero outside D. Consider the following concept, see [14, section 7.6]. We say that a mapping $f\colon D \to \overline{\mathbb{R}^n}$ is a *ring Q-mapping at a point* $x_0 \in \overline{D}$, $x_0 \neq \infty$, if the condition

$$M\big(f(\Gamma(S(x_0, r_1), S(x_0, r_2), D))\big) \leqslant \int\limits_{A(x_0, r_1, r_2)} Q(x) \cdot \eta^n(|x - x_0|)\, dm(x) \qquad (1.1)$$

is fulfilled for some $r_0 = r(x_0) > 0$ and for arbitrary $0 < r_1 < r_2 < r_0$, where $\eta\colon (r_1, r_2) \to [0, \infty]$ is an arbitrary nonnegative Lebesgue measurable function satisfying the inequality

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1. \qquad (1.2)$$

It can be shown that all quasiconformal and quasiregular mappings satisfy relation (1.1), since they also satisfy the relation

$$M\big(f(\Gamma(S(x_0,r_1),S(x_0,r_2),D))\big)\leqslant K\cdot M\big(\Gamma(S(x_0,r_1),S(x_0,r_2),D)\big)$$

when $Q=K=\mathrm{const}$ (see, for example, [17, 22]).

Following [3], a domain G in $\mathbb{R}^n$ is called a *quasiextremal distance domain* (shortened to *QED-domain*) if there is a number $A_0\geqslant 1$ such that the inequality

$$M\big(\Gamma(E,F,\mathbb{R}^n)\big)\leqslant A_0\cdot M\big(\Gamma(E,F,G)\big) \tag{1.3}$$

holds for any continua $E,F\subset G$. Given sets $A,B\subset\mathbb{R}^n$, we put

$$\operatorname{diam} A=\sup_{x,y\in A}|x-y|,\qquad \operatorname{dist}(A,B)=\inf_{x\in A,y\in B}|x-y|.$$

Sometimes, instead of $\operatorname{diam} A$ and $\operatorname{dist}(A,B)$, we write $d(A)$ and $d(A,B)$, respectively.

Recall that a domain $D\subset\mathbb{R}^n$ is called *locally connected at the point* $x_0\in\partial D$ if, for any neighborhood U of the point x_0, there is a neighborhood $V\subset U$ of this point such that $V\cap D$ is connected. A domain D is *locally connected on* ∂D if D is locally connected at every point $x_0\in\partial D$.

Below, the *spherical (chordal)* metric in $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ is defined as follows:

$$h(E):=\sup_{x,y\in E}h(x,y),\qquad h(A,B)=\inf_{x\in A,y\in B}h(x,y), \tag{1.4}$$

$$h(x,\infty)=\frac{1}{\sqrt{1+|x|^2}},$$

$$h(x,y)=\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}},\quad x\neq\infty\neq y.$$

Given a function $Q\colon D\to[0,\infty]$, where $Q(x)\equiv 0$ in $\mathbb{R}^n\backslash D$, denote by $q_{x_0}(r)$ the integral average of the function $Q(x)$ over the sphere $|x-x_0|=r$; that is,

$$q_{x_0}(r):=\frac{1}{\omega_{n-1}r^{n-1}}\int\limits_{|x-x_0|=r}Q(x)\,dS,$$

where dS is the area element of the sphere S, and ω_{n-1} denotes the area of the unit sphere in $\mathbb{R}^n$.

Given a domain $D\subset\mathbb{R}^n$ $(n\geqslant 2)$, points $x_0\in\partial D$ $(x_0\neq\infty)$ and $a\in D$, numbers $A_0>0$ and $\delta>0$, and a function $Q\colon D\to[0,\infty]$, denote by

$$\mathfrak{S}^{A_0}_{Q,a,\delta}(D,x_0)$$

the family of all homeomorphisms $f\colon D\rightarrow {\Bbb R}^n$ satisfying conditions (1.1)–(1.2) at x_0 such that $D_f^{\,\prime}=f(D)$ satisfies condition (1.3) with $G\mapsto D^{\,\prime}_f$, and $h(f(a), \partial D_f^{\,\prime})\geqslant \delta$.

The following statement holds.

Theorem 1.1. *Assume that the following conditions hold:*

1) *D is locally connected at its boundary;*
2) *there is $r^{\,\prime}_0=r^{\,\prime}_0(x_0)>0$ such that the set $B(x_0, r)\cap D$ is connected for all $0<r<r^{\,\prime}_0$;*
3) *there is $\delta_0=\delta_0(x_0)>0$ such that*

$$\int\limits_0^{\delta_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}=\infty. \tag{1.5}$$

Then any $f\in\mathfrak{S}_{Q, a, \delta}^{A_0}(D, x_0)$ has a continuous extension to x_0. Furthermore, there is $\varepsilon_0=\varepsilon_0(x_0)>0$ and a number $\alpha>0$ such that the inequality

$$h(f(x), f(y))\leqslant\alpha\cdot\exp\left\{-\left(A_0\right)^{-\frac{1}{n-1}}\int\limits_{|x-x_0|}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}\right\}$$

holds for all $x, y\in B(x_0, \varepsilon_0(x_0))\cap D$ such that $|x-x_0|\geqslant|y-x_0|$ and for all $f\in\mathfrak{S}_{Q, a, \delta}^{A_0}(D, x_0)$.

Remark 1.2. In particular, condition 1) of Theorem 1.1 is fulfilled if D is a convex domain.

Following [8], we say that a function $\varphi\colon D\rightarrow{\Bbb R}$ has *finite mean oscillation* (FMO) at a point $x_0\in D$ if

$$\limsup\limits_{\varepsilon\rightarrow 0}\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0, \varepsilon)}|\varphi(x)-\widetilde{\varphi_{\varepsilon}}|\, dm(x)<\infty,$$

where

$$\widetilde{\varphi_{\varepsilon}}=\frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0, \varepsilon)}\varphi(x)\, dm(x)$$

is the mean of the function $\varphi(x)$ over the ball $B(x_0, \varepsilon)$. Note that FMO is not BMO_{loc} ([14, p. 211]). It is well known [9] that

$$L^{\infty}(D)\subset BMO(D)\subset L^p_{\rm loc}(D), \qquad 1\leqslant p<\infty,$$

but $FMO(D)\not\subset L^p_{\rm loc}(D)$ for any $p>1$ (see [14]).

Theorem 1.3. *Assume that, under the conditions of Theorem 1.1, instead of* (1.5)*, the following condition holds:*

$$Q \in FMO(x_0).$$

Then there is $\varepsilon_0 = \varepsilon_0(x_0) > 0$ *and a number* $C > 0$ *such that the inequality*

$$h(f(x), f(y)) \leqslant C \left(\frac{1}{\log \frac{1}{|x-x_0|}}\right)^p \tag{1.6}$$

holds for a number $p := \left(\frac{\omega_{n-1}}{A_0 2^n C_1}\right)^{\frac{1}{n-1}}$*, where* $C_1 > 0$ *depends only on the function* Q*, for all* $x, y \in B(x_0, \varepsilon_0(x_0)) \cap D$ *such that* $|x - x_0| \geqslant |y - x_0|$*, and for all* $f \in \mathfrak{S}^{A_0}_{Q,a,\delta}(D, x_0)$*.*

Theorems 1.1 and 1.3 have corresponding counterparts for domains with bad boundaries, see below. The definition of a prime end used below can be found in [7, 6] (cf. [11, 16]).

Let ω be an open set in $\mathbb{R}^k$, $k = 1, \ldots, n-1$. A continuous mapping $\sigma\colon \omega \to \mathbb{R}^n$ is called a *k-dimensional surface* in $\mathbb{R}^n$. A *surface* is an arbitrary $(n-1)$-dimensional surface σ in $\mathbb{R}^n$. A surface σ is called *a Jordan surface* if $\sigma(x) \neq \sigma(y)$ for $x \neq y$. In what follows, we will use σ instead of $\sigma(\omega) \subset \mathbb{R}^n$, $\overline{\sigma}$ instead of $\overline{\sigma(\omega)}$, and $\partial\sigma$ instead of $\overline{\sigma(\omega)}\backslash\sigma(\omega)$.

A Jordan surface $\sigma\colon \omega \to D$ is called a *cut* of D if σ separates D (that is, $D\backslash\sigma$ has more than one component), $\partial\sigma \cap D = \varnothing$, and $\partial\sigma \cap \partial D \neq \varnothing$. A sequence of cuts $\sigma_1, \sigma_2, \ldots, \sigma_m, \ldots$ in D is called *a chain* if:

(i) the set σ_{m+1} is contained in exactly one component d_m of the set $D\backslash\sigma_m$, where $\sigma_{m-1} \subset D\backslash(\sigma_m \cup d_m)$;

(ii) $\bigcap\limits_{m=1}^{\infty} d_m = \varnothing$.

Two chains of cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called *equivalent*, if for each $m = 1, 2, \ldots$ the domain d_m contains all the domains d'_k, except for a finite number, and for each $k = 1, 2, \ldots$ the domain d'_k also contains all domains d_m, except for a finite number. The *end* of the domain D is the class of equivalent chains of cuts in D. Let K be the end of D in $\mathbb{R}^n$, then the set

$$I(K) = \bigcap_{m=1}^{\infty} \overline{d_m}$$

is called *the impression of the end* K. One may to prove that, $I(P) \subset \partial D$ (see, e.g., [11, Proposition 1]).

Following [16], we say that the end K is *a prime end* if K contains a chain of cuts $\{\sigma_m\}$ such that

$$\lim_{m\to\infty} M\big(\Gamma(C,\sigma_m,D)\big)=0 \tag{1.7}$$

for some continuum C in D. We will use the following notation below: the set of prime ends corresponding to the domain D is denoted by E_D, and the completion of the domain D by its prime ends is denoted by $\overline{D}_P$.

Consider the following definition, which goes back to Näkki [16] (see also [11]). We say that the boundary of a domain D in $\mathbb{R}^n$ is *locally quasiconformal* if each point $x_0\in\partial D$ has a neighborhood U in $\mathbb{R}^n$ that can be mapped by a quasiconformal mapping φ onto the unit ball $\mathbb{B}^n\subset\mathbb{R}^n$ so that $\varphi(\partial D\cap U)$ is the intersection of $\mathbb{B}^n$ with the coordinate hyperplane.

The sequence of cuts σ_m, $m=1,2,\ldots$, is called *regular* if $\overline{\sigma_m}\cap\overline{\sigma_{m+1}}=\varnothing$ for $m\in\mathbb{N}$ and, in addition, $d(\sigma_m)\to 0$ as $m\to\infty$. If an end K contains at least one regular chain, then K will be called *regular*. We say that a bounded domain D in $\mathbb{R}^n$ is *regular* if D can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is compact in $\mathbb{R}^n$, and if every prime end in D is regular.

Note that the space $\overline{D}_P=D\cup E_D$ admits a metric defined as follows. If $g\colon D_0\to D$ is a quasiconformal mapping from a domain D_0 with a locally quasiconformal boundary onto a domain D, then for $x,y\in\overline{D}_P$ we put:

$$\rho(x,y):=|g^{-1}(x)-g^{-1}(y)|, \tag{1.8}$$

where the elements $g^{-1}(x)$ and $g^{-1}(y)$ for $x,y\in E_D$ are understood as unique boundary points of the domain D_0. It is easy to verify that ρ in (1.8) is a metric on $\overline{D}_P$, and that the topology on $\overline{D}_P$ defined in this way does not depend on the choice of the mapping g with the indicated property.

We say that a sequence $x_m\in D$, $m=1,2,\ldots$, converges to a prime end $P\in E_D$ as $m\to\infty$ if, for any $k\in\mathbb{N}$, all elements x_m belong to d_k except for a finite number. Here, d_k denotes the sequence of nested domains corresponding to the definition of the prime end P.

Given a domain $D\subset\mathbb{R}^n$ $(n\geqslant 2)$, points $P_0\in E_D$ and $a\in D$, numbers $A_0>0$ and $\delta>0$, and a function $Q\colon D\to[0,\infty]$, denote by

$$\mathfrak{S}^{A_0}_{Q,a,\delta}(D,P_0)$$

the family of all homeomorphisms $f\colon D\to\mathbb{R}^n$ satisfying (1.1)–(1.2) at any $x_0\in I(P_0)$ (where $I(P_0)$ is the impression of P_0) such that $D_f^{\prime}=f(D)$ satisfies (1.3) with $G\mapsto D_f^{\prime}$, and $h(f(a),\partial D_f^{\prime})\geqslant\delta$.

The following statement holds.

Theorem 1.4. *Assume that D is a regular domain and the following conditions are fulfilled:*

1) *for any $y_0 \in \partial D$, there is $r_0' = r_0'(y_0) > 0$ such that $B(y_0, r) \cap D$ is finitely connected for all $0 < r < r_0'$, and for each connected component K of the set $\overline{B(y_0, r)} \cap D$, any pair of points $x, y \in K$ can be joined by a path $\gamma\colon [a, b] \to \mathbb{R}^n$ such that*
$$|\gamma| \subset K \cap \overline{B(x_0, \max\{|x - y_0|, |y - y_0|\})},$$
where
$$|\gamma| = \{x \in \mathbb{R}^n : \exists t \in [a, b] : \gamma(t) = x\};$$
2) *there is $\delta_0 = \delta_0(x_0) > 0$ such that*
$$\int\limits_0^{\delta_0} \frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)} = \infty. \tag{1.9}$$

Then, for any $P \in E_D$, there is a point $y_0 \in \partial D$ such that $I(P) = \{y_0\}$, where $I(P)$ denotes the impression of P. Furthermore, f has a limit as $x \to P_0$, and there exists a neighborhood U of P_0 and a number $\alpha > 0$ such that the relation
$$h(f(x), f(y)) \leqslant \alpha_n \cdot \exp\left\{ -(A_0)^{-\frac{1}{n-1}} \int\limits_{|x-x_0|}^{\varepsilon_0} \frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)} \right\}$$
holds for any $x, y \in U \cap D$ such that $|x - x_0| \geqslant |y - x_0|$ and for all $f \in \mathfrak{S}_{Q,a,\delta}^{A_0}(D, P_0)$, where $I(P_0) = \{x_0\}$.

Theorem 1.5. *Assume that, under the conditions of Theorem 1.4, instead of* (1.9)*, the following condition holds: $Q \in FMO(x_0)$ for any $x_0 \in I(P_0)$. Then f has a limit as $x \to P_0$, and there exists a neighborhood U of P_0 and a number $C > 0$ such that the inequality*
$$h(f(x), f(y)) \leqslant C \left(\frac{1}{\log \frac{1}{|x-x_0|}} \right)^p$$
holds for a number $p := \left(\frac{\omega_{n-1}}{A_0 2^n C_1}\right)^{\frac{1}{n-1}}$, where $C_1 > 0$ depends only on the function Q, for all $x, y \in U \cap D$ such that $|x - x_0| \geqslant |y - x_0|$, and for all $f \in \mathfrak{S}_{Q,a,\delta}^{A_0}(D, P_0)$, where $I(P_0) = \{x_0\}$.

Remark 1.6. In particular, condition 1) of Theorem 1.4 is fulfilled if, for any $y_0 \in \partial D$, there is $r_0' = r_0'(y_0) > 0$ such that the set $B(y_0, r) \cap D$ is finitely connected for all $0 < r < r_0'$, and any component K of the set $B(y_0, r) \cap D$ is convex. Indeed, let $x, y \in K$. Let us join x and y with a segment γ inside K (this is possible since the connected open set K is also path connected; see, e.g., [14, Corollary 13.1]). Assume for definiteness that $|x-y_0|\geqslant |y-y_0|$. Since the ball $\overline{B(x,|x-y_0|)}$ is convex, the entire segment γ belongs to $B(x,|x-y_0|)$. Then γ is the desired path.

2. Mappings of domains locally connected on the boundary

A domain R in $\overline{\mathbb{R}^n}$ ($n\geqslant 2$) is called *a ring* if $\overline{\mathbb{R}^n}\setminus R$ consists of exactly two components, E and F. In this case, we write $R=R(E,F)$. The following statement holds (see [14, ratio (7.29)]).

Proposition 2.1. *If $R=R(E,F)$ is a ring, then*

$$M\big(\Gamma(E,F,\overline{\mathbb{R}^n})\big)\geqslant\frac{\omega_{n-1}}{\left(\log\frac{2\lambda_n^2}{h(E)h(F)}\right)^{n-1}},$$

where $\lambda_n\in[4,2e^{n-1})$, $\lambda_2=4$, $\lambda_n^{1/n}\to e$ as $n\to\infty$, and $h(E)$ denotes the chordal diameter of the set E defined in (1.4).

The following statement holds.

Lemma 2.2. *Assume that, under the conditions of Theorem 1.1, instead of condition* 3)*, the following holds: there are numbers $p<n$ and a Lebesgue measurable function $\psi\colon(\varepsilon,\varepsilon_0)\to[0,\infty]$, defined for $\varepsilon\in(0,\varepsilon_0)$, such that the relation*

$$\int\limits_{\varepsilon<|x-x_0|<\varepsilon_0} Q(x)\cdot\psi^n(|x-x_0|)\,dm(x)\leqslant K_0\cdot I^p(\varepsilon,\varepsilon_0) \tag{2.1}$$

holds as $\varepsilon\to 0$ for some $K_0>0$, where

$$0<I(\varepsilon,\varepsilon_0):=\int\limits_{\varepsilon}^{\varepsilon_0}\psi(t)\,dt<\infty \tag{2.2}$$

for sufficiently small $\varepsilon>0$. Assume that $|x-x_0|\geqslant|y-y_0|$ and that $I(\varepsilon,\varepsilon_0)\to\infty$ as $\varepsilon\to 0$. Then there is a number $\lambda_n>0$ such that the inequality

$$h(f(x),f(y))\leqslant\frac{\lambda_n^2}{\delta}\exp\left\{-\left(\frac{\omega_{n-1}}{A_0K_0}\right)^{\frac{1}{n-1}}I^{\frac{n-p}{n-1}}(|x-x_0|,\varepsilon_0)\right\} \tag{2.3}$$

holds for all $x,y\in B(x_0,\varepsilon_0(x_0))\cap D$ and for all $f\in\mathfrak{S}^{A_0}_{Q,a,\delta}(D,x_0)$.

Proof. Let us now make the following auxiliary considerations. If ∂D contains at least one finite point other than x_0, denote it by y_0. Otherwise, put $y_0:=\infty$. Since D is locally connected on its boundary, the point a can be joined to the point y_0 by a path that belongs entirely to D, excluding the point y_0 itself (see [14, Proposition 13.2]). Let us denote this path by E. Without limiting generality, we may assume that $E\subset D\backslash B(x_0,\varepsilon_0)$, where ε_0 is the number defined above, and also that E is a Jordan path.

Now fix $f\in\mathfrak{S}^{A_0}_{Q,a,\delta}(D,x_0)$ and $x,y\in B(x_0,\varepsilon_0)$. Note that

$$C(y_0,f)\subset\partial D^{\prime}_f,\qquad\qquad D^{\prime}_f:=f(D),$$

where $C(y_0,f)$ is the cluster set of f at the point y_0 (see, e.g., [14, Proposition 13.5]). Let us prove the existence of a point $a_f\in E$ such that

$$h(f(a),f(a_f))\geqslant(1/2)\cdot h(f(a),\partial D^{\prime}_f)\geqslant\delta/2. \tag{2.4}$$

Indeed, let $z_k\in E$, $k=1,2\ldots,$ be an arbitrary sequence such that $z_k\rightarrow y_0$ as $k\rightarrow\infty$. Since the space $\overline{\mathbb{R}^n}$ is compact, we may assume that the sequence $f(z_k)$ also converges to some point $y_0^{\prime}$ as $k\rightarrow\infty$ with respect to the chordal metric. Since $C(y_0,f)\subset\partial D^{\prime}_f$, we obtain that $y_0^{\prime}\in\partial D^{\prime}_f$. Also, since $f(z_k)\rightarrow y_0^{\prime}$ as $k\rightarrow\infty$, for the number $\delta/2$, we may find an integer $k_0=k_0(f)\in\mathbb{N}$ such that

$$h(f(z_k),y_0^{\prime})<\delta/2\quad\forall k\geqslant k_0. \tag{2.5}$$

Put $a_f:=z_{k_0}$. Then, by the triangle inequality, relation (2.5), and the definition of the class $\mathfrak{S}^{A_0}_{Q,a,\delta}(D,x_0)$, we obtain that

$$\begin{aligned}\delta&\leqslant h(f(a),\partial D^{\prime}_f)\\&\leqslant h(f(a),y_0^{\prime})\\&\leqslant h(f(a),f(a_f))+h(f(a_f),y_0^{\prime})\\&\leqslant h(f(a),f(a_f))+\delta/2,\end{aligned} \tag{2.6}$$

or, moving $\delta/2$ to the left side of (2.6),

$$\delta/2\leqslant h(f(a),\partial D^{\prime}_f)\leqslant h(f(a),f(a_f)).$$

This final relation proves (2.4).

Now, let A_f be the continuum in D consisting of the part of E from the point a to the point a_f. Also, let $x,y\in B(x_0,\varepsilon_0)$. By the definition of ε_0,

$$A_f\subset D\backslash B(x_0,\varepsilon_0). \tag{2.7}$$

Since the points of the sphere

$$S(x_0,r)\cap D,\qquad 0<r<r_0^{\prime},$$

are accessible from within the domain D using some path γ, and the set $B(x_0,r)\cap D$ is connected for all $0<r<r_0^{\prime}$, the points x and y can be joined by a path K that belongs entirely to the closed ball $\overline{B(x_0,|x-x_0|)}$ and lies in D. This path K may also be considered a Jordan path.

Note that the paths $f(K)$ and $f(A_f)$ are also Jordan paths and do not split $\mathbb{R}^n$. Indeed, for $n \geqslant 3$, the set

$$f(|K|) \cup f(A_f)$$

has a topological dimension of 1, as it is the union of two closed sets of topological dimension 1 (see [5, Theorem III.2.3]). Then $f(|K|) \cup f(A_f)$ does not split $\mathbb{R}^n$ (see [5, Corollary 1.5.IV]).

Let us now consider $n = 2$. According to Antoine's theorem on the absence of wild arcs (see [10, Theorem II.4.3]), there exists a homeomorphism

$$\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$$

that maps $f(|K|)$ onto some segment I. It follows that any two points $x, y \in \mathbb{R}^2 \backslash f(|K|)$ can be joined by a path γ in $\mathbb{R}^2 \backslash f(|K|)$. Reasoning similarly, it can be shown that any two points $x, y \in \mathbb{R}^2 \backslash (f(|K|) \cup f(A_f))$ can be joined by a path γ in $\mathbb{R}^2 \backslash (f(|K|) \cup f(A_f))$. Therefore,

$$R = R(f(|K|), f(A_f))$$

is a ring domain.

In this case, we put

$$\Gamma = \Gamma(f(|K|), f(A_f), \mathbb{R}^n).$$

Then by Proposition 2.1,

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big) \geqslant \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2}{h(f(|K|))h(f(A_f))}\right)^{n-1}}. \tag{2.8}$$

Due to (2.4) and (2.8), and by the definition of K, we obtain that

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big) \geqslant \frac{\omega_{n-1}}{\left(\log \frac{\lambda_n^2}{\delta \cdot h(f(x), f(y))}\right)^{n-1}}. \tag{2.9}$$

Since D_f' is a QED-domain with constant A_0 from (1.3), condition (2.9) yields

$$M\big(\Gamma(f(|K|), f(A_f), D_f')\big) \geqslant \frac{\omega_{n-1}}{A_0\left(\log \frac{\lambda_n^2}{\delta \cdot h(f(x), f(y))}\right)^{n-1}}. \tag{2.10}$$

Note that

$$\Gamma(|K|, A_f, D) > \Gamma(S(x_0, |x - x_0|), S(x_0, \varepsilon_0), D). \tag{2.11}$$

Indeed, let

$$\gamma \in \Gamma(|K|, A_f, D), \qquad \gamma\colon [0,1] \to D, \qquad \gamma(0) \in |K|, \qquad \gamma(1) \in A_f.$$

Since

$$|K| \subset B(x_0, |x - x_0|), \qquad\qquad A_f \subset B(x_0, \varepsilon_0),$$

it follows from (2.7) that $A_f \subset D\backslash B(x_0, \varepsilon_0)$. Then

$$\gamma \cap B(x_0, |x-x_0|) \neq \varnothing \neq \gamma \cap (D\backslash B(x_0, |x-x_0|)).$$

By [12, Theorem 1.I.5.46], there exists

$$t_1 \in (0,1) \quad \text{such that} \quad \gamma(t_1) \in S(x_0, |x-x_0|).$$

Consider the subpath $\gamma_1 := \gamma|_{[t_1,1]}$. Since

$$|K| \subset B(x_0, \varepsilon_0), \qquad\qquad A_f \subset D\backslash B(x_0, \varepsilon_0),$$

we get from (2.7) that

$$\gamma_1 \cap B(x_0, \varepsilon_0) \neq \varnothing \neq \gamma \cap (D\backslash B(x_0, \varepsilon_0)).$$

Now, by [12, Theorem 1.I.5.46], there exists $t_2 \in (t_1, 1)$ such that

$$\gamma_1(t_2) = \gamma(t_2) \in S(x_0, \varepsilon_0).$$

Let us denote $\gamma_2 := \gamma|_{[t_1,t_2]}$. Then γ_2 is a subpath of γ, and

$$\gamma_2 \in \Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D).$$

This proves (2.11). In this case, by the minorization principle of the modulus of families of paths (see, e.g., [22, Theorem 6.4]), due to (2.11) and (1.1) we obtain that

$$\begin{aligned} M\big(\Gamma(f(|K|), f(A_f), D_f')\big) &= M\big(f(\Gamma(|K|, A_f, D))\big) \\ &\leqslant M\big(f(\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D))\big) \\ &\leqslant \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} Q(z) \cdot \eta^n(|z-x_0|)\, dm(z), \end{aligned} \tag{2.12}$$

where η is an arbitrary Lebesgue measurable function satisfying condition (1.2) for $r_1 = |x-x_0|$ and $r_2 = \varepsilon_0$.

Since $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, there is a $\sigma > 0$ such that $I(\varepsilon, \varepsilon_0) > 0$ for $0 < \varepsilon < \sigma$. If $|x-x_0| < \sigma$, we put

$$\eta(t) := \begin{cases} \frac{\psi(t)}{I(|x-x_0|, \varepsilon_0)}, & t \in (|x-x_0|, \varepsilon_0); \\ 0, & t \notin (|x-x_0|, \varepsilon_0). \end{cases}$$

Note that the function η satisfies condition (1.2) for $r_1 = |x-x_0|$ and $r_2 = \varepsilon_0$. Then, due to (2.12) and (2.1), we obtain that

$$\begin{aligned} &M\big(\Gamma(f(|K|), f(A_f), D_f')\big) \leqslant \\ &\qquad \leqslant \frac{1}{I^n(|x-x_0|, \varepsilon_0)} \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} \frac{Q(z)}{|z-x_0|^n}\, dm(z) \\ &\qquad \leqslant \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}. \end{aligned}$$

Combining (2.10) and (**??**), we have that

$$\frac{\omega_{n-1}}{A_0\left(\log\frac{\lambda_n^2}{\delta\cdot h(f(x),f(y))}\right)^{n-1}}\leqslant\frac{K_0}{I^{n-p}(|x-x_0|,\varepsilon_0)}.$$

The last relation implies that

$$h(f(x),f(y))\leqslant\frac{\lambda_n^2}{\delta}\exp\left\{-\left(\frac{\omega_{n-1}}{A_0K_0}\right)^{\frac{1}{n-1}}I^{\frac{n-p}{n-1}}(|x-x_0|,\varepsilon_0)\right\}.$$

Observe that the latter relation holds for $\sigma<|x-x_0|<\varepsilon_0$ with some (possibly larger) constant $\frac{1}{C_2}>0$ instead of $\frac{\lambda_n^2}{\delta}$. Indeed, by setting

$$C_2:=\exp\left\{-\left(\frac{\omega_{n-1}}{A_0K_0}\right)^{\frac{1}{n-1}}I^{\frac{n-p}{n-1}}(\sigma,\varepsilon_0)\right\},$$

we obtain that

$$h(f(x),f(y))\leqslant 1=\frac{C_2}{C_2}\leqslant\frac{1}{C_2}\exp\left\{-\left(\frac{\omega_{n-1}}{A_0K_0}\right)^{\frac{1}{n-1}}I^{\frac{n-p}{n-1}}(|x-x_0|,\varepsilon_0)\right\}$$

whenever $\sigma<|x-x_0|<\varepsilon_0$. The lemma is proved. □

Proof of Theorem 1.1. Letting $\varepsilon_0=\delta_0$, where δ_0 is from (1.5), and considering $\varepsilon<\varepsilon_0$, we analyze the function $I(\varepsilon,\varepsilon_0)=\int_\varepsilon^{\varepsilon_0}\psi(t)\,dt$ with

$$\psi(t)=\begin{cases}\frac{1}{tq_{x_0}^{\frac{1}{n-1}}(t)}, & t\in(\varepsilon,\varepsilon_0),\\ 0, & t\notin(\varepsilon,\varepsilon_0)\end{cases}$$

(as usual, $a/\infty=0$ for $a\neq\infty$, $a/0=\infty$ for $a>0$, and $0\cdot\infty=0$; see [19, Ch. I]). Since f is a homeomorphism, $I(\varepsilon,\varepsilon_0)<\infty$ (see, e.g., [14, Remark 7.2]). In addition, relation (1.5) implies that $I(\varepsilon,\varepsilon_0)>0$ for some $\varepsilon_1\in(0,\varepsilon_0)$ and any $\varepsilon\in(0,\varepsilon_1)$.

Let us prove that ψ also satisfies relation (2.1). Indeed, applying Fubini's theorem (see, e.g., [19, Theorem 8.1.III]), we obtain that

$$\begin{aligned}\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0}Q(z)\cdot\psi^n(|z-x_0|)\,dm(z)&=\omega_{n-1}\cdot\int\limits_{\varepsilon}^{\varepsilon_0}\frac{dt}{tq_{x_0}^{\frac{1}{n-1}}(t)}\\&=\omega_{n-1}\cdot I(\varepsilon,\varepsilon_0).\end{aligned}$$

Consequently, relation (1.5) implies relation (2.1) with $p=1$ and $K_0=\omega_{n-1}$. The desired inequality (2.3) then follows from Lemma 2.2.

Let us prove that f has a limit as $x\to x_0$. We can argue by contradiction. Indeed, if f does not have a limit as $x\to x_0$, then we can construct at least two sequences $x_m \to x_0$ and $y_m \to x_0$ as $m \to \infty$ such that $h(f(x_m), f(y_m)) \geqslant \delta > 0$ for some positive $\delta > 0$ and for all $m = 1, 2, \ldots$. Let $|x_m - x_0| \geqslant |y - y_m|$. Then this contradicts inequality (2.3), because the right-hand side of this inequality tends to zero as $m \to \infty$ due to (1.5). Therefore, the limit of the mapping f as $x \to x_0$ exists. $\square$

Proof of Theorem 1.3. Due to [14, Corollary 6.3, Ch. 6], the condition $Q \in FMO(x_0)$ implies that, for some small $\varepsilon < \varepsilon_0 < 1$,

$$\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0} Q(z) \cdot \psi^n(|z - x_0|)\, dm(z) = O\left(\log\log\frac{1}{\varepsilon}\right), \quad \varepsilon \to 0, \tag{2.13}$$

where $\psi(t) = \frac{1}{t\log\frac{1}{t}} > 0$. Note that the quantity $I(\varepsilon, \varepsilon_0)$ defined in (2.2) can be calculated as follows:

$$I(\varepsilon, \varepsilon_0) = \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt = \log\frac{\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon_0}}.$$

Thus, estimate (2.13) yields

$$\frac{1}{I^n(\varepsilon, \varepsilon_0)} \int\limits_{\varepsilon<|z-x_0|<\varepsilon_0} Q(z) \cdot \psi^n(|z - x_0|)\, dm(z) \leqslant C_1 \left(\log\log\frac{1}{\varepsilon}\right)^{1-n}.$$

In particular, the condition $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$ holds as well. Now, by Lemma 2.2,

$$\begin{aligned} h(f(x), f(y)) &\leqslant \frac{\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}} \log\frac{\log\frac{1}{|x-x_0|}}{\log\frac{1}{\varepsilon_0}}\right\} \\ &= \frac{\lambda_n^2}{\delta}\left(\frac{\log\frac{1}{\varepsilon_0}}{\log\frac{1}{|x-x_0|}}\right)^{\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}} \\ &\leqslant C\cdot\left(\frac{1}{\log\frac{1}{|x-x_0|}}\right)^{\left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}}. \end{aligned}$$

Thus, relation (1.6) is proved for $p = \left(\frac{\omega_{n-1}}{A_0C_1}\right)^{\frac{1}{n-1}}$. $\square$

In the theory of conformal and quasiconformal mappings, mappings with a fixed point are of particular importance. Based on this, let us consider the following class.

3. Mappings of domains with complex boundaries

In order to prove the statements of Theorems 1.4 and 1.5 concerning domains with complex geometry, we formulate and prove a lemma similar to Lemma 2.2 for this case. The following statement holds.

Lemma 3.1. *Assume that D is a regular domain and the following conditions are fulfilled:*

1) *for any $y_0 \in \partial D$, there is $r_0' = r_0'(y_0) > 0$ such that $B(y_0, r) \cap D$ is finitely connected for all $0 < r < r_0'$, and for each connected component K of the set $\overline{B(y_0, r)} \cap D$, any pair of points $x, y \in K$ can be joined by a path $\gamma\colon [a, b] \to \mathbb{R}^n$ such that*
$$|\gamma| \subset K \cap \overline{B(x_0, \max\{|x - y_0|, |y - y_0|\})},$$
where
$$|\gamma| = \{x \in \mathbb{R}^n : \exists t \in [a, b] : \gamma(t) = x\};$$
2) *for any $x_0 \in \partial D$, there are numbers $p < n$, $0 < \varepsilon_0 < \min\{1, r_0'\}$, and a Lebesgue measurable function $\psi\colon (\varepsilon, \varepsilon_0) \to [0, \infty]$, defined for $\varepsilon \in (0, \varepsilon_0)$, such that the relation*
$$\int\limits_{\varepsilon<|z-x_0|<\varepsilon_0} Q(z) \cdot \psi^n(|z - x_0|)\, dm(z) \leqslant K_0 \cdot I^p(\varepsilon, \varepsilon_0) \tag{3.1}$$
holds for any $x_0 \in I(P_0)$ as $\varepsilon \to 0$ for some $K_0 > 0$, where
$$0 < I(\varepsilon, \varepsilon_0) := \int\limits_{\varepsilon}^{\varepsilon_0} \psi(t)\, dt < \infty$$
for sufficiently small $\varepsilon > 0$ with $\varepsilon < \varepsilon_0$.

Assume that $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$. Then, for any $P \in E_D$, there is a point $y_0 \in \partial D$ such that $I(P) = \{y_0\}$, where $I(P)$ denotes the impression of P. Furthermore, f has a limit as $x \to P_0$, and there exists a neighborhood U of P_0 in $\overline{D}_P$ and a number $\lambda_n > 0$ such that the relation
$$h(f(x), f(y)) \leqslant \frac{\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(|x - x_0|, \varepsilon_0)\right\} \tag{3.2}$$
holds for any $x, y \in U \cap D$ with $|x - x_0| \geqslant |y - x_0|$, and for all $f \in \mathfrak{S}_{Q,a,\delta}^{A_0}(D, P_0)$, where $I(P_0) = \{x_0\}$.

Proof. **I.** Let $f \in \mathfrak{S}_{Q,a,\delta}^{A_0}(D, P_0)$. Since the set $B(y_0, r) \cap D$ is finitely connected for all $y_0 \in \partial D$ and $0 < r < r_0'(y_0)$, D is finitely connected at the boundary. Therefore, D is a uniform domain (see [15, Theorem 3.2]).

In other words, for every $r>0$, there exists a number $\delta>0$ such that the inequality

$$M\big(\Gamma(F^*, F, D)\big) \geqslant \delta$$

holds for any continua $F, F^*\subset D$ such that $h(F)\geqslant r$ and $h(F^*)\geqslant r$.

II. Let us prove that for any $P\in E_D$, there exists $y_0\in\partial D$ such that $I(P)=\{y_0\}$. We prove this statement by contradiction. Suppose that there exists a prime end $P\in E_D$ whose impression contains two points $x, y\in\partial D$, with $x\ne y$. In this case, there are at least two sequences $x_m, y_m\in d_m$, $m=1,2,\ldots$, that converge to x and y as $m\to\infty$, respectively (here d_m denotes the decreasing sequence of domains formed by the corresponding sequence of cuts σ_m in P).

Let us join the points x_m and y_m with a path γ_m in d_m. Since $x\ne y$, there exists $m_0\in\mathbb{N}$ such that $h(|\gamma_m|)\geqslant d(x,y)/2$ for $m>m_0$. Choose any nondegenerate continuum $C\subset D\backslash d_1$. Then, due to the uniformity of the domain D,

$$M\big(\Gamma(|\gamma_m|, C, D)\big) \geqslant \delta_0 > 0 \qquad (3.3)$$

for some $\delta_0>0$ and all $m>m_0$.

The inequality (3.3) contradicts the definition of a prime end P. Indeed, by the definition of σ_m, we have that $\Gamma(|\gamma_m|, C, D)>\Gamma(\sigma_m, C, D)$. Then, due to (1.7), we obtain that

$$M\big(\Gamma(|\gamma_m|, C, D)\big) \leqslant M\big(\Gamma(\sigma_m, C, D)\big) \to 0$$

as $m\to\infty$. This final relation contradicts (3.3), so $I(P)=\{y_0\}$ for some $y_0\in\partial D$.

III. It remains to prove relation (3.2). Let $I(P_0)=\{x_0\}\in\partial D$. Let d_m be a sequence of domains corresponding to cuts σ_m in P_0 such that $\sigma_m\subset S(x_0, r_m)$, where $x_0\in\partial D$ and $r_m\to 0$ as $m\to\infty$ (see [11, Lemma 2]). Since D is regular, the space $\overline{D}_P$ contains no less than two prime ends $P_1, P_2\in E_D$. Let $P_1\subset E_D$ be a prime end that does not coincide with P_0. Suppose that G_m, $m=1,2,\ldots$, is a sequence of domains that corresponds to P_1.

Let us construct a sequence of continua K_m, $m=1,2,\ldots$, as follows. Join the points ζ_1 and a by an arbitrary path in D, which we denote by K_1. Next, join the points ζ_2 and ζ_1 by the path K_1' in G_1. By combining paths K_1 and K_1', we obtain a path K_2 joining points a and ζ_2. And so on. Let K_m be a path joining the points ζ_m and a. Let us join the points ζ_{m+1} and ζ_m by a path K_m', which lies in G_m. By uniting the paths K_m and K_m', we obtain a path K_{m+1}. And so on.

Let us show that there is a number $m_1 \in \mathbb{N}$ such that

$$d_m \cap K_m = \varnothing \quad \forall \quad m \geqslant m_1. \tag{3.4}$$

Let us prove this by contradiction; suppose that (3.4) is not true. Then there is an increasing sequence of numbers $m_k \to \infty$ as $k \to \infty$, and points $\xi_k \in K_{m_k} \cap d_{m_k}$ for $k = 1, 2, \ldots$. Then $\xi_k \to P_0$ as $k \to \infty$.

Note that two cases are possible: either all ξ_k belong to $D\backslash G_1$ for $k = 1, 2, \ldots$, or there is $k_1 \in \mathbb{N}$ such that $\xi_{k_1} \in G_1$. In the second case, consider the sequence ξ_k for $k > k_1$. Note that two cases are possible again: either ξ_k for $k > k_1$ belong to $D\backslash G_2$, or there is $k_2 > k_1$ such that $\xi_{k_2} \in G_2$. In the second case, consider the sequence ξ_k for $k > k_2$, and so on.

Suppose that the element $\xi_{k_{l-1}} \in G_{l-1}$ has already been constructed. Note that there are two cases: either ξ_k belong to $D\backslash G_l$ for $k > k_{l-1}$, or there is a number $k_l > k_{l-1}$ such that $\xi_{k_l} \in G_l$, and so forth. This procedure may be finite or infinite, meaning we have two possible situations:

1) there are $n_0 \in \mathbb{N}$ and $l_0 \in \mathbb{N}$ such that $\xi_k \in D\backslash G_{n_0}$ for all $k > l_0$;
2) for each $l \in \mathbb{N}$, there is ξ_{k_l} such that $\xi_{k_l} \in G_l$, and the sequence k_l is increasing in $l \in \mathbb{N}$.

We consider each of these cases separately and show that they both lead to a contradiction. Suppose that case 1) holds. Observe that all elements of the sequence ξ_k belong to K_{n_0}, which implies the existence of a subsequence ξ_{k_r}, $r = 1, 2, \ldots$, that converges as $r \to \infty$ to some point $\xi_0 \in D$. However, $\xi_k \in d_{m_k}$, and therefore (see [11, Proposition 1])

$$\xi_0 \in \bigcap_{m=1}^{\infty} \overline{d_m} \subset \partial D.$$

This resulting contradiction indicates the impossibility of case 1).

Consider now case 2). Then $\xi_k \to P_0$ and $\xi_k \to P_1$ simultaneously as $k \to \infty$. Since the space $\overline{D}_P$ is a metric space (see [11, Remark 2]), the triangle inequality implies that $P_1 = P_0$, which contradicts our choice of P_1. This resulting contradiction demonstrates the validity of relation (3.4).

IV. As proved above, $I(P_1) = \{z_0\}$ with $z_0 \in \partial D$. Therefore, by construction, there exists $z_m \to z_0$ as $m \to \infty$, where $z_m \in K$, and K is a path originating at point a formed by uniting the paths K_m constructed above ($m = 1, 2, \ldots$). Recall that

$$C(y_0, f) \subset \partial D'_f, \qquad\qquad D'_f := f(D),$$

where $C(y_0, f)$ is the cluster set of f at the point y_0 (see, e.g., [14, Proposition 13.5]). Reasoning in the same way as when proving Theorem 1.1, we can prove the existence of a point $a_f \in K$ such that

$$h(f(a), f(a_f)) \geqslant (1/2) \cdot h(f(a), \partial D'_f) \geqslant \delta/2. \qquad (3.5)$$

Now let A_f be the continuum in D that forms the part of the path E from point a to point a_f.

V. Since $I(P_0) = \{x_0\}$, we can find a neighborhood U of P_0 in $\overline{D}_P$ such that

$$U \cap D \subset B(x_0, \varepsilon_0).$$

By the definition of a regular domain, we may assume that $U \cap D$ is connected. Then $U \cap D$ belongs to one and only one component K of $B(x_0, \varepsilon_0) \cap D$. Let $x, y \in U \cap D$ such that $|x - x_0| \geqslant |y - x_0|$. By the definition of r'_0, the points x and y can be joined by a path K contained in the closed ball $\overline{B(x_0, |x - x_0|)}$.

In what follows, the proof is very similar to the proof of Theorem 1.1. We may assume that K and A_f are Jordan paths. Note also that the paths $f(K)$ and $f(A_f)$ are also Jordan paths and do not divide the space $\mathbb{R}^n$ (see the proof of Theorem 1.1). Therefore,

$$R = R(f(|K|), f(A_f))$$

is a ring domain. Let us denote $\Gamma = \Gamma(f(|K|), f(A_f), \mathbb{R}^n)$. Then, by Proposition 2.1,

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big) \geqslant \frac{\omega_{n-1}}{\left(\log \frac{2\lambda_n^2}{h(f(|K|))h(A_f)}\right)^{n-1}}. \qquad (3.6)$$

Due to (3.5) and (3.6), by the definition of A_f,

$$M\big(\Gamma(f(|K|), f(A_f), \overline{\mathbb{R}^n})\big) \geqslant \frac{\omega_{n-1}}{\left(\log \frac{\lambda_n^2}{\delta \cdot h(f(x), f(y))}\right)^{n-1}}. \qquad (3.7)$$

Since D'_f is a QED-domain with constant A_0 from (1.3), under condition (3.7) we obtain that

$$M\big(\Gamma(f(|K|), f(f(A_f)), D'_f)\big) \geqslant \frac{\omega_{n-1}}{A_0\left(\log \frac{\lambda_n^2}{\delta \cdot h(f(x), f(y))}\right)^{n-1}}. \qquad (3.8)$$

Note that

$$\Gamma(|K|, A_f, D) > \Gamma(S(x_0, |x - x_0|), S(x_0, \varepsilon_0), D). \qquad (3.9)$$

Relation (3.9) can be proved in the same way as (2.11), which was established during the proof of Theorem 1.1. In this case, by the minorization of the modulus (see, e.g., [22, Theorem 6.4]), and due to (2.11) and (1.1), we obtain that

$$\begin{aligned} M\big(\Gamma(f(|K|), f(A_f), D'_f)\big) &= M\big(f(\Gamma(|K|, A_f, D))\big) \\ &\leqslant M\big(f(\Gamma(S(x_0, |x-x_0|), S(x_0, \varepsilon_0), D))\big) \\ &\leqslant \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} Q(z)\cdot \eta^n(|z-x_0|)\, dm(z), \end{aligned}$$

where η is an arbitrary Lebesgue measurable function satisfying relation (1.2) for $r_1 = |x-x_0|$ and $r_2 = \varepsilon_0$.

Since $I(\varepsilon, \varepsilon_0) \to \infty$ as $\varepsilon \to 0$, there is a $\sigma > 0$ such that $I(\varepsilon, \varepsilon_0) > 0$ for $0 < \varepsilon < \sigma$. If $|x - x_0| < \sigma$, we put

$$\eta(t) := \begin{cases} \frac{\psi(t)}{I(|x-x_0|, \varepsilon_0)}, & t \in (|x-x_0|, \varepsilon_0), \\ 0, & t \notin (|x-x_0|, \varepsilon_0). \end{cases}$$

Note that the function η satisfies condition (1.2) for $r_1 = |x - x_0|$ and $r_2 = \varepsilon_0$. Then, due to (3.9) and (3.1), we obtain that

$$\begin{aligned} M\big(\Gamma(f(|K|), f(A_f), D'_f)\big) &\leqslant \\ &\leqslant \frac{1}{I^n(|x-x_0|, \varepsilon_0)} \int\limits_{A(x_0, |x-x_0|, \varepsilon_0)} \frac{Q(z)}{|z-x_0|^n}\, dm(z) \\ &\leqslant \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}. \end{aligned} \tag{3.10}$$

Combining (3.8) and (3.10), we have that

$$\frac{\omega_{n-1}}{A_0\left(\log \frac{\lambda_n^2}{\delta\cdot h(f(x), f(y))}\right)^{n-1}} \leqslant \frac{K_0}{I^{n-p}(|x-x_0|, \varepsilon_0)}.$$

This latter relation implies that

$$h(f(x), f(y)) \leqslant \frac{\lambda_n^2}{\delta} \exp\left\{-\left(\frac{\omega_{n-1}}{A_0 K_0}\right)^{\frac{1}{n-1}} I^{\frac{n-p}{n-1}}(|x-x_0|, \varepsilon_0)\right\}.$$

Arguing similarly to the last part of the proof of Lemma 2.2, we can prove that this relation holds for $\sigma < |x - x_0| < \varepsilon_0$ with some (possibly larger) constant $\frac{1}{C_2} > 0$ instead of $\frac{\lambda_n^2}{\delta}$. The lemma is proved. $\square$

The proofs of Theorems 1.4 and 1.5 follow from Lemma 3.1 similarly to the proofs of Theorems 1.1 and 1.3.

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