Proceedings of the
International Geometry Center
Vol. 17, no. 2 (2024) pp. 158–170

On exponential asymptotics of one class of homeomorphisms at a point of the complex plane

Mariia Stefanchuk

Abstract. The presented paper investigates the exponential asymptotics at a point of the complex plane of the ring Q-homeomorphisms with respect to p-modulus for $p > 2$. Examples showing sharpness of the obtained results have been constructed.

Анотація. Досліджено експоненціальну асимптотику у точці комплексної площини кільцевих Q-гомеоморфізмів відносно p-модуля при $p > 2$. Побудовано приклади, що показують точність отриманих результатів.

1. Introduction

Let Γ be a family of curves γ in the complex plane $\mathbb{C}$. The Borel function $\rho\colon \mathbb{C} \to [0, \infty]$ is called *admissible for* Γ, written $\rho \in \operatorname{adm} \Gamma$, if

$$\int\limits_{\gamma} \rho(z)|dz| \geqslant 1$$

for every (locally rectified) curve $\gamma \in \Gamma$.

Let $p \in (1, \infty)$. Then by the following quantity:

$$\mathrm{M}_p(\Gamma) = \inf_{\rho \in \operatorname{adm} \Gamma} \int\limits_{\mathbb{C}} \rho^p(z)\, dxdy,$$

is called the *p-modulus* of the family Γ.

For arbitrary sets E, F, and G in $\mathbb{C}$, we denote by $\Delta(E, F; G)$ the family of all curves $\gamma\colon [a, b] \to \mathbb{C}$ connecting E and F in G, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$.

This work was supported by a grant from the Simons Foundation (1290607, M.V.S).
2020 Mathematics Subject Classification: 30C65, 30C75
Keywords: p-modulus of a family of curves, ring Q-homeomorphism with respect to p-modulus, condenser, p-capacity of a condenser
DOI: http://dx.doi.org/10.15673/pigc.v17i2.2870

Let $z_0 \in \mathbb{C}$. Then we put

$$\mathbb{A}(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\},$$
$$S_i = S(z_0, r_i) = \{z \in \mathbb{C} : |z - z_0| = r_i\}, \quad i = 1, 2.$$

The following statement provides the formula for calculating p-modulus of a family of curves which connect circles in every ring of the complex plane $\mathbb{C}$ as $p > 2$, see, e.g., [4, Eq. (2)].

Proposition 1.1. *Let $z_0 \in \mathbb{C}$ and $\Delta(S_1, S_2; \mathbb{A})$ be the family of all curves connecting the circles $S_1 = S(z_0, r_1)$ and $S_2 = S(z_0, r_2)$ in the ring $\mathbb{A} = \mathbb{A}(z_0, r_1, r_2)$. Then for $p > 2$ the following itentity holds:*

$$\mathrm{M}_p(\Delta(S_1, S_2; \mathbb{A})) = 2\pi \left(\frac{p-2}{p-1}\right)^{p-1} \left(r_2^{\frac{p-2}{p-1}} - r_1^{\frac{p-2}{p-1}}\right)^{1-p}. \tag{1.1}$$

Let D be a domain in the complex plane $\mathbb{C}$ and $Q\colon D \to [0, \infty]$ be a Lebesgue measurable function. We say that the homeomorphism $f\colon D \to \mathbb{C}$ is a *ring Q-homeomorphism with respect to p-modulus* at the point $z_0 \in D$ if the following inequality

$$\mathrm{M}_p(\Delta(fS_1, fS_2; f\mathbb{A})) \leqslant \int\limits_{\mathbb{A}} Q(z)\, \eta^p(|z - z_0|)\, dxdy$$

holds for any ring $\mathbb{A} = \mathbb{A}(z_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$, $d_0 = \operatorname{dist}(z_0, \partial D)$, and every measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) dr = 1.$$

We will assume that

$$q_{z_0}(r) = \frac{1}{2\pi r} \int\limits_{S(z_0, r)} Q(z)|dz|$$

is the mean integral value of the function $Q(z)$ over the circle $S(z_0, r)$.

Then following statement is a criterion for a homeomorphism to belong to the class of ring Q-homeomorphisms with respect to p-modulus as $p > 1$ on the complex plane $\mathbb{C}$, see [30, Theorem 2.3] for $n = 2$.

Proposition 1.2. *Let D be a domain in the complex plane $\mathbb{C}$, $z_0 \in D$ a point, and $Q\colon D \to [0, \infty]$ a Lebesgue measurable function such that the mean integral value $q_{z_0}(r)$ is finite for a.e. (almost every) $r \in (0, d_0)$, where $d_0 = \operatorname{dist}(z_0, \partial D)$. A homeomorphism $f\colon D \to \mathbb{C}$ is a ring Q-homeomorphism with respect to p-modulus as $p > 1$ at the point z_0 if and* *only if*

$$\mathrm{M}_p(\Delta(fS_1, fS_2; f\mathbb{A})) \leqslant \frac{2\pi}{\left(\int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{1}{p-1}} q_{z_0}^{\frac{1}{p-1}}(r)}\right)^{p-1}}$$

for any r_1, r_2 such that $0 < r_1 < r_2 < d_0$.

Ring Q-homeomorphisms in the space $\mathbb{R}^n$ for $p = n$ were studied in [21–23,27,28], for $1 < p < n$ in [5–7,9,29–32], and for $p > n$ in [17,18,33–35]. For $p > 2$ ring Q-homeomorphisms on the complex plane considered in [25,47]. More general mapping classes were studied in [1–3,10–13,36,43,44].

The present paper investigates the exponential asymptotics at a point of the complex plane of ring Q-homeomorphisms with respect to p-modulus, $p > 2$. As a consequence, an analogue of the famous Ikoma–Schwarz Lemma, [15,16], has been obtained, see Theorem 3.1 and Corollaries 3.2, 3.3 and 3.5 below. The results of this work can be used to study extremal and asymptotic properties of solutions of nonlinear Beltrami equations, see [8,14,19,26,37–42,46].

Below is some information about condensers and p-capacities of condensers. A pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{C}$ is an open set and C is an nonempty compact set contained in A, is called a *condenser*, see [20]. A condenser $\mathcal{E}$ is called a *ring condenser* if $\mathfrak{R} = A \backslash C$ is a ring domain, i.e. $\mathfrak{R}$ is a domain whose complement $\overline{\mathbb{C}} \backslash \mathfrak{R}$ consist of exactly two components. A condenser $\mathcal{E} = (A, C)$ is said to *lie in a domain* D if $A \subset D$. It is obvious that if $f\colon D \to \mathbb{C}$ is a homeomorphism and $\mathcal{E} = (A, C)$ is a condenser in D, then $(f(A), f(C))$ is also a condenser in $f(D)$. We will also put $f(\mathcal{E}) = (f(A), f(C))$.

Let $\mathcal{E} = (A, C)$ be a condenser. We denote by $\mathcal{C}_0(A)$ a set of continuous functions $u\colon A \to \mathbb{R}$ with compact support. Let also $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ be a family of non-negative functions $u\colon A \to \mathbb{R}$ such that

1) $u \in \mathcal{C}_0(A)$,
2) $u(x, y) \geqslant 1$ for $(x, y) \in C$,
3) u belongs to the class ACL.

As $p \geqslant 1$ the quantity

$$\operatorname{cap}_p \mathcal{E} = \operatorname{cap}_p(A, C) = \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int\limits_A |\nabla u|^p \, dxdy\,,$$

where

$$|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$

is called *p-capacity of a condenser $\mathcal{E}$.*

It is known that for $p > 1$

$$\operatorname{cap}_p \mathcal{E} = \mathrm{M}_p(\Delta(\partial A, \partial C; A \backslash C)), \tag{1.2}$$

see [45, Theorem 1].

In what follows, $|E|$ denotes a Lebesgue measure of a set E in $\mathbb{C}$.

The following formula gives the lower estimate of p-capacity of a condenser on the complex plane $\mathbb{C}$ as $p > 2$ (see, e.g. [24, Inequality (8.7)]):

$$\operatorname{cap}_p \mathcal{E} = \operatorname{cap}_p(A, C) \geqslant 2\pi^{\frac{p}{2}} \left(\frac{p-2}{p-1}\right)^{p-1} \left(|A|^{\frac{p-2}{2(p-1)}} - |C|^{\frac{p-2}{2(p-1)}}\right)^{1-p}. \tag{1.3}$$

2. Estimate of a disk image area

In this section, the lower estimate of the exponential type of a disk image area for ring Q-homeomorphisms with respect to p-modulus as $p > 2$ is obtained.

We put

$$B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}, \qquad B_r = \{z \in \mathbb{C} : |z| < r\},$$
$$\overline{B_r} = \{z \in \mathbb{C} : |z| \leqslant r\}, \qquad \mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}.$$

Lemma 2.1. *Let D be a domain in the complex plane $\mathbb{C}$ and $z_0 \in D$ be a point. Let also $f\colon D \to \mathbb{C}$ be a ring Q-homeomorphism with respect to p-modulus at the point z_0, $p > 2$, and for some numbers $r_0 \in (0, d_0)$, $d_0 = \operatorname{dist}(z_0, \partial D)$, $\kappa = \kappa(z_0) > 0$, and $\alpha > 0$ the following condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}} \tag{2.1}$$

is fulfilled for a.e. $t \in (0, r_0)$. Then for any r_1, r_2 such that $0 < r_1 < r_2 < r_0$ the following estimate holds

$$|f(B(z_0, r_2))|^{\frac{p-2}{2(p-1)}} \geqslant$$
$$\geqslant |f(\overline{B(z_0, r_1)})|^{\frac{p-2}{2(p-1)}} + \lambda_p\, \kappa^{-\frac{1}{p-1}} \alpha^{-1} \left(e^{-\frac{\alpha}{(p-1)r_2}} - e^{-\frac{\alpha}{(p-1)r_1}}\right),$$

where $\lambda_p = \pi^{\frac{p-2}{2(p-1)}}(p-2)$.

Proof. Consider the condenser $\mathcal{E} = (A, C)$ in D, where

$$A = \{z \in \mathbb{C} : |z - z_0| < r_2\}, \qquad C = \{z \in \mathbb{C} : |z - z_0| \leqslant r_1\},$$

$0 < r_1 < r_2 < r_0$, $r_0 \in (0, d_0)$, and $d_0 = \operatorname{dist}(z_0, \partial D)$. Since the mapping f is a homeomorphism, the pair $f(\mathcal{E}) = (f(A), f(C))$ is a ring condenser in $\mathbb{C}$ and from equality (1.2) the following relationship holds

$$\operatorname{cap}_p\big(f(A), f(C)\big) = \mathrm{M}_p\left(\Delta\left(\partial f(A), \partial f(C); f(A \backslash C)\right)\right).$$

Whence and from estimate (1.3) we have the following inequality:

$$\operatorname{cap}_p\big(f(A), f(C)\big) \geqslant$$
$$\geqslant 2\,\pi^{\frac{p}{2}} \left(\frac{p-2}{p-1}\right)^{p-1} \left(|f(A)|^{\frac{p-2}{2(p-1)}} - |f(C)|^{\frac{p-2}{2(p-1)}}\right)^{1-p}. \tag{2.2}$$

On the other hand, Proposition 1.2 implies the estimate

$$\operatorname{cap}_p\left(f(A), f(C)\right) \leqslant \frac{2\pi}{\left(\int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{1}{p-1}} q_{z_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}, \tag{2.3}$$

where $q_{z_0}(t) = \frac{1}{2\pi t} \int\limits_{S(z_0,t)} Q(z)|dz|$. Combining inequalities (2.2) and (2.3), we obtain

$$\pi^{\frac{p-2}{2}} \left(\frac{p-2}{p-1}\right)^{p-1} \left(|f(A)|^{\frac{p-2}{2(p-1)}} - |f(C)|^{\frac{p-2}{2(p-1)}}\right)^{1-p} \leqslant \left(\int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{1}{p-1}} q_{z_0}^{\frac{1}{p-1}}(t)}\right)^{1-p}.$$

Then, using condition (2.1), we get the following inequality:

$$\pi^{\frac{p-2}{2}} \left(\frac{p-2}{p-1}\right)^{p-1} \left(|f(A)|^{\frac{p-2}{2(p-1)}} - |f(C)|^{\frac{p-2}{2(p-1)}}\right)^{1-p} \leqslant$$
$$\leqslant \kappa\,\alpha^{p-1}(p-1)^{-(p-1)} \left(e^{-\frac{\alpha}{(p-1)r_2}} - e^{-\frac{\alpha}{(p-1)r_1}}\right)^{1-p}.$$

After elementary transformations in the last estimate, we arrive at the final result

$$|f(A)|^{\frac{p-2}{2(p-1)}} \geqslant$$
$$\geqslant |f(C)|^{\frac{p-2}{2(p-1)}} + \pi^{\frac{p-2}{2(p-1)}}(p-2)\kappa^{-\frac{1}{p-1}}\alpha^{-1} \left(e^{-\frac{\alpha}{(p-1)r_2}} - e^{-\frac{\alpha}{(p-1)r_1}}\right),$$

where $A = B(z_0, r_2)$, $C = \overline{B(z_0, r_1)}$. Lemma 2.1 is proved. □

Putting in Lemma 2.1 $D = \mathbb{B}$ and $z_0 = 0$, we arrive to the following statement.

Corollary 2.2. *Let $f\colon \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to p-modulus at the point $z_0 = 0$, and $p > 2$. Suppose that for some numbers $r_0 \in (0,1)$, $\kappa > 0$ and $\alpha > 0$ the condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

is satisfied for a.e. $t \in (0, r_0)$. *Then for any* r_1, r_2 *such that* $0 < r_1 < r_2 < r_0$ *the following estimate holds:*

$$|f(B_{r_2})|^{\frac{p-2}{2(p-1)}} \geqslant |f(\overline{B_{r_1}})|^{\frac{p-2}{2(p-1)}} + \lambda_p\, \kappa^{-\frac{1}{p-1}} \alpha^{-1} \left(e^{-\frac{\alpha}{(p-1)r_2}} - e^{-\frac{\alpha}{(p-1)r_1}}\right),$$

where $\lambda_p = \pi^{\frac{p-2}{2(p-1)}}(p-2)$.

Theorem 2.3. *Let* D *be a domain in the complex plane* $\mathbb{C}$ *and* $z_0 \in D$. *Assume that* $f\colon D \to \mathbb{C}$ *is a ring* Q*-homeomorphism with respect to* p*-modulus at the point* z_0, $p > 2$, *and for some numbers* $r_0 \in (0, d_0)$, $d_0 = \operatorname{dist}(z_0, \partial D)$, $\kappa = \kappa(z_0) > 0$ *and* $\alpha > 0$ *the condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

is fulfilled for a.e. $t \in (0, r_0)$. *Then for any* $r \in (0, r_0)$ *the following estimate holds:*

$$|f(B(z_0, r))| \geqslant \pi (p-2)^{\frac{2(p-1)}{p-2}} \kappa^{-\frac{2}{p-2}} \alpha^{-\frac{2(p-1)}{p-2}} e^{-\frac{2\alpha}{(p-2)r}}. \tag{2.4}$$

Proof. Putting in Lemma 2.1 $r_1 = \varepsilon$ and $r_2 = r$, $0 < \varepsilon < r < r_0$, $r_0 \in (0, d_0)$, $d_0 = \operatorname{dist}(z_0, \partial D)$, we arrive at the inequality

$$\begin{aligned} |f(B(z_0, r))|^{\frac{p-2}{2(p-1)}} &\geqslant |f(\overline{B(z_0, \varepsilon)})|^{\frac{p-2}{2(p-1)}} + \lambda_p \kappa^{-\frac{1}{p-1}} \alpha^{-1} \left(e^{-\frac{\alpha}{(p-1)r}} - e^{-\frac{\alpha}{(p-1)\varepsilon}}\right) \\ &\geqslant \lambda_p\, \kappa^{-\frac{1}{p-1}} \alpha^{-1} \left(e^{-\frac{\alpha}{(p-1)r}} - e^{-\frac{\alpha}{(p-1)\varepsilon}}\right), \end{aligned}$$

where $\lambda_p = \pi^{\frac{p-2}{2(p-1)}}(p-2)$.

Passing to the limit as $\varepsilon \to 0$ and performing elementary transformations in the last inequality, we obtain the desired estimate (2.4). Theorem 2.3 is proved. □

Corollary 2.4. *Let* $f\colon \mathbb{B} \to \mathbb{C}$ *be a ring* Q*-homeomorphism with respect to* p*-modulus at the point* $z_0 = 0$, $p > 2$, *and for some numbers* $r_0 \in (0, 1)$, $\kappa > 0$ *and* $\alpha > 0$ *the following condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

is fulfilled for a.e. $t \in (0, r_0)$. *Then for any* $r \in (0, r_0)$ *the following relation holds:*

$$|f(B_r)| \geqslant \pi (p-2)^{\frac{2(p-1)}{p-2}} \kappa^{-\frac{2}{p-2}} \alpha^{-\frac{2(p-1)}{p-2}} e^{-\frac{2\alpha}{(p-2)r}}. \tag{2.5}$$

Remark 2.5. The estimate (2.5) is exact and is achieved on the following mapping $f_1\colon \mathbb{B} \to \mathbb{C}$,

$$f_1(z) = \begin{cases} k_0\, e^{-\frac{\alpha}{(p-2)|z|}} \frac{z}{|z|}, & 0 < |z| < 1, \\ 0, & z = 0, \end{cases} \tag{2.6}$$

where $k_0 = (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}$, $\kappa > 0$, $\alpha > 0$. Evidently, f_1 is a homeomorphism which maps the disk B_r to the disk $B_{\tilde{r}}$, where $\tilde{r} = k_0\, e^{-\frac{\alpha}{(p-2)r}}$, $r \in (0,1)$.

It is easy to see that for any $r \in (0,1)$

$$|f(B_r)| = |B_{\tilde{r}}| = \pi(p-2)^{\frac{2(p-1)}{p-2}} \kappa^{-\frac{2}{p-2}} \alpha^{-\frac{2(p-1)}{p-2}} e^{-\frac{2\alpha}{(p-2)r}}.$$

Let us show that the mapping f_1 is a ring Q-homeomorphism with respect to p-modulus at the point $z_0 = 0$ as $p > 2$ with the function

$$Q(z) = \kappa |z|^{2p-3} e^{\frac{\alpha}{|z|}}.$$

Indeed, it is obvious that $q_{z_0}(t) = \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$. Consider the ring $\mathbb{A}(0, r_1, r_2)$, $0 < r_1 < r_2 < 1$. Then the homeomorphism f_1 transforms the ring $\mathbb{A}(0, r_1, r_2)$ into the ring $\mathbb{A}(0, \tilde{r}_1, \tilde{r}_2)$, where $\tilde{r}_i = k_0\, e^{-\frac{\alpha}{(p-2)r_i}}$, $i = 1, 2$.

Denote by Γ the family of all curves connecting the circles $S(0, r_1)$ and $S(0, r_2)$ in the ring $\mathbb{A}(0, r_1, r_2)$. By Proposition 1.1, p-modulus of the family of curves $f_1(\Gamma)$ is calculated by the formula

$$\mathrm{M}_p\big(f_1(\Gamma)\big) = 2\pi (p-1)^{-(p-1)} \kappa\, \alpha^{p-1} \left(e^{-\frac{\alpha}{(p-1)r_2}} - e^{-\frac{\alpha}{(p-1)r_1}} \right)^{1-p}$$

The last equality can be rewritten in the following form

$$\mathrm{M}_p\big(f_1(\Gamma)\big) = \frac{2\pi}{\left(\int\limits_{r_1}^{r_2} \frac{dt}{t^{\frac{1}{p-1}} q_{z_0}^{\frac{1}{p-1}}(t)} \right)^{p-1}},$$

where $q_{z_0}(t)$ is the mean integral value of the function $Q(z)$ over the circle

$$S(0,t) = \{z \in \mathbb{C} \colon |z| = t\}$$

and is determined by the formula $q_{z_0}(t) = \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$.

Therefore, according to Proposition 1.2, the homeomorphism f_1 is a ring Q-homeomorphism with respect to p-modulus at the point $z_0 = 0$ with the function $Q(z) = \kappa |z|^{2p-3} e^{\frac{\alpha}{|z|}}$.

3. Exponential asymptotics at a point

In this section, the exponential asymptotics in terms of upper limits of ring Q-homeomorphisms with respect to p-modulus at the point z_0 as $p > 2$ is investigated. These results are analogues of the famous Ikoma–Schwarz Lemma, [15, 16].

Theorem 3.1. *Let D be a domain in the complex plane $\mathbb{C}$ and $z_0 \in D$. Let also $f \colon D \to \mathbb{C}$ is a ring Q-homeomorphism with respect to p-modulus at* *the point* z_0*,* $p > 2$*, and for some numbers* $r_0 \in (0, d_0)$*,* $d_0 = \mathrm{dist}(z_0, \partial D)$*,* $\kappa = \kappa(z_0) > 0$ *and* $\alpha > 0$ *the following condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

is fulfilled for a.e. $t \in (0, r_0)$*. Then the following asymptotic estimate holds*

$$\limsup_{z\to z_0} |f(z) - f(z_0)|\, e^{\frac{\alpha}{(p-2)|z-z_0|}} \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}.$$

Proof. By Theorem 2.3 we have the estimate

$$|f(B(z_0, r))| \geqslant \pi (p-2)^{\frac{2(p-1)}{p-2}} \kappa^{-\frac{2}{p-2}} \alpha^{-\frac{2(p-1)}{p-2}} e^{-\frac{2\alpha}{(p-2)r}} \tag{3.1}$$

for any $r \in (0, r_0)$.

Put $L_f(z_0, r) = \max\limits_{|z-z_0|=r} |f(z) - f(z_0)|$. Since the mapping f is a homeomorphism, we get the following inequality:

$$\pi L_f^2(z_0, r) \geqslant |f(B(z_0, r))|.$$

From the last estimate and (3.1) it follows that

$$L_f(z_0, r) \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}} e^{-\frac{\alpha}{(p-2)r}}$$

for any $r \in (0, r_0)$.

Multiplying the last inequality by $e^{\frac{\alpha}{(p-2)r}}$ and passing to the upper limit as $r \to 0$, we obtain the following estimate:

$$\limsup_{z\to z_0} |f(z) - f(z_0)|\, e^{\frac{\alpha}{(p-2)|z-z_0|}} =$$

$$= \limsup_{r\to 0} L_f(z_0, r)\, e^{\frac{\alpha}{(p-2)r}} \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}.$$

Theorem 3.1 is completed. □

Corollary 3.2. *Let* D *be a domain in the complex plane* $\mathbb{C}$*,* $z_0 \in D$*,* $f\colon D \to \mathbb{C}$ *be a ring* Q*-homeomorphism with respect to* p*-modulus at the point* z_0*,* $p > 2$*, and for some numbers* $r_0 \in (0, d_0)$*,* $d_0 = \mathrm{dist}(z_0, \partial D)$*,* $\kappa = \kappa(z_0) > 0$ *and* $\alpha > 0$

$$Q(z) \leqslant \kappa\, |z - z_0|^{2p-3} e^{\frac{\alpha}{|z-z_0|}} \tag{3.2}$$

for a.e. $z \in B(z_0, r_0)$*. Then*

$$\limsup_{z\to z_0} |f(z) - f(z_0)|\, e^{\frac{\alpha}{(p-2)|z-z_0|}} \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}. \tag{3.3}$$

Proof. Indeed, using condition (3.2), we have

$$q_{z_0}(t) = \frac{1}{2\pi t} \int\limits_{S(z_0,t)} Q(z)|dz| \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

for a.e. $t \in (0, r_0)$. Applying further Theorem 3.1 we get (3.3). □

Corollary 3.3. *Let $f\colon \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to p-modulus at the point $z_0 = 0$ as $p > 2$ satisfying $f(0) = 0$. Suppose that for some numbers $r_0 \in (0,1)$, $\kappa > 0$ and $\alpha > 0$ the following condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

is fulfilled for a.e. $t \in (0, r_0)$. Then

$$\limsup_{z \to 0} |f(z)| e^{\frac{\alpha}{(p-2)|z|}} \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}. \tag{3.4}$$

Remark 3.4. The estimate (3.4) is exact and is achieved on the ring Q-homeomorphism $f_1\colon \mathbb{B} \to \mathbb{C}$ given by formula (2.6) with respect to p-modulus at the point $z_0 = 0$ as $p > 2$ with the function $Q(z) = \kappa |z|^{2p-3} e^{\frac{\alpha}{|z|}}$. Namely, it is easy to see that $q_{z_0}(t) = \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$, $t \in (0,1)$, and

$$\lim_{z \to 0} |f_1(z)| e^{\frac{\alpha}{(p-2)|z|}} = (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}.$$

Corollary 3.5. *Let $f\colon \mathbb{B} \to \mathbb{C}$ be a ring Q-homeomorphism with respect to p-modulus at the point $z_0 = 0$ as $p > 2$ satisfying $f(0) = 0$. Suppose that for some numbers $r_0 \in (0,1)$, $\kappa > 0$ and $\alpha > 0$*

$$Q(z) \leqslant \kappa\, |z|^{2p-3} e^{\frac{\alpha}{|z|}}$$

for a.e. $z \in B_{r_0}$. Then

$$\limsup_{z \to 0} |f(z)| e^{\frac{\alpha}{(p-2)|z|}} \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}}.$$

Proposition 3.6. *Let $\alpha > 0$, $p > 2$. Then there exists a ring Q-homeomorphism $f\colon \mathbb{B} \to \mathbb{C}$ with respect to p-modulus at the point $z_0 = 0$ such that $f(0) = 0$ and $|f(z)| e^{\frac{\alpha}{(p-2)|z|}} \to 0$ as $z \to 0$. Furthermore,*

$$t^{-(2p-3)} e^{-\frac{\alpha}{t}} q_{z_0}(t) \to \infty$$

as $t \to 0$.

Proof. Let $\alpha > 0$, $p > 2$, $\kappa > 0$ and $l > 1$. Consider the mapping

$$f_2(z) = \begin{cases} \kappa\, e^{-\frac{l\alpha}{(p-2)|z|}} \frac{z}{|z|}, & 0 < |z| < 1, \\ 0, & z = 0. \end{cases}$$

Note that the homeomorphism f_2 maps the disk B_r to the disk $B_{\tilde{r}}$, where

$$\tilde{r} = \kappa\, e^{-\frac{l\alpha}{(p-2)r}}, \quad r \in (0,1).$$

Similarly to Remark 2.5, it can be shown that the mapping f_2 is a ring Q-homeomorphism with respect to p-modulus at the point $z_0 = 0$ as $p > 2$ with the function

$$Q(z) = l_0 |z|^{2p-3} e^{\frac{l\alpha}{|z|}},$$

where $l_0 = (p-2)^{p-1}\kappa^{-(p-2)} l^{-(p-1)} \alpha^{-(p-1)}$. Whence $q_{z_0}(t) = l_0\, t^{2p-3} e^{\frac{l\alpha}{t}}$. It is obvious that

$$\lim_{t\to 0} t^{-(2p-3)} e^{-\frac{\alpha}{t}} q_{z_0}(t) = l_0 \lim_{t\to 0} e^{\frac{(l-1)\alpha}{t}} = \infty.$$

On the other hand, it is easy to see that

$$\lim_{z\to 0} |f_2(z)| e^{\frac{\alpha}{(p-2)|z|}} = \kappa \lim_{z\to 0} e^{\frac{(1-l)\alpha}{(p-2)|z|}} = 0. \qquad \square$$

The following statement gives sufficient conditions for nonexistence of ring Q-homeomorphisms with respect to p-modulus at a point of the complex plane as $p > 2$.

Proposition 3.7. *Let D be a domain in the complex plane $\mathbb{C}$, $z_0 \in D$. Let for some numbers $r_0 \in (0, d_0)$, $d_0 = \operatorname{dist}(z_0, \partial D)$, $\kappa = \kappa(z_0) > 0$, $\alpha > 0$ and $p > 2$ the condition*

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

be fulfilled for a.e. $t \in (0, r_0)$. Then there is no ring Q-homeomorphism $f\colon D \to \mathbb{C}$ with respect to p-modulus at the point z_0 with the asymptotic condition

$$\limsup_{z\to z_0} |f(z) - f(z_0)|\, e^{\frac{\alpha}{(p-2)|z-z_0|}} = 0. \tag{3.5}$$

Proof. Assume that there exists a ring Q-homeomorphism $f\colon D \to \mathbb{C}$ with respect to p-modulus at the point z_0 for which

$$q_{z_0}(t) \leqslant \kappa\, t^{2p-3} e^{\frac{\alpha}{t}}$$

for a.e. $t \in (0, r_0)$ and the asymptotic condition (3.5) is fulfilled. Then, by Theorem 3.1, we have the estimate

$$\limsup_{z\to z_0} |f(z) - f(z_0)|\, e^{\frac{\alpha}{(p-2)|z-z_0|}} \geqslant (p-2)^{\frac{p-1}{p-2}} \kappa^{-\frac{1}{p-2}} \alpha^{-\frac{p-1}{p-2}} > 0,$$

which contradicts the asymptotic condition (3.5). $\square$

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Received: August 16, 2024, accepted: September 26, 2024.

Mariia V. Stefanchuk
Department of Complex Analysis and Potential Theory
Institute of Mathematics of NAS of Ukraine
Tereshchenkivs'ka str., 3, Kyiv, 01024, Ukraine
Email: stefanmv43@gmail.com
ORCID: 0000-0003-3672-8346