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On the existence of a minimal time-like surface of the Minkowski space with constant curvature of its Grassmann image

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Abstract. We investigate the curvature of the Grassmann manifold along planes tangent to the Grassmann image of two-dimensional time-like minimal surfaces in four-dimensional Minkowski space. We establish the existence of two-dimensional time-like minimal surfaces whose Grassmann images exhibit constant curvature $\overline{K} = 1$. Furthermore, we demonstrate that no time-like minimal surfaces exist with a non-degenerate Grassmann image of constant curvature $\overline{K} \neq 1$.

Анотація. В роботі знайдено значення, які може приймати кривина грассманового многовиду вздовж площин, дотичних до грассманового образу двовимірної часоподібної мінімальної поверхні в чотиривимірному просторі Мінковського. Доведено існування двовимірних часоподібних мінімальних поверхонь зі сталою кривиною $\overline{K} = 1$ їх грассманового образу. Показано, що не існує часоподібних мінімальних поверхонь із невиродженим грассмановим образом постійної кривини $\overline{K} \neq 1$.

1. Introduction

The theory of minimal surfaces arose due to the problem connected with finding the surface of minimal area among all surfaces with a common boundary. This theory developed rapidly during the 19th and 20th centuries (reviewed in [3, 4, 14]). Numerous studies are related to generalizations of the theory on multidimensional Riemannian and pseudo-Riemannian spaces ([6, 11, 12]). Of course, the name of the object and its definition have also been generalized: a submanifold F^n of the Euclidean space E^m is called *minimal* if for any sufficiently small circle U on F^n the variation n-dimensional volume on a fixed boundary is equal to zero. Equality to zero

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of the vector of mean curvature is a necessary condition for minimality of the submanifold [1].

The connection between the theory of minimal submanifolds and the theory of analytic functions is considered in [1].

Minimal surfaces can be immersed in the three-dimensional Euclidean space E^3 in the form of soap films. But such an implementation is possible only for so-called stable minimal surfaces. A large number of works, for example, [9, 13], are devoted to the stability of minimal surfaces.

The theory of minimal submanifolds is also extended to non-Euclidean spaces, in particular to spheres. In the monograph [1], minimal surfaces in the three-dimensional sphere of the four-dimensional Euclidean space were investigated, and meaningful examples were constructed.

In the article [5] it is proved that there are no two-dimensional minimal surfaces of constant non-zero Gauss curvature in the pseudo-Euclidean space ${}^1R_4$.

Studies, related to this article, are associated with Grassmann image of minimal surfaces, which is a generalization to high dimensions of the Gauss image of a two-dimensional surface in a three-dimensional Euclidean space. So, the results on the stability of minimal surfaces obtained using the Grassmann image are given, for example, in the work [8].

The review article [2] gives a number of theorems on the properties of minimal surfaces related to their Grassmann image. In particular, it is proved that when the Grassmann image of the surface $F^l \subset E^{l+p}$ has the maximum possible constant curvature $\overline{K} \equiv 2$, then this surface is a two-dimensional minimal surface, whose ellipse of normal curvature is a circle centered on the surface, and for $p = 2$, $l = 2$ this surface is a complex curve in the space E^4.

In the paper [7] it is shown that the Grassmann mapping of the minimal surface F^2 of an arbitrary Riemannian space M^4 is conformal. Moreover, if M^4 is a space of constant curvature, then the inverse image of any conformal Grassmann mapping is a minimal surface.

Note that in Riemannian spaces, surfaces with a constant nonzero mean curvature, with a parallel mean curvature vector, were also studied.

In this article, we applied the properties of the Grassmann image of the minimal time-like surface of the Minkowski space to the study of its differential geometry, in particular, the question connected with the existence of such surfaces. The existence of minimal time-like surfaces whose non-degenerate Grassmann image has constant curvature is proved. A complete description of all such surfaces and the example are given.

2. METRIC AND CURVATURE OF THE GRASSMANN IMAGE OF THE MINIMAL TIME-LIKE SURFACE

Let V^2 be a two-dimensional surface of class C^k, $k \geqslant 1$, in the space ${}^1R_4$. The surface of the space ${}^1R_4$ is called *space-like* (*time-like, isotropic*) if its tangent plane is space-like (time-like, isotropic) at every point. We will consider such two-dimensional surfaces of space ${}^1R_4$ or such areas on these surfaces, in which the type of tangent plane at each point is the same.

In the Minkowski space ${}^1R_4$, there is a coordinate system in which the metric of the space has the form

$$ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

Suppose that an equation $r = r\left(u^1, u^2\right)$ defines a two-dimensional time-like surface V^2. We will assume that the tangent vectors r_1, r_2 are orthogonal at the given point $x \in V^2$. Let us choose linearly independent unit vectors ξ_1 and ξ_2 in the normal (space-like) plane N_x so that four vectors r_1, r_2, ξ_1, ξ_2 are orthogonal in ${}^1R_4$. With the help of each basis vector of the normal plane, we determine the second quadratic forms

$$\mathrm{I\!I}^k = L^k_{ij} du^i du^j, \quad i, j, k = 1, 2,$$

where i, j are summation indices, $L^k_{ij} = \langle \xi_k, r_{ij} \rangle$, $\langle \cdot, \cdot \rangle$ is a scalar product of vectors.

The number $H^k = \frac{1}{2} g^{ij} L^k_{ij}$ is called the *mean curvature* of the surface for the direction of the normal vector ξ_k, while the vector $H = H^1 \xi_1 + H^2 \xi_2$ is the *mean curvature vector.* The time-like surfaces of Minkowski space with zero mean curvature vector will be called *minimal surfaces*, as in the Euclidean space.

Choose the coordinates $\tilde{u}_1$ and $\tilde{u}_2$ on the time-like surface V^2 in which the coefficients of the first quadratic form satisfy the following condition

$$\tilde{g}_{11} = -\tilde{g}_{22}, \qquad \tilde{g}_{12} = 0.$$

We will call them *isothermal* by an analogy with Euclidean space. Let us replace the coordinates according to the formulas $u_1 = \tilde{u_1} + \tilde{u_2}$ and $u_2 = \tilde{u_1} - \tilde{u_2}$. Then the first quadratic form is given by $ds^2 = 2 g_{12} du^1 du^2$. It follows from the minimality condition of the surface $g^{ij} L^k_{ij} = 0$ that $L^k_{12} = 0$.

To each point of the surface V^2 let us associate the plane that passes through a fixed point O of the space ${}^1R_4$ and is parallel to the normal plane at this point. This establishes the mapping of the surface V^2 into the Grassmann manifold $PG(2,4)$, which is a disjunctive union of three submanifolds: ${}^S PG(2,4)$ (a submanifold of space-like planes), ${}^T PG(2,4)$ (a submanifold of time-like planes), and ${}^{Is} PG(2,4)$ (a submanifold of isotropic planes) [15]. The image of this map is called the *Grassmann image* of the surface V^2.

The Grassmann image of a time-like (space-like) two-dimensional surface in the space ${}^1R_4$ is a two-dimensional submanifold of the manifold ${}^SPG(2,4)$ (or ${}^TPG(2,4)$ respectively). The induced metric of the Grassmann image can be sign-defined, sign-undefined, or degenerate, and thus the Grassmann image can be a two-dimensional space-like, a time-like or an isotropic surface. There are surfaces whose the Grassmann image can degenerate into a line. In this paper, we will consider only the non-degenerate Grassmann image and denote it by Γ^2.

The Grassmann image of a two-dimensional surface is an important geometric characteristic. In the paper [15] it is own that the non-degenerate Grassmann image Γ^2 of the Minkowski space surface is a two-dimensional surface $p = p\left(u^1, u^2\right)$, which belongs to the four-dimensional Grassmann submanifold $PG(2,4)$ of the six-dimensional pseudo-Euclidean space ${}^3R_6$ of index 3. Tangent vectors to Γ^2 can be written in the form

$$p_{u_i} = -L^1_{ik} g^{kl} [r_l, \xi_2] - L^2_{ik} g^{kl} [\xi_1, r_l], \qquad l \in \{1,2\}.$$

Let us find the metric of the Grassmann image of the minimal time-like surface of the space ${}^1R_4$. We will use the basis of space, which is determined by formulas

$$e_1 = \frac{r_1 - r_2}{\sqrt{2g_{12}}}, \qquad e_2 = \frac{r_1 + r_2}{\sqrt{2g_{12}}}, \qquad e_3 = \xi_1, \qquad e_4 = \xi_2,$$

where r_1, r_2 are such that $ds^2 = 2g_{12}du^1du^2$. Then

$$e_1^2 = -1, \qquad e_2^2 = 1, \qquad \langle e_1, e_2 \rangle = 0.$$

The basis of the space ${}^3R_6$ is defined by the vectors

$$[e_1, e_2], \quad [e_1, e_3], \quad [e_1, e_4], \quad [e_2, e_3], \quad [e_2, e_4], \quad [e_3, e_4],$$

which can be expressed as follows:

$$[e_1, e_2] = \frac{[r_1, r_2]}{g_{12}}, \qquad [e_1, e_3] = \frac{[r_1, \xi_1] - [r_2, \xi_1]}{\sqrt{2g_{12}}},$$

$$[e_1, e_4] = \frac{[r_1, \xi_2] - [r_2, \xi_2]}{\sqrt{2g_{12}}}, \qquad [e_2, e_3] = \frac{[r_1, \xi_1] + [r_2, \xi_1]}{\sqrt{2g_{12}}},$$

$$[e_2, e_4] = \frac{[r_1 \xi_2] + [r_2, \xi_2]}{\sqrt{2g_{12}}}, \qquad [e_3, e_4] = [\xi_1, \xi_2],$$

where

$$[r_1, \xi_1] = \sqrt{\frac{g_{12}}{2}}([e_1, e_3] + [e_2, e_3]), \quad [r_1, \xi_2] = \sqrt{\frac{g_{12}}{2}}([e_1, e_4] + [e_2, e_4]),$$

$$[r_2,\xi_1]=\sqrt{\frac{g_{12}}{2}}([e_2,e_3]-[e_1,e_3]),\quad [r_2,\xi_2]=\sqrt{\frac{g_{12}}{2}}([e_2,e_4]-[e_1,e_4]).$$

Let us find the vectors tangent to the Grassmann image:

$$\begin{aligned}p_{u^1}&=-L^1_{11}g^{12}[r_2,\xi_2]-L^1_{12}g^{21}[r_1,\xi_2]-L^2_{11}g^{12}[\xi_1,r_2]-L^2_{12}g^{21}[\xi_1,r_1]\\&=[e_1,e_3]\left(\frac{-L^2_{11}+L^2_{12}}{\sqrt{2g_{12}}}\right)+[e_1,e_4]\left(\frac{L^1_{11}-L^1_{12}}{\sqrt{2g_{12}}}\right)+\\&\quad+[e_2,e_3]\left(\frac{L^2_{11}+L^2_{12}}{\sqrt{2g_{12}}}\right)+[e_2,e_4]\left(\frac{-L^1_{11}-L^1_{12}}{\sqrt{2g_{12}}}\right),\\p_{u^2}&=-L^1_{21}g^{12}[r_2,\xi_2]-L^1_{22}g^{21}[r_1,\xi_2]-L^2_{21}g^{12}[r_2,\xi_1]-L^2_{22}g^{21}[\xi_1,r_1]\\&=[e_1,e_3]\left(\frac{-L^2_{12}+L^2_{22}}{\sqrt{2g_{12}}}\right)+[e_1,e_4]\left(\frac{L^1_{12}-L^1_{22}}{\sqrt{2g_{12}}}\right)+\\&\quad+[e_2,e_3]\left(\frac{L^2_{12}+L^2_{22}}{\sqrt{2g_{12}}}\right)+[e_2,e_4]\left(\frac{-L^1_{12}-L^1_{22}}{\sqrt{2g_{12}}}\right).\end{aligned}$$

If V^2 is a minimal surface, then $L^i_{12}=0$, whence the tangent vectors to Γ^2 will have the following coordinates:

$$\begin{aligned}X&=\left(-\frac{L^2_{11}}{\sqrt{2g_{12}}},\ \frac{L^1_{11}}{\sqrt{2g_{12}}},\ \frac{L^2_{11}}{\sqrt{2g_{12}}},\ -\frac{L^1_{11}}{\sqrt{2g_{12}}}\right),\\Y&=\left(\frac{L^2_{22}}{\sqrt{2g_{12}}},\ -\frac{L^1_{22}}{\sqrt{2g_{12}}},\ \frac{L^2_{22}}{\sqrt{2g_{12}}},\ -\frac{L^1_{22}}{\sqrt{2g_{12}}}\right).\end{aligned}\tag{2.1}$$

Therefore, the metric of the Grassmann image of the minimal time-like surface has the form

$$ds^2=\frac{2}{g_{12}}\left(L^1_{11}L^1_{22}+L^2_{11}L^2_{22}\right)du^1du^2,$$

which implies the following

Proposition 2.1. *The Grassmann image of a minimal time-like surface is also a time-like surface.*

Note that the Grassmann image will be degenerate if

$$L^1_{11}L^1_{22}+L^2_{11}L^2_{22}=0.$$

Each two-dimensional subspace of the four-dimensional tangent space of the manifolds ${}^{T}PG(2,4)$ and ${}^{S}PG(2,4)$ will be defined by the bivector

$$\sigma=\left(\sigma^{12},\sigma^{13},\sigma^{14},\sigma^{23},\sigma^{24},\sigma^{34}\right),$$

whose coordinates can be considered as the coordinates of a point of some six-dimensional space. The pseudo-Riemannian metric of Grassmann manifolds gives rise to the signature metric $(+,-,-,-,-,+)$ in this six-dimensional space. Then the expression

$$\left(\sigma^{12}\right)^2-\left(\sigma^{13}\right)^2-\left(\sigma^{14}\right)^2-\left(\sigma^{23}\right)^2-\left(\sigma^{24}\right)^2+\left(\sigma^{34}\right)^2$$

is the scalar square of the bivector σ.

Among all these two-dimensional spaces consider those that are tangent to the Grassmann image of the minimal time-like surface V^2. Then the formula (2.1) gives the coordinates of the direction vectors $X=\frac{\partial p}{\partial u^1}$ and $Y=\frac{\partial p}{\partial u^2}$ of these spaces. The coordinates of the bivector $\sigma=[X,Y]$ will have the form

$$\sigma^{12}=\frac{L_{11}^2L_{22}^1-L_{11}^1L_{22}^2}{2g_{12}},\qquad \sigma^{13}=-\frac{L_{11}^2L_{22}^2}{g_{12}},$$

$$\sigma^{14}=\sigma^{23}=\frac{L_{11}^2L_{22}^1+L_{11}^1L_{22}^2}{2g_{12}},$$

$$\sigma^{24}=-\frac{L_{11}^1L_{22}^1}{g_{12}},\qquad \sigma^{34}=\frac{-L_{11}^2L_{22}^1+L_{11}^1L_{22}^2}{2g_{12}}.$$

The sectional curvature of Grassmann submanifolds of pseudo-Euclidean spaces is calculated by the formula

$$\overline{K}(\sigma)=\frac{\overline{R}_{abcd}\sigma^{ab}\sigma^{cd}}{\left(m_{ac}m_{bd}-m_{ab}m_{cd}\right)\sigma^{ab}\sigma^{cd}},$$

where $\overline{R}_{abcd}$ is the curvature tensor of the submanifold, m_{pq} is the metric tensor of the Grassmann manifold [10]. In expanded form for the submanifold ${}^SPG(2,4)$ we have

$$\overline{K}(\sigma)=\frac{(-\sigma^{12}+\sigma^{34})^2-(\sigma^{13}+\sigma^{24})^2}{(\sigma^{12})^2-(\sigma^{13})^2-(\sigma^{14})^2-(\sigma^{23})^2-(\sigma^{24})^2+(\sigma^{34})^2}.$$

The sectional curvature of the Grassmann manifold along the planes tangent to the Grassmann image of the surface will be called the *curvature of the Grassmann image of the surface.* For a minimal time-like surface, it is calculated by the following formula

$$\overline{K}=1-\frac{(L_{11}^1L_{22}^2-L_{22}^1L_{11}^2)^2}{(L_{11}^2L_{22}^2+L_{11}^1L_{22}^1)^2}. \tag{2.2}$$

It is obvious that the sectional curvature can take values in $(-\infty,1]$.

3. Formulation and proof of results

Let us turn to the question of the existence of minimal time-like surfaces in the Minkowski space whose Grassmann image has a constant curvature. We will also find out for which possible values of this curvature such surfaces exist. A similar problem was solved in [16] for surfaces with flat normal connection of the Minkowski space.

Theorem 3.1. *A time-like minimal surface has a non-degenerate Grassmann image of constant curvature $\overline{K} = 1$ if and only if it is a hypersurface of a three-dimensional subspace of the Minkowski space. Time-like minimal surfaces with a non-degenerate Grassmann image of constant curvature $\overline{K} = 1$ exist.*

Proof. It follows from formula (2.2) for $\overline{K} = 1$ that

$$L^1_{11}L^2_{22} - L^1_{22}L^2_{11} = 0,$$

since the non-degeneracy of the Grassmann image ensures that the denominator of the fraction is non-zero. That equality means that the surface has a codimension of 1, so it is a hypersurface of a three-dimensional subspace. The converse statement is obvious.

The existence in the Minkowski space of minimal time-like surfaces whose Grassmann image has a constant curvature will follow from the existence of a solution of the Gauss-Codazzi-Ricci system supplemented by the condition of constancy of the curvature of the Grassmann image.

We will assume that the minimal time-like surface F^2 is parameterized so that $g_{11} = g_{22} = 0$, $L^1_{12} = L^2_{12} = 0$. Then the Gauss-Codazzi-Ricci system of equations takes the form

$$\begin{cases} R_{1212} = L^1_{11}L^1_{22} + L^2_{11}L^2_{22}, \\ \left(L^1_{11}\right)'_{u^2} = -\mu_{12/2}L^2_{11,}, \\ \left(L^2_{11}\right)'_{u^2} = \mu_{12/2}L^1_{11}, \\ \left(L^1_{22}\right)'_{u^1} = -\mu_{12/1}L^2_{22}, \\ \left(L^2_{22}\right)'_{u^1} = \mu_{12/1}L^1_{22}, \\ \left(\mu_{12/2}\right)'_{u^1} - \left(\mu_{12/1}\right)'_{u^2} + \left(L^1_{11}L^2_{22} - L^2_{11}L^1_{22}\right)\frac{1}{g_{12}} = 0, \end{cases}$$

where $\mu_{12/i}$ are the torsion coefficients. These equations coincide with the equations from [5] up to notation.

It follows from the first and second Codazzi equations that

$$(L^1_{11})^2 + (L^2_{11})^2 = \varphi^2\left(u^1\right),$$

while the third and fourth equations imply that

$$(L^1_{22})^2 + (L^2_{22})^2 = \psi^2\left(u^2\right),$$

where $\varphi(u^1), \psi(u^2)$ are arbitrary nonzero smooth functions.

These relations allow to find the coefficients of second quadratic forms as follows:

$$\begin{aligned} L^1_{11} &= \varphi(u^1)\cos\xi(u^1,u^2),\\ L^2_{11} &= \varphi(u^1)\sin\xi(u^1,u^2),\\ L^1_{22} &= \psi(u^2)\cos\theta(u^1,u^2),\\ L^2_{22} &= \psi(u^2)\sin\theta(u^1,u^2). \end{aligned} \tag{3.1}$$

Note that the condition $L^1_{11}L^2_{22} - L^1_{22}L^2_{11} = 0$ reduces to the following one $\varphi\psi\sin(\theta-\xi) = 0$. Therefore, in the case of a non-degenerate Grassmann image, only the case $\theta = \xi$ is possible.

Hence,

$$\begin{aligned} L^1_{11} &= \varphi(u^1)\cos\xi(u^1,u^2),\\ L^2_{11} &= \varphi(u^1)\sin\xi(u^1,u^2),\\ L^1_{22} &= \psi(u^2)\cos\xi(u^1,u^2),\\ L^2_{22} &= \psi(u^2)\sin\xi(u^1,u^2). \end{aligned}$$

It follows from the first and third Codazzi equation that

$$\partial_2\xi = \mu_{12/2}, \qquad \partial_1\xi = \mu_{12/1}.$$

One easily verifies that then the Ricci equation holds.

It remains to consider the Gauss equation which is written as follows:

$$\partial_{12}g_{12} - \frac{1}{g_{12}}\cdot\partial_1 g_{12}\cdot\partial_2 g_{12} = \varphi(u^1)\psi(u^2)$$

Note that this euqation is hyperbolic with respect to the function $g_{12}(u^1,u^2)$, and any partial solution of this equation will satisfy us.

Since $g_{12} = -2\,\mathrm{sh}^2\left(u^1+u^2\right)$, the left side of the Gauss equation equals 4. Hence, if the functions $\varphi(u^1)$ and $\psi(u^2)$ are chosen to be equal to, for example, 1 and 4, then one of the solutions of the Gauss equation will be the function $g_{12} = -2\,\mathrm{sh}^2\left(u^1+u^2\right)$. Theorem is completed. □

Example 3.2. Consider the following hypersurface

$$r(u^1,u^2) = (\mathrm{sh}\,u^1\,\mathrm{ch}\,u^2, \mathrm{sh}\,u^1\,\mathrm{sh}\,u^2, u^1, 0)$$

in the space ${}^1R_4$. Its metric has the form $ds^2 = -\operatorname{sh}^2 u^1 (du^1)^2 + \operatorname{sh}^2 u^1 (du^2)^2$, and the unit normals

$$\xi_1 = \left(\frac{\operatorname{ch} u^2}{\operatorname{sh} u^1}, \frac{\operatorname{sh} u^2}{\operatorname{sh} u^1}, \frac{\operatorname{ch} u^1}{\operatorname{sh} u^1}, 0\right), \qquad \xi_2 = (0,0,0,1)$$

correspond to the second quadratic forms

$$\mathrm{I\!I}^1 = -(du^1)^2 - (du^2)^2, \qquad \mathrm{I\!I}^2 \equiv 0.$$

Hence, Gauss curvature, mean curvature of the surface and curvature of the Grassmann image are equal, respectively

$$K = -\frac{1}{\operatorname{sh}^4 u^1}, \qquad H = 0, \qquad \overline{K} = 1.$$

As mentioned above, it follows from the formula (2.2) that the curvature of the non-degenerate Grassmann image of the minimal time-like surface does not exceed 1. Let us clarify the question of the existence of a minimal time-like surface with a non-degenerate Grassmann image of constant curvature $\overline{K} \neq 1$.

The conditions for the constancy of the curvature of the Grassmann image can be written as follows:

$$\begin{cases} \partial_1 \overline{K} = 0, \\ \partial_2 \overline{K} = 0. \end{cases}$$

They are equivalent either to the equation $L^1_{11}L^2_{22} - L^1_{22}L^2_{11} = 0$ or to the system of equations

$$\begin{cases} ((L^1_{22})^2 + (L^2_{22})^2)(L^2_{11}\partial_1 L^1_{11} - L^1_{11}\partial_1 L^2_{11}) + \\ \quad +((L^1_{11})^2 + (L^2_{11})^2)(L^1_{22}\partial_1 L^2_{22} - L^2_{22}\partial_1 L^1_{22}) = 0, \\ ((L^1_{22})^2 + (L^2_{22})^2)(L^2_{11}\partial_2 L^1_{11} - L^1_{11}\partial_2 L^2_{11}) + \\ \quad +((L^1_{11})^2 + (L^2_{11})^2)(L^1_{22}\partial_2 L^2_{22} - L^2_{22}\partial_2 L^1_{22}) = 0. \end{cases} \tag{3.2}$$

Since $\overline{K} \neq 1$, we have that $L^1_{11}L^2_{22} - L^1_{22}L^2_{11} \neq 0$. Therefore, we need to consider the system (3.2). As the Grassmann image of the surface is non-degenerate, it is sufficient to consider only the following two cases.

1) Only one of the coefficients of the second quadratic forms is zero, for example $L^1_{11} = 0$, while $L^1_{22}L^2_{11}L^2_{22} \neq 0$. Then the system (3.2) will be written as follows:

$$\begin{cases} (L^2_{11})^2(L^1_{22}\partial_1 L^2_{22} - L^2_{22}\partial_1 L^1_{22}) = 0, \\ (L^2_{11})^2(L^1_{22}\partial_2 L^2_{22} - L^2_{22}\partial_2 L^1_{22}) = 0. \end{cases}$$

Hence $\frac{L^1_{22}}{L^2_{22}} = c_0 = \mathit{const} \neq 0$. Under this condition, it follows from the first and second Codazzi equations that $\mu_{12/2} = 0$ and $L^2_{11} = L^2_{11}(u^1)$. Also the third and fourth equations imply that either $c_0^2 = -1$ (which is a contradiction) or $\mu_{12/1} = 0$ (this case does not satisfy the Ricci equation).

2) All coefficients of the second quadratic forms are non-zero:

$$L^1_{11}L^1_{22}L^2_{11}L^2_{22} \neq 0.$$

Then the system (3.2) can be written in the form

$$\begin{cases} \partial_1\left(\operatorname{arctg}\frac{L^1_{11}}{L^2_{11}}\right) = \partial_1\left(\operatorname{arctg}\frac{L^1_{22}}{L^2_{22}}\right), \\ \partial_2\left(\operatorname{arctg}\frac{L^1_{11}}{L^2_{11}}\right) = \partial_2\left(\operatorname{arctg}\frac{L^1_{22}}{L^2_{22}}\right). \end{cases}$$

Therefore,

$$\begin{cases} \operatorname{arctg}\left(\frac{L^1_{11}}{L^2_{11}}\right) = \operatorname{arctg}\left(\frac{L^1_{22}}{L^2_{22}}\right) + \operatorname{arctg}\alpha(u^2), \\ \operatorname{arctg}\left(\frac{L^1_{11}}{L^2_{11}}\right) = \operatorname{arctg}\left(\frac{L^1_{22}}{L^2_{22}}\right) + \operatorname{arctg}\beta(u^1). \end{cases}$$

This implies that $\alpha(u^2) = \beta(u^1) = \mathit{const} \neq 0$. Denote that constant by the symbol α_0. Then

$$\frac{L^1_{11}}{L^2_{11}} = \frac{\frac{L^1_{22}}{L^2_{22}} + \alpha_0}{1 - \frac{L^1_{22}\alpha_0}{L^2_{22}}},$$

whence

$$L^1_{11}L^2_{22} - L^1_{22}L^2_{11} = \alpha_0(L^1_{11}L^1_{22} + L^2_{11}L^2_{22}), \qquad \alpha_0 = \mathit{const} \neq 0. \tag{3.3}$$

From the Codazzi equations, we can find again the coefficients of quadratic forms in the form (3.1). Then the relation (3.3) reduces to the following one:

$$\sin(\theta - \xi) = \alpha_0 \cos(\theta - \xi).$$

This implies that the difference $\theta - \xi$ is a constant. Let us denote that constant by c_0, $c_0 \neq 0$. Then

$$\begin{aligned} L^1_{11} &= \varphi(u^1)\cos\xi, \\ L^2_{11} &= \varphi(u^1)\sin\xi, \\ L^1_{22} &= \psi(u^2)\cos(\xi + c_0), \\ L^2_{22} &= \psi(u^2)\sin(\xi + c_0), \end{aligned} \tag{3.4}$$

where $\xi = \xi(u^1, u^2)$.

Substituting the obtained expressions for the coefficients of the second quadratic forms into the Ricci equation we get

$$\sin c_0 \varphi\left(u^1\right)\psi\left(u^2\right)\frac{1}{g_{12}}=0,$$

which is contradictory.

Thus we proved the following theorem.

Theorem 3.3. *In the Minkowski space there are no time-like minimal surfaces with a non-degenerate Grassmann image of constant curvature distinct from* 1*.*

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