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Topology of the punctual Hilbert schemes of plane curve singularities with a single Puiseux pair

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Dedicated to Professor Fumio Sakai on his 70th birthday

Abstract. Piontkowski showed the existence of affine cell decompositions of Jacobian factors of plane curve singularities with a single Puiseux pair. He also provided a combinatorial description of the Euler numbers and Betti numbers of these Jacobian factors. Following his results, Oblomkov, Rasmussen, and Shende demonstrated the existence of affine cell decompositions of punctual Hilbert schemes for the same type of singularity. In the present paper, we revisit their theorem from a computational perspective and describe the Euler numbers and Betti numbers of the punctual Hilbert schemes.

Анотація. Піонтковський показав існування афінних клітинних розкладів факторів Якобія сингулярностей плоскої кривої з однією парою Пюїзе. Він також надав комбінаторний опис чисел Ейлера та чисел Бетті цих факторів Якобі. За допомогою його результатів, Обломков, Расмуссен та Шенде продемонстрували існування афінних клітинних розкладів точкових схем Гільберта для того ж типу сингулярності. У цій статті ми переглядаємо їхню теорему з обчислювальної точки зору та описуємо числа Ейлера та числа Бетті точкових схем Гільберта.

1. Introduction

Let X be a singular plane curve over $\mathbb{C}$ with its singular point o. We refer to the pair (X, o) as a *plane curve singularity*. In this paper, we consider the case where (X, o) is reduced and irreducible. Let $R \coloneqq \widehat{\mathcal{O}}_{X,o}$ denote the completion of the local ring $\mathcal{O}_{X,o}$ of X at o, and let Γ be the semigroup of R. We denote by $\mathrm{Hilb}^r(X, o)$ the punctual Hilbert scheme of degree

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r for the given singularity (X,o). It is known that $\mathrm{Hilb}^r(X,o)$ admits a decomposition of the form

$$\mathrm{Hilb}^r(X,o) = \bigsqcup_{\Delta \in \mathrm{Mod_r}(\Gamma)} H(\Delta) \tag{1.1}$$

where $\mathrm{Mod_r}(\Gamma) \coloneqq \{\Delta : \Delta \text{ is a } \Gamma\text{-semimodule with } \#(\Gamma \backslash \Delta) = r\}$. In general, some of $H(\Delta)$ are empty. We call the component $H(\Delta)$ the Δ-*subset* of $\mathrm{Hilb}^r(X,o)$.

Piontkowski [5] studied the topology of the Jacobian factors of irreducible plane curve singularities with a single Puiseux pair. It is known that a Jacobian factor $J_{X,o}$ of an irreducible plane curve singularity (X,o) is isomorphic to certain punctual Hilbert schemes (see Corollary 2.3 below). Therefore, $J_{X,o}$ also admits a decomposition (1.1). By showing that each Δ-subset $H(\Delta)$ in the decomposition is isomorphic to an affine space, he described the Euler number and the Betti numbers of $J_{X,o}$. Following his result, Oblomkov, Rasmussen, and Shende proved the following theorem:

Theorem 1.1 ([3], Theorem 13)**.** *Let p and q be coprime positive integers with $p < q$. For an irreducible plane curve singularity (X,o) with $\Gamma = \langle p, q \rangle$, each Δ-subset $H(\Delta)$ of $\mathrm{Hilb}^r(X,o)$ is isomorphic to an affine space of dimension*

$$\sum_{i=1}^{p-1} \#\big[\{(-\min\Delta + \Gamma) \cap [a_i, a_i + q)\} \backslash \Delta^{(0)}\big] \tag{1.2}$$

where $\{a_0, \dots, a_{p-1}\}$ is the p-basis of the 0-normalization

$$\Delta^{(0)} \coloneqq -\min\Delta + \Delta$$

of Δ.

Remark 1.2. Although the dimension (1.2) was originally expressed differently in [3], we re-express it here using the p-basis, as in [5].

In this paper, we apply Piontokowski's method, emphasizing the computational aspects, to prove Theorem 1.1. The following result immediately follows:

Corollary 1.3. *Under the same assumption as in Theorem 1.1, the Euler number of $\mathrm{Hilb}^r(X,o)$ is equal to $\#\mathrm{Mod_r}(\Gamma)$.*

We also give a combinatorial description of the Betti numbers of punctual Hilbert schemes, which generalize Piontkowski's result. For $\Delta \in \mathrm{Mod_r}(\Gamma)$, define the codimension of $H(\Delta)$ as

$$\operatorname{codim} H(\Delta) \coloneqq \dim \mathrm{Hilb}^r(X,o) - \dim H(\Delta).$$

We also set

$$\mathcal{H}_{r,d} := \{H(\Delta) : \Delta \in \mathrm{Mod_r}(\Gamma) \text{ and } \dim H(\Delta) = d\},$$
$$\mathcal{H}_r^d := \{H(\Delta) : \Delta \in \mathrm{Mod_r}(\Gamma) \text{ and } \operatorname{codim} H(\Delta) = d\}.$$

Theorem 1.4. *Let (X, o) be the same as in Theorem 1.1. The odd (co-) homology groups of $\mathrm{Hilb}^r(X, o)$ vanish. The even (co-) homology groups of $\mathrm{Hilb}^r(X, o)$ are free abelian groups with Betti numbers*

$$h_{2d}(\mathrm{Hilb}^r(X, o)) = \#\mathcal{H}_{r,d} \qquad \textit{and} \qquad h^{2d}(\mathrm{Hilb}^r(X, o)) = \#\mathcal{H}_r^d.$$

This paper is structured as follows: Sections 2, 3, and 4 summarize the foundational concepts required for the proofs. Section 2 recalls the properties of punctual Hilbert schemes of curve singularities. Section 3 introduces definitions and facts related to Γ-semimodules. Section 4 discusses computational techniques involving Gröbner bases. In Section 5, we prove the main theorems. Section 6 presents examples, and Section 7 provides remarks on related results.

2. Punctual Hilbert schemes of curve singularities

In this section, we recall properties of punctual Hilbert schemes established in [4]. Although we focus on irreducible plane curve singularities here, the notions presented here hold in more general situations.

Let (X, o) be an irreducible plane curve singularity. By Puiseux's theorem, there exist coprime positive integers p and q such that $R = \mathbb{C}[\![t^p, \phi]\!]$ where $\phi = t^q +$ higher order terms and $p < q$. The normalization $\overline{R}$ of R is isomorphic to $\mathbb{C}[\![t]\!]$. Let ν be the natural valuation

$$\nu \colon \overline{R} \backslash \{0\} \to \mathbb{Z}_{\geqslant 0}$$

defined by $\nu(f) = \mathrm{ord}_t(f)$, and set $\nu(0) = \infty$.

We call $\Gamma := \nu(R)$ the *semigroup* of R. A semigroup Γ generated by positive integers $\gamma_1, \gamma_2, \ldots, \gamma_l$ is denoted as $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_l \rangle$. The *δ-invariant* of R is defined as $\delta := \dim_{\mathbb{C}}(\overline{R}/R)$. The *conductor* c of Γ is the smallest integer in Γ such that $c - 1 \notin \Gamma$ and $c + n \in \Gamma$ for any $n \in \mathbb{N}$. It satisfies $\delta + 1 \leqslant c \leqslant 2\delta$ and $c = 2\delta$ if and only if R is Gorenstein (see [7]).

For a positive integer a, we denote by (t^a) an ideal of $\overline{R}$. Let

$$\mathrm{Gr}\big(\delta, \overline{R}/(t^{2\delta})\big)$$

be the Grassmannian which consists of δ-dimensional linear subspaces of $\overline{R}/(t^{2\delta})$. For $M \in \mathrm{Gr}\big(\delta, \overline{R}/(t^{2\delta})\big)$, we define a multiplication by

$$R \times M \ni (f, m + (t^{2\delta})) \longmapsto fm + (t^{2\delta}) \in M.$$

The set

$$J_{X,o} \coloneqq \left\{ M \in \mathrm{Gr}\big(\delta, \overline{R}/(t^{2\delta})\big) : \begin{array}{l} M \text{ is an } R\text{-submodule with respect to} \\ \text{the above multiplication.} \end{array} \right\}$$

is called the *Jacobian factor* of (X, o), which was introduced by Rego in [6]. For any non-negative integer r, set

$$\mathcal{I}_r \coloneqq \{I \subset R : I \text{ is an ideal of } R \text{ with } \dim R/I = r\}.$$

Observe that any element I in $\mathcal{I}_r$ satisfies

$$(t^{r+2\delta}) \cap R \subseteq I \subseteq (t^r) \cap R.$$

It follows that

$$(t^{2\delta}) \subseteq t^{-r}I \subseteq \overline{R} \qquad \text{and} \qquad \dim_k(\overline{R}/t^{-r}I) = \dim_k(\overline{R}/I) - r = \delta.$$

Using these properties, Pfister and Steenbrink defined a map

$$\varphi_r : \mathcal{I}_r \to J_{X,o} \subset \mathrm{Gr}(\delta, \overline{R}/(t^{2\delta})) \tag{2.1}$$

by $\varphi_r(I) = t^{-r}I/(t^{2\delta})$ (see [4]). We call φ_r the *δ-normalized embedding.* It has the following properties:

Proposition 2.1 ([4], Theorem 3)**.** *The δ-normalized embedding φ_r is injective for any non-negative integer r. It is also a bijection for $r \geqslant c$. The punctual Hilbert scheme $\mathrm{Hilb}^r(X, o)$ is Zariski closed in $J_{X,o}$.*

Definition 2.2. We call $\mathrm{Hilb}^r(X, o) \coloneqq \varphi_r(\mathcal{I}_r)$ the *punctual Hilbert scheme of degree r for (X, o).*

The following fact follows from Proposition 2.1:

Corollary 2.3. *For any positive integer r with $r \geqslant c$, we have*

$$\mathrm{Hilb}^r(X, o) \cong \mathrm{Hilb}^c(X, o) \cong J_{X,o}.$$

Remark 2.4. In [5], Piontkowski originally proved Theorem 1.1, Corollary 1.3, and Theorem 1.4 for the case where the degrees $r \geqslant c$.

3. Γ-semimodules

Let Γ be a semigroup. A subset $\Delta \subset \mathbb{Z}$ is called a Γ-*semimodule* if it satisfies

$$\Delta + \Gamma \subseteq \Delta.$$

It is straightforward to see that, for any ideal I of R, the set

$$\Gamma(I) \coloneqq \{\nu(f) : f \in I\}$$

is a Γ-semimodule. If integers $a_1, \ldots, a_s$ generate Δ (i.e., $\Delta = \sum_{i=1}^{s}(a_i + \Gamma)$ holds), we write $\Gamma = \langle a_1, \ldots, a_s \rangle_\Gamma$.

Two Γ-semimodules Δ_1 and Δ_2 are *isomorphic* if there exists an integer a such that

$$\Delta_1 = a + \Delta_2 = \{a + d : d \in \Delta_2\}.$$

We define two special normalizations of a Γ-semimodule Δ: We define the 0-*normalization* of Δ as $\Delta^{(0)} \coloneqq -\min \Delta + \Delta$. In general, Δ is called 0-*normalized* if $\min \Delta = 0$. On the other hand, Δ is said to be δ-*normalized* if $\#(\mathbb{N} \backslash \Delta) = \delta$. For $I \in \mathcal{I}_r$, there exists an element Δ of $\mathrm{Mod_r}(\Gamma)$ such that $\Delta = \Gamma(I)$. From the inclusion $(t^{2\delta}) \subset t^{-r} I$ and the definition of φ_r in (2.1)), it follows that $-r + \Delta$ is δ-normalized, and $\#\{(-r + \Delta) \cap [0, 2\delta - 1]\} = \delta$ holds. We call $-r + \Delta$ the δ-*normalization* of Δ and denote it by $\Delta^{(\delta)}$.

In [5], Piontkowski introduced special generators of a Γ-semimodule.

Definition 3.1 ([5])**.** Let $\Gamma = \langle p, q \rangle$ where $p < q$ and $\gcd(p, q) = 1$. The *p-basis* of a Γ-semimodule Δ is the unique set $\{a_0, a_1, \dots, a_{p-1}\}$ satisfying

$$\Delta = \bigcup_{i=0}^{p-1} (a_i + p\mathbb{N}) \qquad \text{and} \qquad a_i \equiv iq \pmod{p}.$$

In particular, we have $\Delta = \langle a_0, \dots, a_{p-1} \rangle_\Gamma$.

Throughout this paper, we consider the p-basis only for 0-normalized Γ-semimodules and therefore we always set $a_0 = 0$. Using the inclusion $\mathbb{N}q \subset \Gamma \subset \Delta^{(0)}$, we find non-negative integers $\alpha_1, \dots, \alpha_{p-1} \in \mathbb{N}$ such that

$$\begin{aligned} a_0 &= 0, \\ a_1 &= q - \alpha_1 p, \\ a_2 &= 2q - \alpha_2 p, \\ &\dots \\ a_{p-1} &= (p-1)q - \alpha_{p-1} p. \end{aligned} \tag{3.1}$$

We also set $\alpha_0 = 0$. These satisfy $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \dots \leqslant \alpha_{p-1} < q$.

The set $\mathcal{I}_r$ can be decomposed in terms of Γ-semimodules.

Lemma 3.2 ([8, Proposition 6])**.** *We have*

$$\mathcal{I}_r = \bigsqcup_{\Delta \in \mathrm{Mod_r}(\Gamma)} \mathcal{I}(\Delta) \tag{3.2}$$

where $\mathcal{I}(\Delta) \coloneqq \{I \in \mathcal{I}_r : \Gamma(I) = \Delta\}$.

By setting $H(\Delta) \coloneqq \varphi_r(\mathcal{I}(\Delta))$, the stratification (1.1) of $\mathrm{Hilb}^r(X, o)$ follows from Lemma 3.2.

Remark 3.3. In general, some of the components in (3.2) may be empty. However, in [5], Piontkowski showed that all components are not empty for an irreducible curve singularity with $\Gamma = \langle p, q \rangle$.

4. Gröbner bases

We begin by recalling some key facts about Gröbner bases, primarily following [2]. Let $R = \mathbb{C}[\![x_1(t), \ldots, x_l(t)]\!]$ be a subring of $\mathbb{C}[\![t]\!]$ such that $\dim_{\mathbb{C}} \mathbb{C}[\![t]\!]/R < \infty$, and let $M \subset \mathbb{C}[\![t]\!]$ be an R-module. As in Section 2, let ν be the natural valuation $\nu \colon \mathbb{C}[\![t]\!] \backslash \{0\} \to \mathbb{Z}_{\geqslant 0}$. We also set $\nu(0) = \infty$. Put $\Gamma \coloneqq \nu(R)$ and $\Gamma(M) \coloneqq \nu(M)$. Clearly, $\Gamma(M)$ is a Γ-semimodule. We consider the local order

$$\nu(1) \succ \nu(t) \succ \nu(t^2) \succ \cdots .$$

For $f \in M$, we denote by $\mathrm{LC}(f)$ (resp. $\mathrm{LT}(f)$) the leading coefficient of f (resp. the leading term of f) with respect to this order.

Definition 4.1. A subset $G = \{g_1, \ldots, g_m\}$ of R is called a *SAGBI*[1] *basis* if, for any $f \in R$, there exists a multi-index $(\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m_{\geqslant 0}$ such that

$$\mathrm{LT}(f) = \mathrm{LT}(g_1^{\beta_1} \cdots g_m^{\beta_m}).$$

Definition 4.1 implies the following:

Theorem 4.2. *A SAGBI basis G of R generates R as $\mathbb{C}$-algebra.*

Definition 4.3. Let $G = \{g_1, \ldots, g_m\}$ be a SAGBI basis for R and let H be a subset of M. The pair (G, H) is called a *standard basis* of M if, for any $f \in M$, there exist $h \in H$ and a multi-index $(\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m_{\geqslant 0}$ such that $\mathrm{LT}(f) = \mathrm{LT}(g_1^{\beta_1} \cdots g_m^{\beta_m} h)$.

Similar to the above, the following theorem holds:

Theorem 4.4. *Let (G, H) be a standard basis of M. Then the set H generates M as R-module.*

The following proposition follows from Definition 4.3.

Proposition 4.5. *Let $G = \{g_1, \ldots, g_m\} \subset R$ be a SAGBI basis for R and let $H = \{h_1, \ldots, h_n\}$ be a subset of M. Then (G, H) is a standard basis of M if and only if $\Gamma = \langle \nu(g_1), \ldots, \nu(g_m) \rangle$ and $\Gamma(M) = \langle \nu(h_1), \ldots, \nu(h_n) \rangle_\Gamma$.*

[1]Subalgebra Analogue to Gröbner Bases for Ideals

Theorem 4.6. *Let G and H be as in Proposition* 4.5. *If (G, H) is a standard basis of M, then any $f \in \mathbb{C}[\![t]\!]$ can be written as*

$$f = q_1 h_1 + \cdots + q_n h_n + r \tag{4.1}$$

where $q_1, \ldots, q_n \in \mathbb{C}[\![g_1, \ldots, g_m]\!]$ and $r = \sum_{j \notin \Gamma(M)} c_j t^j$.

We call r in (4.1) the *reduction* of f modulo (G, H), and denote it by $R(f, G, H)$.

Proof. We claim that the following algorithm yields (4.1):

Division Algorithm
`Input:` $f \in R$, $G = \{g_1, \ldots, g_m\}$, $H = \{h_1, \ldots, h_n\}$
`Output:` $q_1, \ldots, q_n$, r
`Define:` $q_1 := 0, \ldots, q_n := 0$, $r = 0$, $p := f$
`WHILE` $p \neq 0$ `DO`
  If $\nu(p) \in \Gamma(M)$, `THEN` find smallest l such that
  $\mathrm{LT}(p) = \mathrm{LT}(g_1^{\alpha_1} \cdots g_m^{\beta_m} h_l)$ for some $(\beta_1, \ldots, \beta_m) \in \mathbb{Z}_{\geqslant 0}^m$
   $q_l := q_l + g_1^{\beta_1} \cdots g_m^{\beta_m}$
   $p := p - g_1^{\beta_1} \cdots g_m^{\beta_m} h_l$
   $r := r$
  `ELSE`
   $p := p - \mathrm{LT}(p)$
   $r := r + \mathrm{LT} \in p$

We allow the algorithm to proceed infinitely many steps. Since the order $\nu(p)$ strictly increases at every step, p converges to 0. It is also obvious that the orders of all terms in the final reduction r are not in $\Gamma(M)$. For further details, refer to the proof of Theorem 3 in Chapter 2, Section 3 of [1]. □

The reduction $R(f, G, H)$ has the following property:

Lemma 4.7. *Let (G, H) be a standard basis of M. Then for any $f \in M$, the reduction $R(f, G, H)$ is unique no matter how the elements of G and H are listed.*

Proof. The proof of this lemma is similar to that of Proposition 1 in Chapter 2, Section 6 of [1]. So we omit it. □

For $f_1, f_2 \in M$, there exist multi-indices

$$(\beta_1, \ldots, \beta_m), (\gamma_1, \ldots, \gamma_m) \in \mathbb{Z}_{\geqslant 0}^m$$

and elements $h_1, h_2 \in H$ such that

$$\mathrm{LT}(f_1) = \mathrm{LT}(g_1^{\beta_1} \cdots g_m^{\beta_m} h_1) \qquad \text{and} \qquad \mathrm{LT}(f_2) = \mathrm{LT}(g_1^{\gamma_1} \cdots g_m^{\gamma_m} h_2).$$

The *S-process* for $f_1, f_2 \in M$ is defined as

$$S(f_1, f_2) \coloneqq g_1^{\gamma_1} \cdots g_m^{\gamma_m} h_2 f_1 - g_1^{\beta_1} \cdots g_m^{\beta_m} h_1 f_2.$$

Among all such expressions, we define the one with minimal order as the *minimal S-process* of f_1 and f_2, denoted by $S_{\min}(f_1, f_2)$.

Proposition 4.8 ([2, Theorem 2.3]). *Let $G = \{g_1, \ldots, g_m\} \subset R$ be a SAGBI basis for R, and let $H = \{h_1, \ldots, h_n\}$ be a set of generators of M. The pair (G, H) is a standard basis of M if and only if*

$$R(S_{\min}(h_i, h_j), G, H) = 0$$

for all $h_i, h_j \in H$.

5. Proofs of the Main Results

As noted in Remark 2.4, Piontkowski [5] originally proved Theorem 1.1, Corollary 1.3, and Theorem 1.4 in the case $r \geqslant c$. The following result about syzygies was used in his proof:

Lemma 5.1 ([5], Proposition 5). *Let $\Gamma = \langle p, q \rangle$, and let*

$$\Delta = \bigcup_{i=0}^{p-1} (a_i + p\mathbb{N})$$

be a Γ-semimodule with p-basis $\{a_0, \ldots, a_{p-1}\}$. Consider a graded algebra $\mathbb{C}[\Gamma] \coloneqq \bigoplus_{\gamma \in \Gamma} \mathbb{C}t^\gamma$, and a graded $\mathbb{C}[\Gamma]$-algebra $\mathbb{C}[\Delta] \coloneqq \bigoplus_{\gamma \in \Delta} \mathbb{C}t^\gamma$ generated by $(t^{a_0}, \ldots, t^{a_{p-1}})$. Then the syzygies of this p-tuple are minimally generated by the following vectors:

$$\begin{aligned} v_0 &\coloneqq (t^q, -t^{\alpha_1 p}, 0, \ldots, 0), \\ v_1 &\coloneqq (0, t^q, -t^{(\alpha_2 - \alpha_1)p}, 0, \ldots, 0), \\ &\cdots \\ v_{p-2} &\coloneqq (0, \ldots, 0, t^q, -t^{(\alpha_{p-1} - \alpha_{p-2})p}), \\ v_{p-1} &\coloneqq (-t^{(q - \alpha_{p-1})p}, 0, \ldots, 0, t^q) \end{aligned}$$

where $\alpha_1, \ldots, \alpha_{p-1}$ are numbers given in (3.1).

Before proving the main theorems, we briefly summarize Piontkowski's approach.

Outline of Piontkowski's proofs. If $r \geqslant c$, then we have

$$\mathrm{Hilb}^r(X,o) \cong \mathrm{Hilb}^c(X,o) \cong J_{X,o}$$

by Corollary 2.3. Recall that points of $\mathrm{Hilb}^c(X,o)$ correspond bijectively to elements $I \in \mathcal{I}_c$ via φ_c. Moreover, each $I \in \mathcal{I}_c$ corresponds to $t^{-d}I$ where $d \coloneqq \min I$. If $\Delta \coloneqq \nu(I)$, then $\nu(t^{-d}I) = \Delta^{(0)}$. Thus, Piontkowski identified $t^{-d}I$ with a point in the Δ-subset $H(\Delta)$ of $J_{X,o} = \mathrm{Hilb}^c(X,o)$. In his proof, the following generators of $t^{-d}I$ were used:

$$h_i = t^{a_i} + \sum_{\substack{a_i+k \notin \Delta^{(0)} \\ k \geqslant 0}} \lambda_{i,k} t^{a_i+k}, \ (i = 0, \ldots, p-1) \tag{5.1}$$

where $\{a_0, a_1, \ldots, a_{p-1}\}$ is the p-basis of $\Delta^{(0)}$. He verified the conditions for

$$\Delta^{(0)} = \langle \nu(h_0), \ldots, \nu(h_{p-1}) \rangle_\Gamma.$$

Set $G = \{t^p, \phi\}$ where $\phi = t^q$+higher order terms, and $H = \{h_0, \ldots, h_{p-1}\}$. Then $\Delta^{(0)} = \langle \nu(h_0), \ldots, \nu(h_{p-1}) \rangle_\Gamma$ holds if and only if (G, H) is a standard basis of R-module $t^{-d}I$ by Proposition 4.5. Moreover, by Proposition 4.8 and Lemma 5.1, (G, H) is a standard basis of $t^{-d}I$ if and only if

$$R(S_{\min}(h_i, h_{i+1}), G, H) = 0 \ \text{ for } \ i = 0, \ldots, p-2, \tag{5.2}$$

$$R(S_{\min}(h_{p-1}, h_0), G, H) = 0 \tag{5.3}$$

hold. The minimal S-processes are given by

$$S_{\min}(h_i, h_{i+1}) = \phi h_i - t^{(\alpha_{i+1}-\alpha_i)p} h_{i+1} \text{ for } i = 0, \ldots, p-2,$$
$$S_{\min}(h_{p-1}, h_0) = \phi h_{p-1} - t^{(q-\alpha_{p-1})p} h_0.$$

Using (5.2) and (5.3), he analyzed the coefficients of S-processes and concluded that the number of independent coefficients in h_i's such that

$$\Delta^{(0)} = \langle \nu(h_0), \ldots, \nu(h_{p-1}) \rangle_\Gamma$$

is given by

$$\sum_{i=0}^{p-1} \#\{[a_i, a_i + q] \backslash \Delta^{(0)}\}. \tag{5.4}$$

Hence, each Δ-subset $H(\Delta)$ of $J_{X,o} = \mathrm{Hilb}^c(X,o)$ is an affine cell of dimension (5.4). When $r \geqslant c$, we see that $\min \Delta \geqslant c$, and so

$$(-\min \Delta + \Gamma) \cap [a_i, a_i + q] = [a_i, a_i + q],$$

on, making (1.2) equal to (5.4).

Once the decomposition (1.1) is shown to be an affine cell decomposition, it follows that Euler number of $J_{X,o}$ equals the number of Δ-subsets in (1.1).

Moreover, these Δ-subsets form a CW complex. So the Betti numbers follow from standard (co-)homology theory (see [5] for details). $\square$

Proofs of Theorem 1.1, Corollary 1.3 and Theorem 1.4. Piontkowski's method applies directly to our general case without modification. We note one remark about the difference between the dimensions (1.2) and (5.4) of Δ-subsets. Let $I \in \mathcal{I}(\Delta) \subset \mathcal{I}_r$, and set $d = \min \Delta$. Consider the generators $t^d h_i$ $(i = 0, \dots, p-1)$ of I, which can be written as

$$t^d h_i = t^{b_i} + \sum_{\substack{b_i+k \notin \Delta \\ k \geqslant 0}} \lambda_{i,k} t^{b_i+k} \tag{5.5}$$

where $b_i := a_i + d$ and $(i = 0, \dots, p-1)$. Every ideal in $I(\Delta)$ has generators of the same form as (5.5). The numbers of coefficients in (5.1) and (5.5) are same, since the sets $\mathbb{N} \backslash \Delta^{(0)}$ and $\{n \in \mathbb{N} \backslash \Delta : n > d\}$ are in one-to-one correspondence.

Piontkowski's idea was to count the number of independent coefficients in (5.1) (equivalently, in (5.5)). This count gives the dimension of $H(\Delta)$ as an affine space. If $r < c$, then (5.5) may contain some terms whose orders are not in Γ. We claim that the coefficients of such terms are not independent. Setting

$$A := \{\beta_i \notin \Gamma : \beta_i > q\} := \{\beta_1, \dots, \beta_s\},$$

we may assume that ϕ is of form $t^q + \sum_{\beta_i \in A} c_i t^{\beta_i}$. The set $\{t^p, \phi\}$ is a standard basis of R in ordinary sense. Reducing $t^d h_i$ by $\{t^p, \phi\}$, the final reduction of $t^d h_i$ must be 0, since $t^d h_i \in R$.

This implies that, for each term $\lambda_{i,k} t^{b_i+k}$ with $b_i + k \notin \Gamma$, there exists polynomial $f_{i,k}(x_1, \dots, x_s)$ in $\mathbb{C}[x_1, \dots, x_s]$ such that $\lambda_{i,k} = f_{i,k}(c_1, \dots, c_s)$ (i.e. the coefficient $\lambda_{i,k}$ is determined by ϕ, and hence are not independent). This fact explains the difference between (1.2) and (5.4). $\square$

6. Examples

The punctual Hilbert schemes of curve singularity of type A_{2l} (i.e., the curve singularity with $R = \mathbb{C}[\![t^2, t^{2l+1}]\!]$) was studied in [9]. For this singularity, we have $\Gamma = \langle 2, 2l+1 \rangle$, $\delta = l$, and $c = 2l$. Let $e(\mathrm{Hilb}^r(X, o))$ denote the Euler number of $\mathrm{Hilb}^r(X, o)$. Let also $[\,\cdot\,]$ denote the greatest integer function, where for a real number a, the value $[a]$ is the largest integer satisfying $[a] \leqslant a$. The following example is obtained from the results in [9]:

Example 6.1. Let (X, o) be the A_{2l}-singularity. The Euler numbers of the punctual Hilbert schemes $\mathrm{Hilb}^r(X, o)$ are given in the following table:

r	$0 \leqslant r \leqslant 2l-1$	$r \geqslant 2l$
$e(\mathrm{Hilb}^r(X,o))$	$[r/2]+1$	$l+1$

The Betti numbers of $\mathrm{Hilb}^r(X,o)$ are as follows:

$$h_{2d}(\mathrm{Hilb}^r(X,o)) = h^{2d}(\mathrm{Hilb}^r(X,o)) = 1 \quad (d \in \{0,1,\ldots,[r/2]\})$$

The punctual Hilbert schemes of curve singularities of type E_6 and E_8 (i.e. the curve singularities with $R = \mathbb{C}[\![t^3,t^4]\!]$ and $R = \mathbb{C}[\![t^3,t^5]\!]$ respectively) were studied in [8]. For the E_6-singularity, the fundamental invariants are $\Gamma = \langle 3,4\rangle$, $\delta = 3$ and $c = 6$. On the other hand, we have $\Gamma = \langle 3,5\rangle$, $\delta = 4$ and $c = 8$. The following examples are derived from the results in [8]:

Example 6.2. Let (X,o) be the E_6-singularity. The Euler numbers of the punctual Hilbert schemes $\mathrm{Hilb}^r(X,o)$ are given in the following table:

r	0	1	2	3	4	5	$r \geqslant 6$
$e(\mathrm{Hilb}^r(X,o))$	1	1	2	3	4	4	5

The Betti numbers of $\mathrm{Hilb}^r(X,o)$ are also given in the following tables:

r	h_0	h_2	h_4	h_6
0	1			
1	1			
2	1	1		
3	1	1	1	
4	1	1	2	
5	1	1	2	
6	1	1	2	1

r	h^0	h^2	h^4	h^6
0	1			
1	1			
2	1	1		
3	1	1	1	
4	2	1	1	
5	2	1	1	
6	1	2	1	1

Example 6.3. Let (X,o) be the E_8-singularity. The Euler numbers of the punctual Hilbert schemes $\mathrm{Hilb}^r(X,o)$ are given in the following table:

r	0	1	2	3	4	5	6	7	$r \geqslant 8$
$e(\mathrm{Hilb}^r(X,o))$	1	1	2	3	4	5	6	6	7

The Betti numbers of $\mathrm{Hilb}^r(X,o)$ are also given in the following tables:

7. Remarks on the results similar to Theorem 1.1

In this section, we discuss the results of Pfister and Steenbrink in [4]. Here we consider a monomial curve singularity (X,o).

Definition 7.1 ([4], Definition 8)**.** We call a curve singularity (X,o) with $R = \mathbb{C}[\![t^{a_1},\cdots,t^{a_n}]\!]$ for some positive integers $a_1,\ldots,a_n$ a *monomial curve singularity.*

Without loss of generality, we assume that $\gcd(a_1,\ldots,a_n) = 1$ in Definition 7.1. Additionally, we focus on special semigroups defined as follows:

r	h_0	h_2	h_4	h_6	h_8
0	1				
1	1				
2	1	1			
3	1	1	1		
4	1	1	2		
5	1	1	2	1	
6	1	2	1	2	
7	1	1	2	2	
8	1	1	2	2	1

r	h^0	h^2	h^4	h^6	h^8
0	1				
1	1				
2	1	1			
3	1	1	1		
4	2	1	1		
5	1	2	1	1	
6	2	1	2	1	
7	2	2	1	1	
8	1	2	2	1	1

Definition 7.2 ([4, Definition 9]). A semigroup Γ is called *monomial* if $0 \in \Gamma$, $\#(\mathbb{N}\backslash\Gamma) < \infty$, and any reduced, irreducible curve singularity with Γ is a monomial curve singularity.

The monomial semigroups were completely determined in [4].

Theorem 7.3 ([4, Theorem 10]). *A monomial semigroup is monomial if and only if it is one of the following three types:*

$$\{imi = 0, 1, \ldots, s\} \cup [sm + b, \infty) \text{ with } 1 \leqslant b < m,\ s \geqslant 1,$$
$$\{0\} \cup [m, m + r - 1] \cup [m + r + 1, \infty) \text{ with } 2 \leqslant r \leqslant m - 1,$$
$$\{0, m\} \cup [m + 2, 2m] \cup [2m + 2, \infty) \text{ with } m \geqslant 3$$

Example 7.4. Semigroups $\langle 2, 2l + 1\rangle$ with $l \geqslant 1$, $\langle 3, 4\rangle$, and $\langle 3, 5\rangle$ are all monomial semigroups that can be realized as plane curve singularities.

For a Γ-semimodule Δ, let S be the set of minimal generators of $\Delta^{(\delta)}$. Set $S' := S \cap [0, 2\delta - 1]$. For each $\gamma \in S'$, define

$$J_\gamma := [\gamma + 1, 2\delta - 1]\backslash\Delta^{(\delta)}.$$

Theorem 7.5 ([4, Theorem 11]). *Let (X, o) be a curve singularity with a monomial semigroup. For $I \in \mathcal{I}_r$, there exist uniquely determined $u_{\gamma,j} \in \mathbb{C}$ such that, as R-submodule, $\varphi_r(I)$ is generated by*

$$g_\gamma = t^\gamma + \sum_{j \in J_\gamma} u_{\gamma,j} t^j, (\gamma \in S').$$

Corollary 7.6 ([4, Corollary of Theorem 11]). *For a curve singularity (X, o) with a monomial semigroup, a Δ-subset $H(\Delta)$ of* $\mathrm{Hilb}^r(X, o)$ *is an affine space of dimension $\sum_{\gamma \in S'} \#J_\gamma$.*

Remark 7.7. Theorem 1.1 is stated in terms of $\Delta^{(0)}$, whereas Theorem 7.5 and Corollary 7.6 are expressed using $\Delta^{(\delta)}$.

However, Theorem 7.5 and Corollary 7.6 do not hold in general. Suppose that Theorem 7.5 and Corollary 7.6 do hold for the E_6-singularity. Recall that $R = \mathbb{C}[\![t^3, t^4]\!]$, $\Gamma = \langle 3, 4\rangle$ and $\delta = 3$ for this singularity. Consider a Γ-semimodule $\Delta = \langle 4, 6, 7\rangle_\Gamma$. It is easy to verify that $\Delta \in \mathrm{Mod}_2(\Gamma)$. So its δ-normalization is given by

$$\Delta^{(\delta)} = -2 + \Delta = \langle 2, 4, 5\rangle_\Gamma.$$

Let I be an element of $\mathcal{I}(\Delta)$. It follows from Theorem 7.5 that $\varphi_2(I)$ is generated by $g_2 = t^2 + u_{2,3}t^3$, $g_4 = t^4$ and $g_5 = t^5$ as R-submodule. By Corollary 7.6, we also have $\dim H(\Delta) = 1$. However, we see that

$$\mathcal{I}(\Delta) = \{(t^4, t^6, t^7)\},$$

which means $H(\Delta)$ consist of a single point. Therefore, $\dim H(\Delta) = 0$. Moreover, it is obvious that $\varphi_2((t^4, t^6, t^7))$ is generated by t^2, t^4, and t^5. These contradict the assumption.

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