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Schwartz-Green's families for quantum mechanics

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Abstract. This paper is devoted to the concept of Green families associated with a linear continuous endomorphism defined on the space of tempered distributions $\mathcal{S}'_n$. Our concept of Green family is an operative and rigorous version (counterpart) of some formal type of Green function used in Theoretical Physics, Quantum Mechanics, Quantum Chemistry and Engineering and sometime also in Mathematical Physics to solve some kind of Linear Differential Equations. The relationships between our method and the approach by fundamental solutions is immediately showed, to solve inhomogeneous linear equations.

Анотація. Стаття присвячена поняттю сімей Гріна, асоційованих з лінійним неперервним ендоморфізмом простору узагальнених функцій повільного росту $\mathcal{S}'_n$. Запропонована концепція сім'ї Гріна є операторною та строгою версією певного формального типу функції Гріна, що використовується в теоретичній фізиці, квантовій механіці, квантовій хімії та інженерії, і іноді також у математичній фізиці для розв'язування деяких лінійних диференціальних рівнянь. В роботі також показано взаємозв'язок між даним методом та методом побудови фундаментальних розв'язків неоднорідних лінійних рівнянь.

1. Introduction: the role of Schwartz Linear Algebra in Quantum Mechanics

The classic Hilbert space methods cannot be used for the definition and resolution of the free relativistic Schrödinger equation because the very fundamental solutions of this equation cannot be framed in a Hilbert or Banach space context. We can justify the necessity to use distribution theory, for many reasons. If we lock down ourselves in separable Hilbert space theory, we cannot hope to solve satisfactorily (from a physical point of view) the relativistic Schrödinger equation for free particles. The simple reason is that the very main (and generating) solutions of the relativistic

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Schrödinger equation for free particles cannot be considered as elements of a separable Hilbert Space.

First of all, if we desire to consider the standard L^2 product, we immediately observe that the de Broglie waves do not belong to the space of square integrable functions.

Moreover, the set of all harmonic waves is a continuous set (not a discrete one) and, if we select its "naturally orthonormal" subfamily (generating the unitary Minkowsky-Fourier transform as integral kernel), we are getting again a continuous family that should be orthogonal by right, from a physical perspective, but cannot be as such in any separable Hilbert space (even different from L^2). We cannot find continuous orthonormal families in a separable Hilbert space, but only discrete orthonormal systems! Furthermore, even forcing the matter and considering a Hilbert space generated by all those "unitary orthogonal waves", we would obtain a non-separable Hilbert space, that would complicate enormously the matter, from a functional calculus point of view, because we have no reasonable or natural spectral theory for non-separable Hilbert spaces. We are not saying, here, that we have not to use nonseparable Hilbert spaces in Quantum mechanics, but we see that they do not help in the formulation and resolution of the relativistic Schrödinger equation

Analogous problems we would risk to face if we lock down ourselves in separable Normed Space theory: it is very hard to keep, in a unique functional theory, a reasonable and convenient separable norm with a good spectral theory and the presence of the continuous family of de Broglie waves. That is a general problem, in quantum mechanics and quantum field theory: when we consider harmonic waves and related differential equations (or operatorial equations), theoretical phisicists, actually, do not use Hilbert Space theory - and we know it - we use, instead, smooth function theory, differentiable function theory, we work, essentially, with calculus techniques and distribution theory. Then, we have another general problem of Hilbert space theory in QM: even when we solve the classic Dirac free equation, the manipulations and resolutions proceed, essentially, in a differentiable theory context; indeed, the basis solutions of the free Dirac equation are bispinors constructed by harmonic waves and then, automatically, we work out of the Hilbert space theory. Moreover, when we work out of the Hilbert space theory, we work also out of the spectral theory on Hilbert spaces.

Consequently, we cannot expect to find a correct and unambiguous definition of the square root of an operator by Hilbert space techniques, if the domain of such operator should contain the de Broglie waves, because in this case we are playing outside of any separable Hilbert space. It's not a case (and do not surprise at all) that Dirac equation was solved by smooth calculus and finite algebraic methods, rather than infinite dimensional Hilbert space techniques. In order to define the square-roots of differential operators, in quantum mechanical context (where we need to manage harmonic waves, eigenstates of position operator, continuous spectra, and so on...), we need a spectral theory constructed elsewhere, not in Hilbert spaces. In some way, quantum mechanics needs to coordinate and put together two apparently incompatible aspects: the state space of a quantum system can be generated (simultaneously) by discrete and continuous bases, at the same time: the position and momentum basis ($|x\rangle$), ($|p\rangle$) are continuous, while the Hermite function basis is discrete.

Very often we read "let's solve the harmonic oscillator problem in the position basis", or "let's solve the harmonic oscillator problem in the momentum basis", which are continuous basis (in some Radon-Pettis-Schwartz sense to be correctly defined) only to see, after a while, that the harmonic oscillator is solved by the discrete Hermite function basis of L^2 (it would be better to say of the Schwartz function Space $\mathcal{S}$)

How the position basis and momentum basis (that completely stay out of L^2) can generate the same state space generated by the L^2 Hermite function basis? In what sense a continuous family of vectors can generate a functional space? The position eigenstates are not even functions, they are measures.

In what space are we moving? The state space is separable or not? What is its Hilbert dimension, $\aleph_0$ or $\aleph_1$? How a separable Hilbert (or Banach) space could contain "non-normalizable" vectors and continuous orthogonal families of non-normalizable vectors that, from a physical point of view, represent simply the certainty to observe a specific result? In tempered distribution spaces, we know that the Hermite function family is a discrete basis in a rightful algebraic-topological sense, it is a total family, it is also a basis in a generalized Hilbert sense (with respect to the tempered distribution topology)

Moreover, position and momentum basis live in $\mathcal{S}'$ and generate $\mathcal{S}'$ in the Schwartz linear algebraic sense S' as a separable topological vector space, it is wonderfully generated by discrete and continuous basis, in two different rigorous and operative meanings and, by the way, exactly the meaning used practically by quantum physicists, in a more heuristic way. In addition to that, in the distribution approach, any quantum mechanics observable is a continuous and everywhere defined operator, while in the Hilbert space approach we almost surely face discontinuous (unbounded) operators and very strange, unnatural domains, even for the most simple observables (position, momentum, ladder operators, number operator and so on and so forth).

Consequently, in Hilbert spaces, we face any kind of difficulties, even to add or multiply two straightforward operators such as a derivative operator and the position operator – which show different domains – that without, and well before, coming to ask "what the principal square root of a discontinuous, non-everywhere defined, not-properly hermitian, densely defined (or perhaps closable) operator is". What our new Schwartz linear Algebra theory clarifies compared to previous methods?

The new theory clarifies definitely where we are working, in quantum theories and quantum field theory, especially in relativistic quantum mechanics. We clarify that the state spaces of quantum objects are not Hilbert spaces (if we exclude the finite dimensional spaces that, anyway, are subspaces of the tempered distribution space $\mathcal{S}'$), but powers of tempered distribution spaces. Fortunately, tempered distribution spaces contain a lot of good inner product spaces, useful in the evaluation of probabilities and expectation values for quantum mechanics. By those inner products, we finally understand the possible way to "normalize" properly the eigenvectors of the position and momentum operators (eigenstates that should be normalizable because of their straightforward physical meaning: they simply represent certainty). First of all, we solve the problem of evolution in the tempered distribution space. Then, when necessary, we go to find the right inner product subspace of tempered distribution space, in which calculate probabilities and expectation values.

2. Background and literature review

The general intent of our study is to start from Laurent Schwartz distribution spaces, based also on the Minkowski space-time, following the spirit of Linear Algebra and Geometry, and offer precise meanings and rigorous support to many calculus methods of Quantum Mechanics.

2.1. **The role of Schwartz Distribution Theory.** The mathematical pre-context of our analysis is, so, the Laurent Schwartz distribution theory – in particular, that based on the Minkowski space-time. For distribution theory, we refer to Schwartz ([26–31]), Barros-Neto ([1]), Dieudonné ([20, 21]), Bourbaki ([3–5]), Horváth ([23]), Kesavan ([24]), Boccara ([2]), Lang ([25]), Trèves ([33]), Yosida ([34]) and Zeidler ([35])

Our approach not only provides a rigorous and efficient justification for the use of many quantum mechanics mathematical tools substantially as they usually show up in the physical practice, but – by a "correct" formulation of the calculus methods in terms of contemporary mathematics – it helps to reach a deeper comprehension of the physical structures studied in Quantum Mechanics.

2.2. **Schwartz Linear Algebra and physical Hilbert space.** Distribution Theory allows us to provide an efficient and deep calculus method for Quantum Mechanics, inspired by the renowned Dirac Calculus ([22]). Specifically, we move inside the realm of Schwartz Linear Algebra, which helps us to reach a deeper comprehension of the physical structures studied in Quantum Mechanics

In particular, in Schwartz Linear Algebra, we consider a new definition of state spaces for quantum systems. These structures are often identified with separable Hilbert spaces, leading immediately to the so-called "non-normalizable" states – which revealed fundamental in the development of the quantum mechanics. Indeed, those particular states should be normalizable, but with respect other convenient scalar products, orthogonal to the initial one. Some researchers in theoretical quantum mechanics already understand that the state space of a Quantum system should be a larger structure than Hilbert spaces, sometimes they call it "physical Hilbert space" ([32]), without introducing a clear definition. We have already shown in the past ([6–19]) that "physical Hilbert spaces" can smoothly be identified with distribution spaces on suitable Euclidean spaces or Minkovski spaces, depending on the nature of the quantum system considered. Such distribution spaces should be endowed with some algebraic-topological structures such as: continuous operations of superposition; extended Dirac products; partial inner products; etc

2.3. **Extended Dirac products and associated non-separable Hilbert structures.** In particular, the extended Dirac product allows us to introduce new usual scalar products upon some distinguished subspaces of distribution spaces, endowing those subspaces with non-separable Hilbert structures, which clarify definitely the role of the so-called non-normalizable states. For example, the singular Dirac distributions and the celebrated de Broglie waves become elements of those new non-separable Hilbert spaces and, consequently, they acquire the status of normalizable states, as it seems completely natural because of the physical usual probability interpretation of such states.

2.4. **Continuous superpositions.** The new operation of continuous superposition revealed the right tool which allows us to build – in a mathematically rigorous way – the extended Linear Algebra of Dirac, in distribution spaces, using the Schwartz natural topological-linear structures of those spaces.

More precisely, we saw that the natural algebraic-topological structure of those spaces allows to define an extension of the finite linear combination, when the sets indexing the families of vectors are continuous sets, even in the case in which the systems of coefficients show a continuous-infinity of terms different from zero. Now, it appears utterly clear how our approach to distribution theory and QM induces a renewed geometrical vision of the functional analysis developments of the two theories themselves: we realize the existence of continuous bases of distributions and corresponding coordinate systems, continuous matrices of continuous operators, tangent and cotangent spaces of infinite dimensional manifolds, modeled upon distribution spaces, endowed with suitable and comfortable continuous bases, leading to a infinite dimensional differential geometry theory much closer to the finite dimensional one

3. Semi-classic definition of Green's function

3.1. **Green's functions.** In this section we give a definition of a Green's function of a linear continuous operator that stays between the classic one (definition which very often in the applications is far from being a rigorous one) and our new definition of ${}^{\mathcal{S}}$Green's family of a linear endomorphism on the space $\mathcal{S}'_n$.

Definition 3.2 (Semi-classic definition of Green's function)**.** Let O be an open subset of the Euclidean space $\mathbb{R}^n$ and let $L\colon \mathcal{D}'(O) \to \mathcal{D}'(O)$ be a linear operator, acting on the space of distributions over the open subset O. Any function $G\colon O^2 \to \mathbb{K}$ (where $\mathbb{K}$ is the real line or the complex plane), such that the section $G(\cdot, s)$ is Lebesgue measurable and locally summable, for every point s of the open subset O, is said to be the *Green's function of the linear operator* L if it satisfies the following equality

$$L([G(\cdot, s)]) = \delta_s,$$

for any point s of the open subset O, where δ_s is the Dirac delta distribution of $\mathcal{D}'(O)$ centered at the point s and the distribution $[G(\cdot, s)]$ is the regular distribution canonically associated with the locally summable function $G(\cdot, s)$.

Remark 3.3 (Fundamental solutions at a point)**.** Let O be an open subset of the n-dimensional Euclidean space $\mathbb{R}^n$ and let $L\colon \mathcal{D}'(O) \to \mathcal{D}'(O)$ be a linear operator. When the equality $L(u) = \delta_s$ is satisfied for some point s of O and some distribution u in $\mathcal{D}'(O)$, we say that the distribution u is a *fundamental solution* of the operator L at the point s. So, in the conditions of the above definition, we can say that the function G is a Green's function for L if the (distribution associated with the) section $G(\cdot, s)$ is a fundamental solution of L at the point s, for every point s of the open subset O.

3.4. **Motivations for Green's families.** We are going to introduce the Green's families because the requirement that the "object" G, such that $L(G_s) = \delta_s$, must be a function is too strict in order to give to the concept itself reasonable manageability and effectiveness in the applications. A way to extend the classic definition can be to consider the Green's function G as a distribution on the product $O \times O$, but this choice gives some problem to the definition itself, since we could no longer consider the parameter s as a "true" parameter

Classically, the definition property of a Green's function can be exploited to solve differential equations of the form

$$L(u) = a,$$

where a is a given regular distribution on the open set O. To solve such equations we need to find a Green's family and in general, an operator L does not need to have a Green's function. On the other hand, if L has a Green's function it can have many Green's functions.

In this paper:

- we shall define the concepts of ${}^{\mathcal{S}}$Green's family and ^{E}Green's family;
- we show how to solve a linear equation (not necessarily a linear differential one) via superpositions; more precisely, a linear equation $L(.) = a$, where L is an endomorphism of $\mathcal{S}'_n$ admitting an ${}^{\mathcal{S}}$Green's family;
- we study the existence of ${}^{\mathcal{S}}$Green's families;
- we shall study the special case of linear operators commuting with the translations.

4. Green's families in $\mathcal{S}'_n$

In this section we introduce the main concept of the paper. Let us start with the classic definition of fundamental solution in our context.

Definition 4.1 (Fundamental solution for a linear operator)**.** Let L be a linear endomorphism on the space of tempered distributions $\mathcal{S}'_n$. When the equality $L(u) = \delta_s$ is verified for some point s and some distribution u, we say that the tempered distribution u is a *fundamental solution of the operator L at the point s.*

Now we can define Green's families for linear operators.

Definition 4.2 (Green's families of tempered distributions)**.** Let L be a linear endomorphism on the space of tempered distributions $\mathcal{S}'_n$. Any family $G = (G_s)_{s\in\mathbb{R}^n}$ of tempered distributions is said to be the *Green's family of* *the linear operator* L if it satisfies the equality

$$L(G_s) = \delta_s,$$

for any point s of the space $\mathbb{R}^n$, where δ_s is the Dirac distribution centered at the point s. In other terms, a family $G = (G_s)_{s\in\mathbb{R}^n}$ of tempered distributions is said to be a *Green's family of the linear operator* L if the distribution G_s is a fundamental solution for the linear operator L at the point s, for every index p of the family.

Note that our definition is not bounded to linear differential operators.

The *Green's family of an endomorphism* L *of* $\mathcal{S}'_n$ is the (ordered) family of fundamental solutions, at each point p in $\mathbb{R}^n$, of the operator L.

Theorem 4.3 (On the existence of Green's families)**.** *Let* L *be a linear endomorphism on the space of tempered distributions* $\mathcal{S}'_n$*. We have that*

- *if* L *is a surjective endomorphism on the space* $\mathcal{S}'_n$*, then* L *admits a Green's family;*
- *if* L *is a strict (topological) endomorphism on the space* $\mathcal{S}'_n$*, then* L *admits an Green family if and only if* L *is surjective.*

Proof. The first assertion is evident, since L is surjective, every element of the Dirac basis must have at least an anti-image. For the second one it is enough to prove that the existence of a Green family implies the surjectivity. Indeed, by the Dieudonné-Schwartz theorem, the image of L must be closed and, by definition of Green family, it must contain the Dirac basis; consequently (since the Dirac basis is dense in $\mathcal{S}'_n$) the image must coincide with the entire $\mathcal{S}'_n$. □

5. ${}^{\mathcal{S}}$Green's families

The ${}^{\mathcal{S}}$Linear Algebra is necessary for the next development, in particular a *family of class* $\mathcal{E}$ (or simply, a *smooth* family) is a family of tempered distributions that is of class C^∞ when viewed as a vector valued function from its index set, which is a Euclidean m-dimensional space, to the space of tempered distributions $\mathcal{S}'_n$, under consideration. Indeed, the families in Schwartz Linear Algebra are mappings from $\mathbb{R}^m$ to $\mathcal{S}'_n$, so it makes perfect sense to ask for differentiability properties, even more than in the case of a generic vector valued functions valued into general topological vector spaces, because of the good topological properties exhibited by the space $\mathcal{S}'_n$. More specifically, the explicit definition of smooth family, with respect to the weak* topology of tempered distribution spaces, is that, for every test function ϕ, the application of the family v to the test function phi should be a smooth function from $\mathbb{R}^m$ to $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). The application of v to ϕ being simply defined pointwisely, as the weak-star topology requires.

In a perfectly analogous way, a family of tempered distribution, defined upon the m-dimensional Euclidean space $\mathbb{R}^m$, is defined of *class* S (*Schwartz class*) if the application of the family v to any test function ϕ (defined on $\mathbb{R}^n$) gives back another Schwartz test function $v(\phi)$ (this time defined on $\mathbb{R}^m$). This means that $v(\phi)$ should be smooth and rapidly decreasing at infinity with all its derivatives.

These two properties allow us to define superpositions $\int av$, of v under a distribution coefficient a (compactly supported or tempered), by means of the simple formula:

$$\Big(\int av\Big)(\phi) = a(v(\phi)),$$

for every test function ϕ.

Now, we can give our main definition.

Definition 5.1 ($^{\mathcal{S}}$Green's families)**.** Let $L\colon \mathcal{S}'_n \to \mathcal{S}'_n$ be a linear operator acting on the space of tempered distributions. An $^{\mathcal{S}}$*Green's family* (respectively, an ^{E}*Green's family*) *of the linear operator* L is a Green's family of the operator L which is of Schwartz class $\mathcal{S}$ (respectively of smooth class E).

Sometimes we will call by $^{\mathcal{S}}$*Green operators* the $^{\mathcal{S}}$linear operators with an $^{\mathcal{S}}$Green family.

Theorem 5.2 (Surjectivity of the $^{\mathcal{S}}$Green operators)**.** *Let L be a continuous endomorphism on the space $\mathcal{S}'_n$ admitting an $^{\mathcal{S}}$Green family, that is an $^{\mathcal{S}}$Green operator. Then*

(i) *the operator L is surjective;*

(ii) *any $^{\mathcal{S}}$Green family of the operator L is $^{\mathcal{S}}$linearly independent.*

Proof. (i) Indeed, if L admits a Green family G then $L(G) = \delta$. Now, for every u in the space $\mathcal{S}'_n$, we have

$$L\Big(\int_{\mathbb{R}^n} uG\Big) = \int_{\mathbb{R}^n} uL(G) = \int_{\mathbb{R}^n} u\delta = u,$$

so that any u is the image of some distribution (namely the superposition $u.G$).

(ii) Let G be any $^{\mathcal{S}}$Green family of the operator L. The image of the family G is an $^{\mathcal{S}}$basis, which is necessarily $^{\mathcal{S}}$linearly independent, consequently the family G is $^{\mathcal{S}}$linearly independent too, because the image of an $^{\mathcal{S}}$linearly dependent family by an $^{\mathcal{S}}$linear operator is $^{\mathcal{S}}$linearly dependent too. $\square$

6. Application to linear equations

In this section we study the linear equation

$$E\colon L(\cdot) = a$$

associated with a linear continuous endomorphism L, on the topological vector space $(\mathcal{S}'_n)_\sigma$, and with a tempered distribution a belonging to that space. Observe that the set of all solutions of the equation E is simply the reciprocal image

$$L^-(a),$$

but this is far from explain us how to determine this set of solutions or even one particular solution of the equation E. We will show that,

- if G is an ${}^{\mathcal{S}}$Green's family of the operator L, then we can find explicitly a distribution u, that is a solution of the linear equation E. Namely, this particular solution will be the superposition

  $$\int_{\mathbb{R}^n} aG,$$

  i.e. the superposition of the family G with respect to the coefficient distribution a.

Indeed the above result follows immediately by the theorem on ${}^{\mathcal{S}}$Green operators of the preceding section and by its proof, but we state and prove again in the more explicit way

Theorem 6.1 (Solution of a linear equation)**.** *Let $L\colon \mathcal{S}'_n \to \mathcal{S}'_n$ be a continuous endomorphism on the space of tempered distributions $\mathcal{S}'_n$. Assume also that there exists an ${}^{\mathcal{S}}$Green's family $G = (G_s)_{s\in\mathbb{R}^n}$ of the operator L. Then, for every tempered distribution a in $\mathcal{S}'_n$, a (particular) solution u of the equation*

$$E\colon L(\cdot) = a$$

can be determined by the following superposition

$$u = \int_{\mathbb{R}^n} aG.$$

Moreover, the datum a is the coordinate system of the particular solution u in the ${}^{\mathcal{S}}$linearly independent family G.

Proof. A verification that the superposition $\int_{\mathbb{R}^n} aG$ is a solution of the above equation is straightforward. Indeed, applying ${}^{\mathcal{S}}$linearity of the operator L, we have

$$L\Big(\int_{\mathbb{R}^n} aG\Big) = \int_{\mathbb{R}^n} aL(G) = \int_{\mathbb{R}^n} a\delta = a,$$

as we desired. $\square$

Remark 6.2. Note that any solution u of the above equation E verifies the relation

$$L(u) = L\Big(\int_{\mathbb{R}^n} aG\Big).$$

Indeed, since G is an ${}^{\mathcal{S}}$Green's family for the operator L, then we have

$$L(G) = \delta,$$

from the very definition of Green's family. By superposition, from the preceding equality, we obtain that

$$\int_{\mathbb{R}^n} aL(G) = \int_{\mathbb{R}^n} a\delta = a.$$

Thus, if u is distribution such that $L(u) = a$, we have

$$L(u) = \int_{\mathbb{R}^n} aL(G).$$

Since the operator L is a linear continuous operator, we can use its ${}^{\mathcal{S}}$linearity, obtaining

$$L(u) = L\Big(\int_{\mathbb{R}^n} aG\Big).$$

In the condition of the above theorem the particular solution $u = a.G$ is said to be the *Green solution of the equation E with respect to the Green family G*.

Note that the ${}^{\mathcal{S}}$linear hull of the family G is exactly the set of all Green solutions obtainable from the family G

7. Interpretation of the solution

Consider the linear continuous equation $L(\cdot) = a$. To find a particular solution of E, we can proceed by steps:

(i) we expand the datum a in the Dirac basis δ, obtaining the same equation in a slightly different form

$$L(\cdot) = \int_{\mathbb{R}^n} a\delta;$$

(ii) then (if possible) we find a solution G_x of the linear equation

$$E_x \colon L(\cdot) = \delta_x,$$

for every index x in $\mathbb{R}^n$ (this is possible, for instance, when L is surjective);

(iii) choice the system $G = (G_x)_{x\in\mathbb{R}^n}$ of particular solutions in such a way that it should be an ${}^{\mathcal{S}}$family (this is possible, for instance, when the operator L is surjective and an ${}^{\mathcal{S}}$homomorphism);

(iv) at last, make the superposition

$$\int_{\mathbb{R}^n} aG$$

of the ${}^{\mathcal{S}}$system G of particular solutions by the weight system a.

Thus, we can obtain a solution u of the equation $L(\cdot) = a$ through the information provided by an ${}^{\mathcal{S}}$Green's family G for the operator L and the source term a. This process results from the ${}^{\mathcal{S}}$linearity of the operator L

If the operator L is injective the solution is unique.

Let us formalize a first existence result.

Theorem 7.1 (on the existence of an ${}^{\mathcal{S}}$Green's family)**.** *Let $L \colon \mathcal{S}'_n \to \mathcal{S}'_n$ be a surjective continuous endomorphism on the space of tempered distributions $\mathcal{S}'_n$ and let assume the operator is an ${}^{\mathcal{S}}$homomorphism. Then, there exists an ${}^{\mathcal{S}}$Green's family G of the operator L.*

Proof. Since L is surjective its image contains the Dirac family. Since L is a weak ${}^{\mathcal{S}}$homomorphism and the Dirac family is an ${}^{\mathcal{S}}$family, there is an ${}^{\mathcal{S}}$family G whose image is the Dirac family, and this implies that G is an ${}^{\mathcal{S}}$Green family of the operator L. □

8. Characterization of Green's families

The problem now lies in finding an ${}^{\mathcal{S}}$Green's family G for an operator L, that is an ${}^{\mathcal{S}}$family G satisfying the equality

$$L(G) = \delta.$$

Not every linear and continuous operator L admits an ${}^{\mathcal{S}}$Green's family, as we shall see later. More precisely, in this section, we shall obtain a characterization of those ${}^{\mathcal{S}}$linear operators admitting an ${}^{\mathcal{S}}$Green's family

The key observation to determine the existence of ${}^{\mathcal{S}}$Green's family is the following one:

- an ${}^{\mathcal{S}}$Green's family determines a continuous right inverse of the operator L and vice versa.

Theorem 8.1. *Let G be an ${}^{\mathcal{S}}$Green's family (necessarily of class $\mathcal{S}$) of a linear continuous operator $L \in \mathcal{L}(\mathcal{S}'_n)$. Then,*

(i) *the superposition operator of the family G is a continuous right inverse of L. In other terms, we have*

$$L \circ {}^t\widehat{G} = (\cdot)_{\mathcal{S}'_n},$$

or equivalently

$$L \circ \int_{\mathbb{R}^n} (\cdot, G) = (\cdot)_{\mathcal{S}'_n}.$$

(ii) *Vice versa, if $R \colon \mathcal{S}'_n \to \mathcal{S}'_n$ is a (not necessarily linear nor continuous) right inverse of a linear operator L then the family $G = R(\delta)$ is a Green's family of the operator L.*

(iii) *Moreover, let R be an ${}^{\mathcal{S}}$operator (not necessarily linear nor continuous) which is a right inverse of a linear operator L, then the image family $G = R(\delta)$ is an ${}^{\mathcal{S}}$Green's family of the operator L.*

Proof. (i) For every a in $\mathcal{S}'_n$, we have

$$L \circ \int_{\mathbb{R}^n} (\cdot, G)(a) = L\Big(\int_{\mathbb{R}^n} aG\Big) = \int_{\mathbb{R}^n} aL(G) = \int_{\mathbb{R}^n} a\delta = a,$$

as we claimed.

(ii) Vice versa, if R is a right inverse of the operator L we have

$$L\big(R(\delta_p)\big) = \delta_p,$$

for every index p of the Dirac family, and this means exactly (by the very definition of Green's family) that the family $R(\delta)$ is a Green's family of the operator L.

(iii) Moreover, if R is an ${}^{\mathcal{S}}$operator, the image $R(\delta)$ of the Dirac ${}^{\mathcal{S}}$family δ is an ${}^{\mathcal{S}}$family too. □

9. Existence of Green's families

In this section we solve the problem of existence of ${}^{\mathcal{S}}$Green's family for a linear and continuous endomorphism on the space $\mathcal{S}'_n$. We start with some elementary considerations.

Proposition 9.1. *Let L be a bijective endomorphism of the space $\mathcal{S}'_n$. Then the reciprocal image $L^{-1}(\delta)$ is the unique Green's family of L.*

Theorem 9.2. *Let L be a continuous and bijective endomorphism of $\mathcal{S}'_n$. Then the inverse image $L^{-1}(\delta)$ is (not only the unique Green's family of the operator L but also) an ${}^{\mathcal{S}}$Green's family of the operator L.*

Proof. By the Banach theorem on the inverse of a continuous operator among dual of Fréchet spaces, the inverse operator L^{-1} is linear and continuous too and so the inverse image $L^{-1}(\delta)$ is an ${}^{\mathcal{S}}$family. □

From the characterization of the preceding section, using the following theorem, we can deduce the conclusive existence theorem for ${}^{\mathcal{S}}$Green's family of an operator.

Theorem 9.3. *Let E and F be two topological vector spaces and let L be a continuous linear map from E into F. Then, there exists a continuous linear map R from F into E such that $L \circ R$ is the identity map $\mathbb{I}_F$ from F into itself if and only if the linear operator L is a surjective topological homomorphism and the kernel* $\ker(L)$ *of the operator L has a topological supplement in E. Precisely, if there exists a continuous right inverse R of the surjection L, a topological supplement of the kernel of the operator L is the image of R.*

Theorem 9.4 (Characterization of the existence of ${}^{\mathcal{S}}$Green's family)**.** *A continuous endomorphism L on the space $\mathcal{S}'_n$ has an ${}^{\mathcal{S}}$Green family if and only if it is a surjective topological homomorphism and its kernel* $\ker(L)$ *has a topological supplement in $\mathcal{S}'_n$.*

Corollary 9.5 (Characterization of the existence of ${}^{\mathcal{S}}$Green's family)**.** *Let L be a continuous endomorphism L on the space $\mathcal{S}'_n$. The following assertions are equivalent:*

- *the operator L is a linear surjective ${}^{\mathcal{S}}$homomorphism on the space $\mathcal{S}'_n$;*
- *the operator L has an ${}^{\mathcal{S}}$Green family;*
- *the operator L is a surjective topological homomorphism and its kernel* $\ker(L)$ *has a topological supplement in $\mathcal{S}'_n$;*
- *there exists a continuous linear right inverse of L.*

10. Translation invariance and convolutions

Theorem 10.1. *Let the linear operator $L\colon \mathcal{S}'_n \to \mathcal{S}'_n$ be translation invariant (in the physical sense), i.e., L commutes with any translations operator τ_p, which means that*

$$\tau_p L(u) = L(\tau_p u),$$

for any p in $\mathbb{R}^n$. Then,

(i) *if the operator L admits a fundamental solution, a Green's family there exists and can be constructed by that solution (by translations), furthermore this Green family is of class $\mathcal{E}$;*
(ii) *if the operator L admits a Green's family, a Gren's family can be generated by only one distribution (by translations);*

(iii) *if the operator L admits a fundamental solution in the space $\mathcal{O}'_C$, an $^{\mathcal{S}}$Green's family there exists and it can be constructed by that fundamental solution (by translations).*

In any of the above cases the superposition operators of the Green's families are convolution operators.

Proof. (i) Let the operator L commute with translations and assume $g \in \mathcal{S}'_n$ be a fundamental solution of the operator L at 0, that is $L(g) = \delta_0$. We put $G_p = \tau_p g$, for each p, so that

$$L(G_p) = L(\tau_p g) = \tau_p L(g) = \tau_p \delta_0 = \delta_p,$$

hence G is a Green's family of the operator L. We already have noted that this family is a smooth family, so we can consider its superposition operator

$$\int_{\mathbb{R}^n} (\cdot, G) : \mathcal{E}'_n \to \mathcal{S}'_n,$$

we have

$$\int_{\mathbb{R}^n} aG = \int_{\mathbb{R}^n} a(\tau_p g)_{p\in\mathbb{R}^n} = a * g,$$

for every distribution a in $\mathcal{E}'_n$, so that

$$\int_{\mathbb{R}^n} (\cdot, G) = (\cdot)_{\mathcal{E}'_n} * g,$$

as we claimed.

(ii) follows immediately by (i), indeed, if L has a Green family then it has a fundamental solution.

(iii) the proof is exactly as for (i), noting that, if the fundamental solution g is in the space $\mathcal{O}'_C(n)$, then the family $(\tau_p g)_{p\in\mathbb{R}^n}$ is a family of class $\mathcal{S}$. Moreover in this case the superposition operator of the Green family is defined on the entire space $\mathcal{S}'_n$. □

10.2. **Example: the Laplacian.** We know that the smooth function

$$-\frac{1}{4\pi\|\cdot\|} : \mathbb{R}^3_{\neq} \to \mathbb{R}$$

defined on the Euclidean 3-dimensional $\mathbb{R}^3$ minus the origin is a locally summable function. Moreover, for any test function ϕ in $\mathcal{S}'_n$, we have

$$\mu\left(-\frac{\nabla\phi}{4\pi\|\cdot\|}\right) = \phi(0),$$

where μ is the Lebesgue-Radon measure on $\mathbb{R}^3$. The above equality can be immediately reread as

$$\nabla\left[-\frac{1}{4\pi\|\cdot\|}\right]=\delta_0,$$

so that the regular distribution

$$\left[-\frac{1}{4\pi\|\cdot\|}\right]$$

is a fundamental solution of the Laplacian operator ∇ at 0. The Laplacian ∇ commutes with any translation, so we can write

$$\nabla\left[\frac{-1}{4\pi d(x,\cdot)}\right]=\delta_x,$$

for every point x of the Euclidean space $\mathbb{R}^3$, where (clearly) d denotes the Euclidean distance of the 3-space. Thus we have found a smooth Green's family of the Laplacian ∇ and the distribution defined by the superposition

$$\int_{\mathbb{R}^3}\rho\left(\left[\frac{-1}{4\pi d(x,\cdot)}\right]\right)_{x\in\mathbb{R}^3},$$

is a solution of the Laplace equation

$$\nabla(\cdot)=\rho,$$

for any distribution ρ with compact support

11. ${}^{\mathcal{S}}$Green's families relative to ${}^{\mathcal{S}}$bases

In this section we introduce a generalization of ${}^{\mathcal{S}}$Green's family

Definition 11.1 (Green's families with respect to a basis)**.** Let $L\colon \mathcal{S}'_n\to\mathcal{S}'_n$ be a linear operator, acting on the space of tempered distributions and let v be an ${}^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$. Any family $G=(G_s)_{s\in\mathbb{R}^n}$ of tempered distributions is said to be the *Green's family of the linear operator L with respect to the ${}^{\mathcal{S}}$basis v* if it satisfy the following equality

$$L(G_s)=v_s,$$

for any point s of the space $\mathbb{R}^n$.

Definition 11.2 (${}^{\mathcal{S}}$Green's families)**.** Let $L\colon \mathcal{S}'_n\to\mathcal{S}'_n$ be a linear operator, acting on the space of tempered distributions. An ${}^{\mathcal{S}}$Green's family of L with respect to a basis is a Green's family for L with respect to that basis of class $\mathcal{S}$.

Theorem 11.3 (Solution of a linear equation). *Let $L\colon \mathcal{S}'_n \to \mathcal{S}'_n$ be a linear continuous operator acting on the space of tempered distributions $\mathcal{S}'_n$ and let assume there exists an ${}^{\mathcal{S}}$Green's family $G = (G_s)_{s\in\mathbb{R}^n}$ for the operator L with respect to an ${}^{\mathcal{S}}$basis v. Then, for every tempered distribution a in $\mathcal{S}'_n$, a solution of the equation*

$$E\colon L(\cdot) = a$$

can be determined by the following superposition

$$u = \int_{\mathbb{R}^n} (a)_v G,$$

where $(a)_v$ is the unique coefficient distribution such that $(a)_v.v = a$.

Proof. To see that the superposition $\int_{\mathbb{R}^n} (a)_v G$ is a solution is straightforward. Indeed, applying ${}^{\mathcal{S}}$linearity of the operator L, we have

$$L\Big(\int_{\mathbb{R}^n} (a)_v G\Big) = \int_{\mathbb{R}^n} (a)_v L(G) = \int_{\mathbb{R}^n} (a)_v v = a,$$

as we desired. □

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