Proceedings of the
International Geometry Center
Vol. 17, no. 3 (2024) pp. 203–217

On curve based ruled affine submanifolds

Olena Shugailo

Abstract. In this paper we consider affine ruled submanifolds of arbitrary dimension and codimension in the classical sense, i.e. curve based ones. For such a submanifold we define the natural parameterization and the natural transversal distribution. We calculate all the affine characteristics for such an affine immersion. We find conditions on the base curve and the directions of the rectilinear generators such that the induced connection is flat and the natural transversal distribution is equiaffine.

Анотація. В роботі розглядаються афінні лінійчаті підмноговиди довільної вимірності і ковимірності в класичному сенсі, тобто побудовані над кривою. Обрано натуральну параметризацію та трансверсальний розподіл, знайдено всі афінні характеристики для такого афінного занурення. Знайдено умови на базову криву та напрямки прямолінійних твірних, щоб індукована зв'язність була пласкою. З'ясовано в якому випадку обраний трансверсальний розподіл є еквіафінним.

1. Introduction

When we study affine submanifolds with given properties of affine fundamental form, connection, or Weingarten mapping. It often turns out that representatives of such submanifolds are linear submanifolds. So they deserve a detailed research.

Let us remind the definition of an affine immersion. Let (M^n, ∇) be an affine n-dimensional manifold with affine connection ∇ and $(\mathbb{R}^{n+m}, D)$ be the standard (arithmetic) affine space with flat connection D. According to K. Nomizu and T. Sasaki [4], the immersion $f\colon (M^n, \nabla) \to (\mathbb{R}^{n+m}, D)$ is said to be *affine* if along f there exists a m-dimensional transversal differentiable distribution $Q\colon x \in M^n \mapsto Q_x$ such that for all $x \in M^n$ and for all smooth tangent vector fields X, Y on M^n we have the following decomposition

$$D_{f_*(X)} f_*(Y) = f_*(\nabla_X Y) + h(X,\ Y), \quad h(X,\ Y) \in Q. \tag{1.1}$$

The research is supported by the Akhiezer Foundation
2020 Mathematics Subject Classification: 53A15
Keywords: affine immersion, equiaffine structure, ruled submanifild, flat connection
DOI: http://dx.doi.org/10.15673/pigc.v17i3.2883

The symmetric bilinear map

$$h(X,\ Y)\colon T(M^n)\times T(M^n)\to Q$$

is called the *affine fundamental form.*

If we have an immersion $f\colon M^n\to(\mathbb{R}^{n+m},D)$ without any prescribed connection, then the choice of a transversal distribution Q defines a torsion-free *induced affine connection* ∇ from (1.1).

For a chosen transversal frame $\xi_1,\dots\xi_m$ we have the affine analogues of Gauss and Weingarten decompositions

$$D_{f_*(X)}f_*(Y)=f_*(\nabla_XY)+h^\alpha(X,\ Y)\xi_\alpha,$$
$$D_{f_*(X)}\xi_\alpha=-f_*(S_\alpha X)+\tau_\alpha^\beta(X)\xi_\beta.$$

The Weingarten mapping $S_x\colon Q_x\times T_x(M^n)\to T_x(M^n)$ is defined [6] as follows: $(\xi,X)\to S_\xi X$ at every point $x\in M^n$.

The *induced volume element* θ on M^n is defined by the formula:

$$\theta(X_1,\dots,X_n)=|f_*(X_1),\dots,f_*(X_n),\xi_1,\dots,\xi_m|.$$

The transversal distribution Q with transversal frame $\{\xi_1,\dots,\xi_m\}$ is called *equiaffine*, if $\nabla_X\theta=0$ for all $X\in T_x(M^n)$, $x\in M^n$. It is known that this condition is equivalent to the following one:

$$\tau_1^1(X)+\tau_2^2(X)+\dots+\tau_m^m(X)\equiv 0.$$

With an equiaffine transversal distribution Q we have an *equiaffine structure* (∇,θ) on M^n.

It is obvious that the transversal distribution for an affine immersion can be chosen in a more than one way. The connection between the affine characteristics of the affine immersion for different transversal distributions can be found in [7].

In particular, it is known that the rank (the pointwise codimension) and the kernel of the affine fundamental form do not depend on the choice of transversal distribution.

Affine immersions with flat connections have been studied by many authors [2–4, 6, 7, 10] and others. Complete affine hypersurfaces with flat connections in standard affine $\mathbb{R}^{n+1}$ have been studied by K. Nomizu and U. Pinkall [3]. They classified immersions as:

(i) a hyperplane ($h\equiv 0$);
(ii) a graph ($S\equiv 0$);
(iii) an affine cylinder with $(n-1)$-dimensional rulings ($h\neq 0,\ S\neq 0$).

In case of higher codimension ($f\colon M^n\to\mathbb{R}^{n+m}$) we have some similar results [7]:

- if $h\equiv 0$ then f is a totally geodesic immersion and $f(M^n)$ is an n-dimensional affine subspace or its part;

- if $S \equiv 0$ then f is affinely equivalent to the graph immersion of some smooth map $F\colon M^n \to \mathbb{R}^m$.

Case of $S \not\equiv 0$ and $h \not\equiv 0$ is much more complicated. In particular, if a manifold M^n is complete and $\dim\ker h = \mu = \mathrm{const} \neq 0$, then $f(M^n)$ is a μ-ruled affine submanifold [7]. In this case we mean a generalized ruled affine submanifold with μ rectilinear generators and $(n-\mu)$-dimensional base.

A $(n-2)$-ruled submanifolds have been classified in [9] with respect to theirs two-dimensional base. With the additional requirements on the affine fundamental form, the ruled submanifold is necessarily a cylinder [6, 8, 9].

In the present paper we study affine $(n-1)$-ruled submanifolds of arbitrary codimension with a nondegenerate curve as the base. For such a submanifold we define the natural parametrization and the natural transversal distribution. Then we find all its affine characteristics (Lemma 3.1). We see that corresponding to directions of rectilinear generators coordinate tangent vectors belong to $\ker S$ and $\ker R$. Also we have the following estimations (Corollary 3.2):

$$\dim\ker S \geqslant n-1, \qquad \dim\ker R \geqslant n-1, \qquad \dim\ker h \leqslant n-1.$$

They mean that the image of the Weingarten mapping is not greater than 1. However, if the condition $S \equiv 0$ defines an affine immersion of a graph [7], then the condition $\dim \operatorname{im} S = 1$ does not define a ruled submanifold. In the case of codimension 2 the classification of immersions with flat connections and one-dimensional Weingarten mapping has been obtained in [10]. Such an immersion can be a peculiar mix of a graph and a $(n-1)$-ruled submanifold.

Finally, we find conditions on the base curve and the directions of the rectilinear generators under which the induced connection is flat and the natural transversal distribution is equiaffine.

In the paper, by *affine completeness* we will mean the completeness of the induced connection [5]. Therefore, an affine manifold (M^n, ∇) is called *complete* if every ∇-geodesic extends infinitely with respect to its affine parameter. In case a complete ruled affine submanifold is not a graph and its codimension is greater than 1, we obtain its parameterization (Theorem 3.3). It is known that a complete ruled affine hypersurface with flat connection is a cylinder [3].

2. Preliminaries

Let us remand equaffine curve theory in $\mathbb{R}^k$ [1]. We consider a smooth curve $\bar\rho(t) = \{\rho^1(t), \rho^2(t), \dots, \rho^k(t)\}$, where $\rho_i(t)$ are smooth functions of t defined on a certain interval G.

We say that the curve is *nondegenerate* if

$$|\bar\rho'_t, \bar\rho''_t, \dots, \bar\rho^{(k)}_t| \neq 0 \quad \text{for all} \quad t \in G.$$

It is clear that a nondegenerate curve $\bar\rho(t)$ admits a parameter σ such that

$$|\bar\rho'_\sigma, \bar\rho''_\sigma, \dots, \bar\rho^{(k)}_\sigma| \equiv 1.$$

Parameter σ is called an *affine arclength parameter.* It is unique up to a constant summand and is defined by

$$\bar\sigma'_t = |\bar\rho'_t, \bar\rho''_t, \dots, \bar\rho^{(k)}_t|^{\frac{2}{k(k+1)}}.$$

Differentiating the identity $(|\bar\rho'_\sigma, \bar\rho''_\sigma, \dots, \bar\rho^{(k)}_\sigma| \equiv 1)'_\sigma$, we obtain that

$$|\bar\rho'_\sigma, \dots, \bar\rho^{(k-1)}_\sigma, \bar\rho^{(k+1)}_\sigma| = 0.$$

The latter identity implies the following decomposition

$$\bar\rho^{(k+1)}_\sigma = -k_1(\sigma)\bar\rho'_\sigma - \dots - k_{k-1}(\sigma)\bar\rho^{(k-1)}_\sigma, \tag{2.1}$$

where $k_1(\sigma), \dots, k_{k-1}(\sigma)$ are called *affine curvatures* of the curve.

We will define the *natural parameterization* of a ruled affine submanifold in $\mathbb{R}^{n+m}$ by the position vector

$$\bar r(u^1, u^2, \dots, u^n) = \bar\rho(u^1) + \sum_{i=2}^{n} u^i \bar a_i(u^1), \tag{2.2}$$

where

$$\bar\rho(u^1) = (\rho^1(u^1), \dots, \rho^{m+1}(u^1),\ 0, \dots\ 0)$$

is a nondegenerate curve in $\mathbb{R}^{m+1}$ with an affine arclength parameter u^1, and

$$\bar a_i(u^1) = \left(a_i^1(u^1), \dots, a_i^{m+1}(u^1), 0, \dots, 1, \dots 0\right), \quad (i = \overline{2, n}),$$

are directions of rectilinear generators. Here 1 stands at the $(i+m)$-th coordinate place of $\bar a_i$.

We define the *natural transversal frame* as follows

$$\xi_1 = \bar\rho''(u^1), \qquad \xi_2 = \bar\rho'''(u^1), \qquad \dots, \qquad \xi_m = \bar\rho^{(m+1)}(u^1). \tag{2.3}$$

Say that a natural parametrization (2.2) of a ruled submanifold is *regular* if it is regular with respect to the corresponding natural transversal frame (2.3), that is the following condition

$$\det(\bar r'_1, \dots, \bar r'_n, \xi_1, \dots, \xi_m) \neq 0 \tag{2.4}$$

holds for all parameters u^i.

3. Main results

We deal with the *natural parameterization* (2.2) of a ruled affine submanifold in $\mathbb{R}^{n+m}$. Denote by $\{e_1, e_2, \ldots, e_n\}$ the corresponding coordinate tangent frame. Here the vector e_1 corresponds to the first tangent vector, i.e. to $\bar{r}_1'$, while vectors e_i $(i = \overline{2,n})$ correspond to the directions of rectilinear generators, i.e. to $\bar{a}_i = \bar{r}_i'$.

Lemma 3.1. *An affine immersion of a complete ruled submanifold with the natural parameterization* (2.2) *and the natural transversal frame* (2.3) *has the following affine characteristics:*

– *the induced affine connection*

$$\nabla_{e_1} e_1 = -k_1 \sum_{i=2}^{n} u^i \lambda_i^{m+1} e_1,$$
$$\nabla_{e_j} e_1 = 0, \quad \nabla_{e_j} e_k = 0, \qquad j, k = \overline{2,n};$$

– *the curvature tensor*

$$R(e_k, e_1)e_j = R(e_j, e_k)e_1 = R(e_k, e_i)e_j = 0,$$
$$R(e_1, e_j)e_1 = k_1(u^1)\lambda_j^{m+1}(u^1)e_1, \qquad i, j, k = \overline{2,n}$$

– *the components of the affine fundamental form*

$$h_{11}^1 = 1 + k_1 \sum_{i,j=2}^{n} u^i u^j \lambda_i^2 \lambda_j^{m+1} + \sum_{i=2}^{n} u^i \left((\lambda_i^2)' - k_2 \lambda_i^{m+1}\right),$$

$$h_{11}^s = k_1 \sum_{i,j=2}^{n} u^i u^j \lambda_i^{s+1} \lambda_j^{m+1} +$$
$$+ \sum_{i=2}^{n} u^i \left(\lambda_i^s + (\lambda_i^{s+1})' - k_{s+1}\lambda_i^{m+1}\right), \qquad s = \overline{2, m-1},$$

$$h_{11}^m = k_1 \sum_{i,j=2}^{n} u^i u^j \lambda_i^{m+1} \lambda_j^{m+1} + \sum_{i=2}^{n} u^i \left(\lambda_i^m + (\lambda_i^{m+1})'\right),$$

$$h_{1j}^l = \lambda_j^{l+1}, \quad h_{ij}^l = 0, \quad l = \overline{1,m}, \quad i, j = \overline{2,n};$$

– *the shape operators*

$$S_l \equiv 0, \; l = \overline{1, m-1}, \qquad S_m(e_j) = 0, \; j = \overline{2,n}, \qquad S_m(e_1) = k_1(u^1)e_1;$$

– *the transversal connection forms*

$$\tau_\alpha^\beta(e_1) = 0, \quad \alpha, \beta = \overline{1, m-1}, \; \beta \neq \alpha + 1, \quad \tau_\alpha^{\alpha+1}(e_1) = 1,$$

$$\tau_m^\beta(e_1) = k_1(u^1)\sum_{i=2}^n u^i\lambda_i^{\beta+1} - k_{\beta+1}, \quad \beta = \overline{1, m-1};$$

$$\tau_m^m(e_1) = k_1(u^1)\sum_{i=2}^n u^i\lambda_i^{m+1}, \qquad \tau_\alpha^\beta(e_j) = 0 \quad \forall\alpha,\beta,\ j = \overline{2,n}.$$

Here $k_1(u^1)$,..., $k_{k-1}(u^1)$ *are* affine curvatures *of the base curve. The functions* $\lambda_i^j(u^1)$ *are obtained from the decompositions*

$$\bar{a}_i'(u^1) = \lambda_i^1(u^1)\bar{\rho}'(u^1) + \lambda_i^2(u^1)\bar{\rho}''(u^1) + \dots + \lambda_i^{m+1}(u^1)\bar{\rho}^{(m+1)}(u^1). \quad (3.1)$$

Proof. For the natural parameterization (2.2) we have

$$\bar{r}_1' = \bar{\rho}'(u^1) + \sum_{i=2}^n u^i\bar{a}_i'(u^1), \qquad \bar{r}_i' = \bar{a}_i(u^1),$$

$$\bar{r}_{11}'' = \bar{\rho}''(u^1) + \sum_{i=2}^n u^i\bar{a}_i''(u^1), \qquad \bar{r}_{1i}'' = \bar{a}_i'(u^1), \qquad \bar{r}_{ij}'' = \bar{0},\ i,j = \overline{2,n}.$$

Due to (3.1), the vectors $\bar{\rho}'(u^1), \bar{\rho}''(u^1), \dots, \bar{\rho}^{(m+1)}(u^1)$ form a basis of the space to which the curve $\bar{\rho}(u^1)$ and the vectors $\bar{a}_i'(u^1)$ belong.

In order to write out the Gauss decomposition we find the relationship between $\bar{\rho}'(u^1)$ and $\bar{r}_1'$:

$$\bar{r}_1' = \bar{\rho}' + \sum_{i=2}^n u^i \sum_{s=1}^{m+1} \lambda_i^s\bar{\rho}^{(s)} = \Big(1 + \sum_{i=2}^n u^i\lambda_i^1\Big)\bar{\rho}' + \sum_{i=2}^n u^i \sum_{s=1}^m \lambda_i^{s+1}\xi_s,$$

$$\Big(1 + \sum_{i=2}^n u^i\lambda_i^1\Big)\bar{\rho}' = \bar{r}_1' - \sum_{i=2}^n u^i \sum_{s=1}^m \lambda_i^{s+1}\xi_s.$$

It is easy to see that the determinant (2.4) is equal to $1 + \sum\limits_{i=2}^n u^i\lambda_i^1$ up to a sign. Therefore, the natural parameterization is regular if

$$\Big(1 + \sum_{i=2}^n u^i\lambda_i^1\Big) \neq 0. \quad (3.2)$$

But let us temporarily omit this requirement and get results in general case.

Since u^1 is an affine arclength parameter of the curve $\bar{\rho}(u^1)$, we have that

$$\bar{\rho}^{(m+2)} = -\sum_{l=1}^m k_l(u^1)\bar{\rho}^{(l)},$$

where $k_1(u^1), \dots, k_m(u^1)$ are the affine curvatures of the curve $\bar{\rho}(u^1)$.

Then the Gauss decomposition for $\bar{r}_{1j}''$ can be written as follows:

$$\bar{r}_{1j}'' = \bar{a}_j' = \lambda_j^1\bar{\rho}' + \lambda_j^2\bar{\rho}'' + \dots + \lambda_j^{m+1}\bar{\rho}^{(m+1)}$$

$$= \frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}\left(\bar{r}_1' - \sum_{i=2}^{n} u^i \sum_{s=1}^{m} \lambda_i^{s+1}\xi_s\right) + \sum_{s=1}^{m} \lambda_j^{s+1}\xi_s$$

$$= \frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}\bar{r}_1' + \sum_{s=1}^{m} \frac{\lambda_j^{s+1} + \sum\limits_{i=2}^{n} u^i\left(\lambda_j^{s+1}\lambda_i^1 - \lambda_j^1\lambda_i^{s+1}\right)}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}\xi_s.$$

Thus, we have the following affine components:

$$\Gamma_{1j}^1 = \frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}, \qquad \Gamma_{1j}^k = 0, \quad k = \overline{2,n};$$

$$h_{1j}^s = \frac{\lambda_j^{s+1} + \sum\limits_{i=2}^{n} u^i\left(\lambda_j^{s+1}\lambda_i^1 - \lambda_j^1\lambda_i^{s+1}\right)}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}.$$

From decomposition (3.1) we obtain that

$$\begin{aligned}\bar{a}_i''(u^1) &= (\lambda_i^1)'\bar{\rho}' + \sum_{l=1}^{m} \lambda_i^l\bar{\rho}^{(l+1)} + \sum_{l=2}^{m+1} (\lambda_i^l)'\bar{\rho}^{(l)} + \lambda_i^{m+1}\bar{\rho}^{(m+2)}\\ &= (\lambda_i^1)'\bar{\rho}' + \sum_{l=1}^{m}\left(\lambda_i^l + (\lambda_i^{l+1})'\right)\bar{\rho}^{(l+1)} - \lambda_i^{m+1}\sum_{l=1}^{m} k_l\bar{\rho}^{(l)}\\ &= \left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)\bar{\rho}' + \sum_{l=1}^{m-1}\left(\lambda_i^l + (\lambda_i^{l+1})' - k_{l+1}\lambda_i^{m+1}\right)\xi_l +\\ &\quad + \left(\lambda_i^m + (\lambda_i^{m+1})'\right)\xi_m.\end{aligned}$$

Then the Gauss decomposition for $\bar{r}_{11}''$ is given by

$$\begin{aligned}\bar{r}_{11}'' &= \bar{\rho}''(u^1) + \sum_{i=2}^{n} u^i\bar{a}_i''(u^1)\\ &= \xi_1 + \sum_{i=2}^{n} u^i\left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)\bar{\rho}'\\ &\quad + \sum_{i=2}^{n} u^i \sum_{l=1}^{m-1}\left(\lambda_i^l + (\lambda_i^{l+1})' - k_{l+1}\lambda_i^{m+1}\right)\xi_l\\ &\quad + \sum_{i=2}^{n} u^i\left(\lambda_i^m + (\lambda_i^{m+1})'\right)\xi_m\end{aligned}$$

$$= \xi_1 + \frac{\sum\limits_{i=2}^{n} u^i \left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{1 + \sum\limits_{i=2}^{n} u^i\lambda_i^1} \left(\bar{r}_1' - \sum_{i=2}^{n} u^i \sum_{s=1}^{m} \lambda_i^{s+1}\xi_s\right) +$$
$$+ \sum_{i=2}^{n} u^i \sum_{l=1}^{m-1} \left(\lambda_i^l + (\lambda_i^{l+1})' - k_{l+1}\lambda_i^{m+1}\right)\xi_l$$
$$+ \sum_{i=2}^{n} u^i \left(\lambda_i^m + (\lambda_i^{m+1})'\right)\xi_m$$

Thus, we have the following affine components:

$$\Gamma_{11}^1 = \frac{\sum\limits_{i=2}^{n} u^i \left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{1 + \sum\limits_{i=2}^{n} u^i\lambda_i^1}, \qquad \Gamma_{11}^k = 0, \quad k = \overline{2,n}.$$

Since $\bar{r}_{jk}'' = \bar{0}$ for $j,k = \overline{2,n}$, we have the following affine connection:

$$\nabla_{e_j} e_k = 0, \quad j,k = \overline{2,n},$$
$$\nabla_{e_j} e_1 = \frac{\lambda_j^1}{1 + \sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1, \qquad \nabla_{e_1} e_1 = \frac{\sum\limits_{i=2}^{n} u^i \left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{1 + \sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1,$$

and the Weingarten decompositions:

$$\frac{\partial \xi_l}{\partial u^1} = \bar{\rho}^{(l+2)}(u^1) = \xi_{l+1}, \quad l = \overline{1,m-1},$$
$$\frac{\partial \xi_s}{\partial u^k} = \bar{0}, \quad s = \overline{1,m},\ k = \overline{2,n},$$
$$\frac{\partial \xi_m}{\partial u^1} = \bar{\rho}^{(m+2)} = -\sum_{l=1}^{m} k_l(u^1)\bar{\rho}^{(l)}$$
$$= \frac{-k_1}{1 + \sum\limits_{i=2}^{n} u^i\lambda_i^1} \left(\bar{r}_1' - \sum_{i=2}^{n} u^i \sum_{s=1}^{m} \lambda_i^{s+1}\xi_s\right) - \sum_{s=1}^{m-1} k_{s+1}\xi_s.$$

Therefore, the corresponding shape operators and transversal connection forms can be written as follows:

$$S_l \equiv 0, \quad l = \overline{1,m-1}; \qquad S_m(e_j) = 0, \quad j = \overline{2,n},$$

$$
\begin{aligned}
S_m(e_1) &= \frac{k_1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1,\\
\tau_\alpha^\beta(e_j) &= 0, \quad \forall \alpha,\beta, \quad j=\overline{2,n};\\
\tau_\alpha^\beta(e_1) &= 0, \quad \alpha,\beta=\overline{1,m-1},\ \beta\neq\alpha+1,\\
\tau_\alpha^{\alpha+1}(e_1) &= 1,\\
\tau_m^\beta(e_1) &= \frac{k_1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}\sum_{i=2}^{n} u^i\lambda_i^{\beta+1} - k_{\beta+1}, \quad \beta=\overline{1,m-1},\\
\tau_m^m(e_1) &= \frac{k_1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}\sum_{i=2}^{n} u^i\lambda_i^{m+1}.
\end{aligned}
$$

Note, that in general case the transversal distribution is not equiaffine.

Finally, we obtain the curvature tensor:

$$
\begin{aligned}
R(e_k,e_1)e_j &= \nabla_{e_k}\nabla_{e_1}e_j \;-\; \nabla_{e_1}\nabla_{e_k}e_j\\
&= \nabla_{e_k}\left(\frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1\right) - \nabla_{e_1}(0)\\
&= \frac{\lambda_j^1\lambda_k^1}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1 - \frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1}\cdot\frac{\lambda_k^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1 = 0,
\end{aligned}
$$

$$
\begin{aligned}
R(e_j,e_k)e_1 &= \nabla_{e_j}\nabla_{e_k}e_1 \;-\; \nabla_{e_k}\nabla_{e_j}e_1\\
&= \nabla_{e_j}\left(\frac{\lambda_k^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1\right) - \nabla_{e_k}\left(\frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1\right) = 0,
\end{aligned}
$$

$$
\begin{aligned}
R(e_1,e_j)e_1 &= \nabla_{e_1}\nabla_{e_j}e_1 \;-\; \nabla_{e_j}\nabla_{e_1}e_1\\
&= \nabla_{e_1}\left(\frac{\lambda_j^1}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1\right) - \nabla_{e_j}\left(\frac{\sum\limits_{i=2}^{n} u^i\left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{1+\sum\limits_{i=2}^{n} u^i\lambda_i^1} e_1\right)
\end{aligned}
$$

$$= \frac{(\lambda_j^1)'\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right) - \lambda_j^1\left(\sum\limits_{i=2}^{n} u^i(\lambda_i^1)'\right)}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1 +$$

$$+ \frac{\lambda_j^1 \sum\limits_{i=2}^{n} u^i\left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1 -$$

$$- \frac{\left((\lambda_j^1)' - k_1\lambda_j^{m+1}\right)\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1 -$$

$$+ \frac{\lambda_j^1 \sum\limits_{i=2}^{n} u^i\left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1 - \frac{\lambda_j^1 \sum\limits_{i=2}^{n} u^i\left((\lambda_i^1)' - k_1\lambda_i^{m+1}\right)}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1$$

$$= \frac{k_1\left(\lambda_j^{m+1} + \sum\limits_{i=2}^{n} u^i(\lambda_j^{m+1}\lambda_i^1 - \lambda_j^1\lambda_i^{m+1})\right)}{\left(1+\sum\limits_{i=2}^{n} u^i\lambda_i^1\right)^2} e_1.$$

It is easy to see that with obtained induced connection ∇ the curves $u^1 = c_i$, $u^i = b_i s + d_i$, $c_i, b_i, d_i \in \mathbb{R}$, $i = \overline{2,n}$ are ∇-geodesic with their affine parameter s. If the induced connection is complete, parameters u^i ($i = \overline{2,n}$) run from $-\infty$ to ∞. Therefore, the regularity condition (3.2) fulfilled for any u^i ($i = \overline{2,n}$), which is true only if $\lambda_i^1 \equiv 0$, $i = \overline{2,n}$. Now we substitute the necessary completeness condition into all the obtained affine characteristics and get the ones formulated in the lemma. □

Corollary 3.2. *An affine immersion of a ruled submanifold with the natural parameterization* (2.2) *and the natural transversal frame* (2.3) *has the following properties:*

$$\dim\ker S \geqslant n-1, \qquad \dim\ker R \geqslant n-1, \qquad \dim\ker h \leqslant n-1.$$

Furthermore, coordinate tangent vectors $e_2, \dots, e_n$ correspond to directions of rectilinear generators,

$$\operatorname{span}(e_2,\dots,e_n) \subseteq \ker S, \qquad \operatorname{span}(e_2,\dots,e_n) \subseteq \ker R,$$

and $\operatorname{span}(e_2,\dots,e_n) \supseteq \ker h$.

As we can see, the curvature tensor is zero in the following two cases:

(a) the affine curvature $k_1(u^1)$ of the base curve is zero;

(b) for any values of the parameters $u^1, u^2, \dots, u^n$, the following system is fulfilled:

$$\lambda_j^{m+1} + \sum_{i=2}^{n} u^i(\lambda_j^{m+1}\lambda_i^1 - \lambda_j^1\lambda_i^{m+1}) = 0, \quad j = \overline{2,n},$$

it is possible only in the case of $\lambda_j^{m+1}(u^1) \equiv 0$, $j = \overline{2,n}$ in (3.1). We see that condition for flat connection is the same for a complete and an incomplete submanifold.

Obviously, that in both cases we have that $\sum_{i=1}^{m} \tau_i^i \equiv 0$. Hence the natural transversal distribution is equiaffine when the induced connection is flat.

Theorem 3.3. *Let* $f\colon (M^n, \nabla) \to \mathbb{R}^{n+m}$, $m > 1$, *be an affine immersion with flat connection* ∇, *satisfying* $\dim \operatorname{im} S = 1$ *and such that* $f(M^n)$ *is a complete* $(n-1)$*-ruled submanifold. If there exist its regular natural parameterization, then it takes the following form*

$$\bar{r}(u^1, \dots, u^n) = \bar{\rho}(u^1) + \sum_{i=2}^{n} u^i \sum_{s=2}^{m} \int \lambda_i^s(u^1)\bar{\rho}^{(s)}(u^1)du^1, \tag{3.3}$$

where $\bar{\rho}(u^1)$ *is a nondegenerate curve in* $\mathbb{R}^{m+1}$ *with an affine arclength parameter* u^1, *and* $\lambda_i^s(u^1)$ *are certain smooth functions.*

Proof. If we choose the natural parameterization (2.2) for a $(n-1)$-ruled submanifold, then according to Lemma 3.1 the curvature tensor will be zero in the following two cases:

(1) the affine curvature $k_1(u^1)$ of the base curve is zero;

(2) $\lambda_j^{m+1}(u^1) \equiv 0$ for $j = \overline{2,n}$ in (3.1).

In the first case ($k_1(u^1) \equiv 0$) the shape operators are zero, therefore such an immersion affinely equivalent to a graph immersion of some smooth map [7]. We have one more proof of this statement. In case $k_1(u^1) \equiv 0$ we get from (2.1) that the parameterization of the base curve is

$$\bar{\rho}(u^1) = (u^1, \rho^2(u^1), \dots, \rho^{m+1}(u^1), 0, \dots 0).$$

This means that we have an affine immersion of a graph in (2.2).

The condition $\dim \operatorname{im} S = 1$ implies $k_1(u^1) \neq 0$, whence we have the second case ($\lambda_j^{m+1}(u^1) \equiv 0$ for $j = \overline{2,n}$ in (3.1)). Let us consider it in more detail. Note that the tangent vectors are:

$$\bar{r}_i' = \bar{a}_i(u^1), \quad i = \overline{2,n},$$

$$\bar{r}_1' = \Big(1 + \sum_{i=2}^{n} \lambda_i^1(u^1)u^i\Big)\bar{\rho}'(u^1) + \sum_{i=2}^{n} u^i \sum_{s=1}^{m} \lambda_i^{s+1}(u^1)\bar{\rho}^{(s+1)}(u^1).$$

The immersion is regular and complete only if $\lambda_i^1 \equiv 0$, $i = \overline{2,n}$. We have the following decomposition for vector $\bar{a}_i$:

$$\bar{a}_i'(u^1) = \sum_{s=2}^{m} \lambda_i^s(u^1)\bar{\rho}^{(s)}(u^1), \qquad \bar{a}_i(u^1) = \sum_{s=2}^{m} \int \lambda_i^s(u^1)\bar{\rho}^{(s)}(u^1)du^1.$$

Thus

$$\bar{r}(u^1, \dots, u^n) = \bar{\rho}(u^1) + \sum_{i=2}^{n} u^i \sum_{s=2}^{m} \int \lambda_i^s(u^1)\bar{\rho}^{(s)}(u^1)du^1$$

is a regular natural parameterization of a complete $(n-1)$-ruled submanifold in $\mathbb{R}^{n+m}$.

In this case the natural transversal distribution is equiaffine. □

Using the previous results we get the following corollary.

Corollary 3.4. *An affine immersion* (3.3) *with the natural transversal distribution has the following general affine characteristics:*

– *the induced affine connection is flat:*

$$\Gamma_{ij}^k = 0 \quad \forall i, j, k;$$

– *the components of the affine fundamental form:*

$$h^1 = \begin{pmatrix} 1 + \sum\limits_{i=2}^{n} u^i(\lambda_i^2)' & \lambda_2^2 & \dots & \lambda_n^2 \\ \lambda_2^2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n^2 & 0 & \dots & 0 \end{pmatrix},$$

$$h^s = \begin{pmatrix} \sum\limits_{i=2}^{n} u^i\left(\lambda_i^s + (\lambda_i^{s+1})'\right) & \lambda_2^{s+1} & \dots & \lambda_n^{s+1} \\ \lambda_2^{s+1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n^{s+1} & 0 & \dots & 0 \end{pmatrix}, \quad s = \overline{2, m-1};$$

$$h^m = \begin{pmatrix} \sum\limits_{i=2}^{n} u^i\lambda_i^m & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix};$$

– *the shape operators:*

$$S_l \equiv 0,\ l = \overline{1, m-1};\ S_m(e_j) = 0,\ j = \overline{2, n},\ S_m(e_1) = k_1(u^1)e_1;$$

– *the transversal connection forms:*

$$\begin{aligned} \tau_\alpha^\beta(e_j) &= 0 \quad \forall \alpha, \beta, \quad j = \overline{2, n}; \\ \tau_\alpha^\beta(e_1) &= 0, \quad \alpha, \beta = \overline{1, m-1},\ \beta \neq \alpha + 1, \\ \tau_\alpha^{\alpha+1}(e_1) &= 1, \\ \tau_m^\beta(e_1) &= k_1(u^1) \sum_{i=2}^n u^i \lambda_i^{\beta+1} - k_{\beta+1}(u^1), \quad \beta = \overline{1, m-1}, \\ \tau_m^m(e_1) &= 0. \end{aligned}$$

4. Examples

Example 4.1. A incomplete affine ruled surface in $\mathbb{R}^3$.

Consider the following surface in $\mathbb{R}^3$ with the position vector:

$$\bar{r}(u, v) = \left(e^u + \tfrac{1}{2} v e^{2u}, \tfrac{1}{2} e^{-u} - \tfrac{1}{2} uv,\ v\right).$$

It is a ruled surface with the base curve $\bar{\rho}(u) = \left(e^u, \frac{1}{2} e^{-u}, 0\right)$. Since

$$\bar{\rho}'(u) = \left(e^u, -\tfrac{1}{2} e^{-u}, 0\right), \qquad \bar{\rho}''(u) = \left(e^u, \tfrac{1}{2} e^{-u}, 0\right),$$

it is easy to see that u is the affine arclength parameter for $\bar{\rho}(u)$. From the equality

$$\bar{\rho}'''(u) = \left(e^u, -\tfrac{1}{2} e^{-u}, 0\right) = \bar{\rho}'(u)$$

we obtain that the affine curvature ia $k = -1$ (the curve is the hyperbola $xy = 1/2$).

The natural transversal vector is $\xi = \bar{\rho}''(u)$.

The direction of rectilinear ruling is $\bar{a}(u) = \left(\frac{1}{2} e^{2u}, -\frac{1}{2} u, 1\right)$. Furthermore,

$$\bar{a}'(u) = \left(e^{2u}, -\tfrac{1}{2}, 0\right) = e^u \bar{\rho}'(u),$$

whence $\lambda^1(u) = e^u, \lambda^2(u) = 0$. Since $\lambda^1(u) = e^u$, the surface is incomplete, and since $\lambda^2(u) = 0$, the induced connection is flat. This surface is an incomplete ruled one and it is not a cylinder. We know that complete affine hypersurface with flat connection can be only a subspace, or a graph, or a cylinder [3].

Example 4.2. A complete 2-ruled 3-dimensional affine submanifold in $\mathbb{R}^5$.

We will construct a complete ruled affine submanifold with 3-dimensional base curve and two rulings. The base curve is $\bar{\rho}(t) = \left(e^t, \frac{1}{30} e^{-3t}, \frac{1}{4} e^{2t}\right)$. We check that t is affine arclength parameter and find affine curvatures k_1, k_2. Evidently,

$$\bar\rho'_t = \left(e^t, -\tfrac{1}{10}e^{-3t}, \tfrac{1}{2}e^{2t}\right), \quad \bar\rho''_t = \left(e^t, \tfrac{3}{10}e^{-3t}, e^{2t}\right), \quad \bar\rho'''_t = \left(e^t, -\tfrac{9}{10}e^{-3t}, 2e^{2t}\right).$$

It is easy to see that $|\bar\rho', \bar\rho'', \bar\rho'''| \equiv 1$, i. e. t is affine arclength parameter for the curve. From the equality

$$\bar\rho^{(4)}(t) = \left(e^t, \tfrac{27}{10}e^{-3t}, 4e^{2t}\right) = -6\bar\rho'(t) + 7\bar\rho''(t)$$

we obtain affine curvatures $k_1 = 6$, $k_2 = -7$.

In parameterization (3.3) we put $\bar\rho(u^1) = \left(e^{u^1}, \tfrac{1}{30}e^{-3u^1}, \tfrac{1}{4}e^{2u^1}, 0, 0\right)$ and, for example, $\lambda_2^2(u^1) = e^{u^1}$, $\lambda_3^2(u^1) = e^{-u^1}$. Then we get the following parameterization:

$$\bar r(u^1, u^2, u^3) = \begin{pmatrix} e^{u^1} \\ \frac{1}{30}e^{-3u^1} \\ \frac{1}{4}e^{2u^1} \\ 0 \\ 0 \end{pmatrix} + u^2 \begin{pmatrix} \frac{1}{2}e^{2u^1} \\ -\frac{3}{20}e^{-2u^1} \\ \frac{1}{3}e^{3u^1} \\ 1 \\ 0 \end{pmatrix} + u^3 \begin{pmatrix} u^1 \\ -\frac{3}{40}e^{-4u^1} \\ e^{u^1} \\ 0 \\ 1 \end{pmatrix}.$$

The natural equiaffine transversal frame is

$$\xi_1 = \bar\rho'' = \left(e^{u^1}, \tfrac{3}{10}e^{-3u^1}, e^{2u^1}, 0, 0\right),$$
$$\xi_2 = \bar\rho''' = \left(e^{u^1}, -\tfrac{9}{10}e^{-3u^1}, 2e^{2u^1}, 0, 0\right).$$

This affine immersion has flat induced connection, one-dimensional Weingarten mapping, and components of affine fundamental form

$$h^1 = \begin{pmatrix} 1 + u^2e^{u^1} - u^3e^{-u^1} & e^{u^1} & e^{-u^1} \\ e^{u^1} & 0 & 0 \\ e^{-u^1} & 0 & 0 \end{pmatrix}, \quad h^2 = \begin{pmatrix} u^2e^{u^1} + u^3e^{-u^1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case we have that $\dim\ker h = 1$, $\ker h = \operatorname{span}(e^{-u^1}e_2 - e^{u^1}e_3)$. Tangent vectors e_2, e_3 correspond to directions of rectilinear rulings $\bar a_2(u^1)$, $\bar a_3(u^1)$ of the submanifold.

References

[1] Declan Davis. Generic affine differential geometry of curves in $\mathbb{R}^n$. *Proc. Roy. Soc. Edinburgh Sect. A*, 136(6):1195–1205, 2006. `doi:10.1017/S0308210500004947`.

[2] Martin Magid and Luc Vrancken. Flat affine surfaces in $\mathbb{R}^4$ with flat normal connection. *Geom. Dedicata*, 81(1-3):19–31, 2000. `doi:10.1023/A:1005267821555`.

[3] Katsumi Nomizu and Ulrich Pinkall. On the geometry of affine immersions. *Math. Z.*, 195(2):165–178, 1987. `doi:10.1007/BF01166455`.

[4] Katsumi Nomizu and Takeshi Sasaki. *Affine differential geometry*, volume 111 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1994. Geometry of affine immersions.

[5] Barbara Opozda. Completeness in affine and statistical geometry. *Ann. Global Anal. Geom.*, 59(3):367–383, 2021. doi:10.1007/s10455-021-09752-x.

[6] E. A. Shugailo. Parallel affine immersions $M^n \to \mathbb{R}^{n+2}$ with flat connection. *Ukrainian Math. J.*, 65(9):1426–1445, 2014. doi:10.1007/s11253-014-0870-6.

[7] O. O. Shugailo. On affine immersions with flat connections. *J. Math. Phys. Anal. Geom.*, 8(1):90–105, 109, 112, 2012.

[8] O. O. Shugailo. On cylinder representation of affine immersions. *Proc. Intern. Geom. Center*, 5(3-4):68–78, 2012.

[9] O. O. Shugailo. Affine submanifolds of rank two. *J. Math. Phys. Anal. Geom.*, 9(2):227–238, 278, 282, 2013.

[10] O. O. Shugailo. Equiaffine immersions of codimension two with flat connection and one-dimensional Weingarten mapping. *Mat. Stud.*, 60(1):99–112, 2023. doi:10.30970/ms.60.1.99-112.

Received: September 2, 2024, accepted: November 11, 2024.

O. O. Shugailo
Department of Pure Mathematics
V. N. Karazin Kharkiv National University
Ukraine
Email: lfisdi@gmail.com
ORCID: 0000-0002-1238-3403