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A characteristic property of Sasakian manifolds

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Abstract. We study the case when a unit vector field ξ on a Riemannian manifold (M, g) defines an isometric embedding $\xi\colon (M, g) \to (T_1M, \tilde{G})$ where $\tilde{G}$ is the Riemannian g-natural metric. The main goal is to find conditions under which the submanifold $\xi(M) \subset (T_1M, \tilde{G})$ can be totally geodesic. It is proved that the Reeb vector field of a K-contact metric structure on M gives rise to totally geodesic $\xi(M)$ if and only if the structure is Sasakian. As a by-product, we find the expression for the second fundamental form of $\xi(M) \subset (T_1M, \tilde{G})$.

Анотація. Ми вивчаємо випадок, коли одиничне векторне поле ξ на рімановому многовиді (M, g) визначає ізометричне вкладення $\xi\colon (M, g) \to (T_1M, \tilde{G})$, де $\tilde{G}$ — g-природна метрика Рімана. Основна мета полягає в тому, щоб знайти умови, за яких підмноговид $\xi(M) \subset (T_1M, \tilde{G})$ є цілком геодезичним. Доведено, що векторне поле Ріба K-контактної метричної структури на M породжує цілком геодезичний підмноговид $\xi(M)$ тоді і тільки тоді, коли структура є сасаківською. Зокрема, ми знаходимо вираз для другої фундаментальної форми $\xi(M) \subset (T_1M, \tilde{G})$.

1. INTRODUCTION

Let (M, g) be the Riemannian manifold and (T_1M, g_S) be the unit tangent bundle of M endowed with the *Sasaki metric* (see [10, 11]). It is well known that the fibers of T_1M as submanifolds in (T_1M, g_S) are totally geodesic ones. The other natural type of submanifolds in (T_1M, g_S) are given by (at least locally) unit vector field ξ being considered as a mapping $\xi\colon M \to T_1M$ given by $\xi(x) = (x, \xi(x))$. The submanifold $\xi(M) \subset (T_1M, g_S)$ is transversal to the fibers, and (locally) homeomorphic to the base manifold but almost never isometric to the base except ξ is a parallel unit vector field. M. T. K. Abbassi and M. Sarih [1,2] defined a family of the so-called Riemannian g-natural metrics on the unit tangent bundle of the Riemannian manifold (M, g) which depends on four constants (a, b, c, d).

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The Sasaki metric belongs to the family with parameters value $(1, 0, 0, 0)$. D. Perrone showed that a *non-parallel* unit vector field can define an *isometric embedding* of the base manifold into its unit tangent bundle endowed with a 2-parametric family of *Riemannian g-natural metrics* (the family does not include the Sasaki metric). This is the case when the base manifold is K-contact and the field ξ is the characteristic vector field of the Killing-contact structure (the so called, *Reeb unit vector field*). Namely,

Theorem 1.1 ([8])**.** *Let* (M, g, η, ξ, ϕ) *be a contact metric manifold with* $\dim M = 2n+1$ *and let* $\tilde{G}$ *be a Riemannian g-natural metric on* T_1M *with* $c = 1-2a$. *Then* $\xi\colon (M, g) \to (T_1M, \tilde{G})$ *is an isometric embedding if and only if* $d = a$ *and* M *is* K*-contact.*

In addition, he proved

Theorem 1.2 ([8])**.** *The Reeb vector field of a* K*-contact manifold* (M, η, g) *defines a harmonic map* $\xi\colon (M, g) \to (T_1M, \tilde{G})$ *for any Riemannian g-natural metric* $\tilde{G}$ *on* T_1M.

In the case of isometric embedding, minimality of embedded manifold implies harmonicity of embedding, and vice-versa [6]. In particular, the Theorem above implies that the image of the Reeb vector field from [8, Proposition 3.5] is *minimal.*

A unit vector field ξ on the Riemannian manifold (M, g) is called *totally geodesic* if the image of (local) embedding $\xi\colon M \to T_1M$ is a totally geodesic submanifold of $(T_1M, \tilde{G})$ (see [12, 13] for the case of the Sasaki metric). In this paper we prove the following Theorem which is a natural addition to D. Perrone's result.

Theorem 3.1. *Let* (M, g, η, ξ, ϕ) *be a* $(2n+1)$*-dimensional* K*-contact metric manifold. Let* $\mathcal{F}$ *be the family of Riemannian g-natural metrics on* T_1M *defined by the parameters*

$$0 < a < 1, \qquad b^2 < a(1-a), \qquad c = 1-2a, \qquad d = a.$$

Then the Reeb vector field ξ *for each* $\tilde{G} \in \mathcal{F}$ *defines totally geodesic isometric embedding* $\xi\colon (M, g) \to (T_1M, \tilde{G})$ *if and only if* M *is a Sasakian manifold.*

Theorem 3.1 distinguishes Sasakian manifolds among K-contact manifolds taking into account the geometry of the unit tangent bundle endowed with g-natural metrics.

2. PRELIMINARIES

2.1. **g-natural metrics on (unit) tangent bundle.** Let (M, g) be an n-dimensional Riemannian manifold with metric g. Denote by $g(\cdot, \cdot)$ the scalar product with respect to g. The *tangent bundle* of a Riemannian manifold (M,g) is a Riemannian manifold TM which is given by

$$TM = \bigsqcup_{x\in M} T_xM = \{(x,\xi) : x \in M,\ \xi \in T_xM\},$$

where T_xM denotes the tangent space to M at the point x. The *point of* TM is a pair (x,ξ), where x is a point in M and ξ is a tangent vector to M at x.

It is well known that at each point $(x,\xi) \in TM$ the tangent space $T_{(x,\xi)}TM$ splits into *vertical* and *horizontal* parts:

$$T_{(x,\xi)}TM = \mathcal{H}_{(x,\xi)}TM \oplus \mathcal{V}_{(x,\xi)}TM.$$

The vertical part $\mathcal{V}_{(x,\xi)}$ is tangent to the fiber, while the horizontal part $\mathcal{H}_{(x,\xi)}$ is transversal to it. Denote by $(x^1,\dots,x^n;\xi^1,\dots,\xi^n)$ the natural induced local coordinate system on TM. Let also

$$\partial_i = \frac{\partial}{\partial x^i} \quad \text{and} \quad \partial_{n+i} = \frac{\partial}{\partial \xi^i}.$$

Then for $\tilde{X} \in T_{(x,\xi)}TM$ we have that

$$\tilde{X} = \tilde{X}^i\partial_i + \tilde{X}^{n+i}\partial_{n+i}.$$

If $X \in T_xM$, then, denoting by Γ^i_{jk} the Christoffel's symbols of g,

$$X^h = X^i\partial_i - \Gamma^i_{jk}\xi^j X^k\partial_{n+i}$$

belongs to $\mathcal{H}_{(x,\xi)}TM$ and is called the *horizontal lift* of X, while

$$X^v = X^i\partial_{n+i}$$

belongs to $\mathcal{V}_{(x,\xi)}TM$ and is called the *vertical lift* of X.

The *unit tangent bundle* T_1M is defined as a hypersurface in TM given by $g_x(\xi,\xi) = 1$, that is

$$T_1M = \{(x,\xi) \mid x \in M,\ \xi \in T_xM,\ g_x(\xi,\xi) = 1\}.$$

The *Riemannian g-natural metric G on TM* [2] is defined by

$$\begin{aligned}
G_{(x,\xi)}(X^h,Y^h) &= (\alpha_1+\alpha_3)(r^2)g_x(X,Y) + (\beta_1+\beta_3)(r^2)g_x(X,\xi)g_x(Y,\xi),\\
G_{(x,\xi)}(X^h,Y^v) &= \alpha_2(r^2)g_x(X,Y) + \beta_2(r^2)g_x(X,\xi)g_x(Y,\xi),\\
G_{(x,\xi)}(X^v,Y^v) &= \alpha_1(r^2)g_x(X,Y) + \beta_1(r^2)g_x(X,\xi)g_x(Y,\xi),
\end{aligned}$$

for all tangent vectors $X,Y \in T_xM$ and $(x,\xi) \in TM$, where

$$\alpha_i,\beta_i\colon [0,+\infty) \to \mathbb{R}, \qquad i = 1,2,3,$$

are smooth functions, and $r^2 = g_x(\xi,\xi)$.

The *Riemannian g-natural metric $\tilde{G}$ on T_1M* [1] is defined by

$$\begin{aligned}
&\tilde{G}_{(x,\xi)}(X^h, Y^h) = (a+c)g_x(X,Y) + dg_x(X,\xi)g_x(Y,\xi),\\
&\tilde{G}_{(x,\xi)}(X^h, Y^t) = bg_x(X,Y),\\
&\tilde{G}_{(x,\xi)}(X^t, Y^t) = ag_x(X,Y) - \frac{\phi}{a+c+d}g_x(X,\xi)g_x(Y,\xi).
\end{aligned}$$

for all tangent vectors $X, Y \in T_xM$ and $(x,\xi) \in T_1M$, where the constants a, b, c, d satisfy the following inequalities:

$$a > 0, \qquad \alpha := a(a+c) - b^2 > 0, \qquad \phi := a(a+c+d) - b^2 > 0;$$

the unit normal at any point of T_1M is given by

$$N^G_{(x,\xi)} = \frac{1}{\sqrt{(a+c+d)\phi}}\Big(-b\xi^h + (a+c+d)\xi^v\Big)\,; \tag{2.1}$$

the "*tangential lift*" X^t, with respect to G, of a chosen vector $X \in T_xM$ to $(x,\xi) \in T_1M$, is defined as the tangential projection of the vertical lift of X to (x,ξ) with respect to N^G, namely

$$\begin{aligned}
X^t &= X^v - G_{(x,\xi)}(X^v, N^G_{(x,\xi)})N^G_{(x,\xi)}\\
&= X^v - \sqrt{\frac{\phi}{a+c+d}}g_x(X,\xi)N^G_{(x,\xi)}.
\end{aligned} \tag{2.2}$$

Note that

$$\xi^t = \frac{b}{a+c+d}\xi^h. \tag{2.3}$$

If $a = 1$ and $b = c = d = 0$, then $\tilde{G}$ is the *Sasaki metric.*

Let ∇ be the *Levi-Civita connection* on M. The *Nomizu operator*

$$A_\xi\colon \mathfrak{X}(M) \to \xi^\perp \subset \mathfrak{X}(M)$$

for a unit smooth vector field ξ is defined by $A_\xi X = -\nabla_X\xi$. It is worth to remark that $A_\xi X$ is orthogonal to ξ and $(A_\xi X)^v = (A_\xi X)^t$.

The *tangent map* $\xi_*\colon \mathfrak{X}(M) \to T\xi(M)$ is defined by

$$\xi_* X = X^h - (A_\xi X)^t. \tag{2.4}$$

The *rough Hessian* and the *ξ-harmonicity* tensor (see [13]) are given by

$$Hess_\xi(X,Y) = \tfrac{1}{2}((\nabla_X A_\xi)Y + (\nabla_Y A_\xi)X), \tag{2.5}$$

$$Hm_\xi(X,Y) = \tfrac{1}{2}(R(\xi, A_\xi X)Y + R(\xi, A_\xi Y)X), \tag{2.6}$$

where $(\nabla_X A_\xi)Y = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y)$ and R is the *curvature tensor* of the base manifold (M,g). The *second fundamental form of the map* $\phi\colon (M,g) \to (N,h)$ between Riemannian manifolds is defined as

$$B_\phi(X,Y) = \nabla^\phi_{\phi_* X}(\phi_* Y) - \phi_*(\nabla^g_X Y), \tag{2.7}$$

where ∇^{ϕ} is the induced Levi-Civita connection on $\phi(M)$ and ∇^{g} is the Levi-Civita connection on M.

2.2. **Contact metric manifolds.** A differentiable $(2n+1)$-dimensional manifold M carries a (ϕ,ξ,η)-*structure* if it admits a field ϕ of endomorphisms of the tangent spaces, a vector field ξ, and a 1-form η satisfying the following identities:

$$\eta(\xi)=1, \qquad \phi^2=-I+\eta\otimes\xi,$$

where I denotes the identity transformation. Note that

$$\phi\xi=0, \qquad \eta\circ\phi=0. \tag{2.8}$$

If the manifold M with a (ϕ,ξ,η)-structure admits a Riemannian metric g such that

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

for any vector fields X, Y, then M is said to have a (ϕ,ξ,η,g)-*structure*, or an *almost contact metric structure*, while g is called a *compatible metric.* Setting $Y=\xi$, an immediate consequence is that η is the covariant form of ξ, that is

$$\eta(X)=g(X,\xi). \tag{2.9}$$

Define the following 2-form Φ on M by

$$\Phi(X,Y)=g(X,\phi Y).$$

Note that

$$g(X,\phi Y)=-g(\phi X,Y). \tag{2.10}$$

We will call Φ the *fundamental* 2-*form* of the almost contact metric structure (ϕ,ξ,η,g). If $\Phi=d\eta$, then an almost contact metric structure (ϕ,ξ,η,g) is called a *contact metric structure*, while (M,ϕ,ξ,η,g) is called a *contact metric manifold*, ξ is called *Reeb vector field* and η uniquely defines ξ by the conditions

$$\eta(\xi)=1, \qquad d\eta(\xi,X)=0.$$

The Reeb vector field of a contact manifold plays a fundamental role in the study of the Riemannian geometry of a contact metric manifold. More detailed information about the contact metric manifolds and the Reeb vector fields can be found in [3,14]. Besides, consider several types of almost contact metric manifolds.

Definition 2.3. A differentiable $(2n+1)$-dimensional contact metric manifold (M,g,η,ξ,ϕ) is called *K-contact manifold* if ξ is Killing vector field.

Equivalently, (M,g,η,ξ,ϕ) is K-contact if and only if

$$\mathcal{L}_\xi\phi = 0, \tag{2.11}$$

where $\mathcal{L}_\xi\phi$ is the *Lie derivative* of ϕ along the vector field ξ. The basic property of a K-contact manifold is that

$$\nabla_X\xi = -\phi X. \tag{2.12}$$

The other important properties of a K-contact manifold are that

$$\eta(R(\xi,X)Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.13}$$

$$(\nabla_X\phi)Y = R(\xi,X)Y, \tag{2.14}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi. \tag{2.15}$$

Note that equation (2.13) is a direct consequence of equation (2.15).

Definition 2.4. A contact metric manifold (M,g,η,ξ,ϕ) is called a *Sasakian manifold* if

$$(\nabla_X\phi)Y = g(X,Y)\xi - \eta(Y)X.$$

Note that any Sasakian manifold is K-contact. An important property of a Sasakian manifold is that

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X.$$

Definition 2.5. A contact metric manifold (M,g,η,ξ,ϕ) is called a *nearly Sasakian manifold* if

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X. \tag{2.16}$$

Note that each Sasakian manifold is nearly Sasakian. From (2.16), we obtain that

$$\nabla_X\xi = -\phi X - HX,$$

where

$$HX = \phi(\nabla_\xi\phi)X. \tag{2.17}$$

The tensor field H is of type $(1,1)$ and satisfies $H\xi = 0$, $\eta\circ H = 0$, and

$$\nabla_\xi H = -\nabla_\xi\phi = \phi H = -\frac{1}{3}\mathcal{L}_\xi\phi. \tag{2.18}$$

If H vanishes, then a nearly Sasakian manifold is Sasakian. Note that, using (2.11), (2.17) and (2.18), we obtain that

Theorem 2.6. *Nearly Sasakian manifold is K-contact if and only if it is Sasakian.*

It is worth to note that in dimension $2n+1>5$ it is known that every nearly Sasakian manifold is Sasakian [5].

More detailed information about the K-contact, Sasakian and nearly Sasakian manifolds can be found in [3–5, 7, 9, 14].

3. MAIN RESULT

Let (M, g, η, ξ, ϕ) be a K-contact metric manifold, $\dim M = 2n+1$, and let $\tilde{G}$ be a Riemannian g-natural metric on T_1M with $c = 1-2a$ and $d = a$. By Theorem 1.1, the *Reeb vector field defines an isometric embedding* $\xi\colon (M,g) \to (T_1M, \tilde{G})$. Under this hypothesis we can prove the following assertions.

Theorem 3.1. *Let (M, g, η, ξ, ϕ) be a $(2n+1)$-dimensional K-contact metric manifold. Let $\mathcal{F}$ be the family of Riemannian g-natural metrics on T_1M defined by the parameters*

$$0 < a < 1, \qquad b^2 < a(1-a), \qquad c = 1-2a, \qquad d = a.$$

Then the Reeb vector field ξ for any $\tilde{G} \in \mathcal{F}$ defines a totally geodesic isometric embedding $\xi\colon (M,g) \to (T_1M, \tilde{G})$ if and only if M is a Sasakian manifold.

The proof is based on the following Lemma.

Lemma 3.2. *Let $\tilde{G}$ be Riemannian g-natural metric on the unit tangent bundle T_1M of Riemannian manifold (M,g) defined by four parameters a, b, c and d. Then for the unit vector field ξ defining the map*

$$\xi\colon (M,g) \to (T_1M, \tilde{G})$$

the second fundamental form of $\xi(M) \subset (T_1M, \tilde{G})$ has the following expression

$$\begin{aligned} B_\xi(X,Y) = \Big\{ &- \tfrac{a^2}{\alpha} Hm_\xi(X,Y) - \tfrac{ab}{\alpha}\rho_\xi(X,Y) \\ &+ \tfrac{bd}{\alpha}\gamma_\xi(X,Y) - \tfrac{ad}{\alpha} A_\xi(\gamma_\xi(X,Y)) \\ &+ \tfrac{d}{(a+c+d)\alpha}\Big[a^2 g(Hm_\xi(X,Y),\xi) + abg(\rho_\xi(X,Y),\xi) \\ &- bdg(\gamma_\xi(X,Y),\xi) - \tfrac{\alpha}{2}(g(X, A_\xi Y) + g(Y, A_\xi X))\Big]\xi\Big\}^h \\ + \Big\{ &- Hess_\xi(X,Y) + \tfrac{b^2}{\alpha}\rho_\xi(X,Y) - \tfrac{(a+c)d}{\alpha}\gamma_\xi(X,Y) \\ &+ \tfrac{bd}{\alpha} A_\xi(\gamma_\xi(X,Y)) + \tfrac{ab}{\alpha} Hm_\xi(X,Y)\Big\}^t, \end{aligned} \tag{3.1}$$

where $\alpha = a(a+c) - b^2$, *and*

$$\rho_\xi(X,Y) = \tfrac{1}{2}(R(X,\xi)Y + R(Y,\xi)X), \tag{3.2}$$

$$\gamma_\xi(X,Y) = \tfrac{1}{2}(g(X,\xi)Y + g(Y,\xi)X). \tag{3.3}$$

Proof. Substituting (2.4) in (2.7), we can calculate the second fundamental form of the map

$$B_\xi(X,Y) = \tilde\nabla_{\xi_* X}(\xi_* Y) - \xi_*(\nabla_X Y).$$

Note that

$$\xi_*(\nabla_X Y) = (\nabla_X Y)^h - (A_\xi(\nabla_X Y))^t = (\nabla_X Y)^h + (\nabla_{\nabla_X Y}\xi)^t$$

and

$$\begin{aligned}\tilde\nabla_{\xi_* X}(\xi_* Y) &= \tilde\nabla_{X^h}Y^h - \tilde\nabla_{X^h}(A_\xi Y)^t - \tilde\nabla_{(A_\xi X)^t}Y^h + \tilde\nabla_{(A_\xi X)^t}(A_\xi Y)^t \\ &= \tilde\nabla_{X^h}Y^h + \tilde\nabla_{X^h}(\nabla_Y\xi)^t + \tilde\nabla_{(\nabla_X\xi)^t}Y^h + \tilde\nabla_{(\nabla_X\xi)^t}(\nabla_Y\xi)^t.\end{aligned}$$

Using (2.5), (2.6), (3.2), (3.3) and [1, Proposition 5], we obtain that

$$\begin{aligned}\tilde\nabla_{X^h}Y^h = \Big\{&\nabla_X Y - \tfrac{ab}{\alpha}\rho_\xi(X,Y) + \tfrac{bd}{\alpha}\gamma_\xi(X,Y) \\ &+ \tfrac{b}{(a+c+d)\alpha}\big[(ad+b^2)g(\rho_\xi(X,Y),\xi) \\ &\qquad - d(a+c+d)g(\gamma_\xi(X,Y),\xi)\big]\xi\Big\}^h \\ &+ \Big\{\tfrac{b^2}{\alpha}R(X,\xi)Y - \tfrac{a(a+c)}{2\alpha}R(X,Y)\xi - \tfrac{(a+c)d}{\alpha}\gamma_\xi(X,Y) \\ &\qquad + \tfrac{1}{\alpha}\big[-b^2 g(\rho_\xi(X,Y),\xi) + d(a+c)g(\gamma_\xi(X,Y),\xi)\big]\xi\Big\}^t,\end{aligned}$$

$$\begin{aligned}\tilde\nabla_{X^h}(\nabla_Y\xi)^t = \Big\{&-\tfrac{a^2}{2\alpha}R(\nabla_Y\xi,\xi)X + \tfrac{ad}{2\alpha}g(X,\xi)\nabla_Y\xi \\ &+ \tfrac{1}{2(a+c+d)\alpha}\big[a(ad+b^2)g(R(X,\xi)\nabla_Y\xi,\xi) \\ &\qquad + d\alpha g(X,\nabla_Y\xi)\big]\xi\Big\}^h \\ &+ \Big\{\nabla_X\nabla_Y\xi + \tfrac{ab}{2\alpha}R(\nabla_Y\xi,\xi)X \\ &\qquad - \tfrac{bd}{2\alpha}g(X,\xi)\nabla_Y\xi - \tfrac{ab}{2\alpha}g(R(X,\xi)\nabla_Y\xi,\xi)\xi\Big\}^t,\end{aligned}$$

$$\begin{aligned}\tilde\nabla_{(\nabla_X\xi)^t}Y^h = \Big\{&-\tfrac{a^2}{2\alpha}R(\nabla_X\xi,\xi)Y + \tfrac{ad}{2\alpha}g(Y,\xi)\nabla_X\xi \\ &+ \tfrac{1}{2(a+c+d)\alpha}\big[a(ad+b^2)g(R(\nabla_X\xi,\xi)Y,\xi) \\ &\qquad + d\alpha g(\nabla_X\xi,Y)\big]\xi\Big\}^h\end{aligned}$$

$$\begin{aligned}
&+\Big\{\tfrac{ab}{2\alpha}R(\nabla_X\xi,\xi)Y-\tfrac{bd}{2\alpha}g(Y,\xi)\nabla_X\xi\\
&\qquad-\tfrac{ab}{2\alpha}g(R(\nabla_X\xi,\xi)Y,\xi)\xi\Big\}^t,
\end{aligned}$$

$$\tilde{\nabla}_{(\nabla_X\xi)^t}(\nabla_Y\xi)^t=0.$$

First, calculate the horizontal component

$$\begin{aligned}
h:\quad &\nabla_XY-\tfrac{ab}{\alpha}\rho_\xi(X,Y)+\tfrac{bd}{\alpha}\gamma_\xi(X,Y)\\
&+\tfrac{b}{(a+c+d)\alpha}\big[(ad+b^2)g(\rho_\xi(X,Y),\xi)-d(a+c+d)g(\gamma_\xi(X,Y),\xi)\big]\xi\\
&-\tfrac{a^2}{2\alpha}R(\nabla_Y\xi,\xi)X+\tfrac{ad}{2\alpha}g(X,\xi)\nabla_Y\xi\\
&+\tfrac{1}{2(a+c+d)\alpha}\big[a(ad+b^2)g(R(X,\xi)\nabla_Y\xi,\xi)+d\alpha g(X,\nabla_Y\xi)\big]\xi\\
&-\tfrac{a^2}{2\alpha}R(\nabla_X\xi,\xi)Y+\tfrac{ad}{2\alpha}g(Y,\xi)\nabla_X\xi\\
&+\tfrac{1}{2(a+c+d)\alpha}\big[a(ad+b^2)g(R(\nabla_X\xi,\xi)Y,\xi)+d\alpha g(\nabla_X\xi,Y)\big]\xi\\
&-\nabla_XY.
\end{aligned}$$

Note that

$$\begin{aligned}
&\tfrac{a}{2}g\big(R(X,\xi)\nabla_Y\xi,\xi\big)+\tfrac{a}{2}g\big(R(\nabla_X\xi,\xi)Y,\xi\big)=\\
&\qquad=\tfrac{a}{2}g\big(R(\xi,A_\xi Y)X,\xi\big)+\tfrac{a}{2}g\big(R(\xi,A_\xi X)Y,\xi\big)=ag\big(Hm_\xi(X,Y),\xi\big),\\
&\tfrac{ad}{2\alpha}g(X,\xi)\nabla_Y\xi+\tfrac{ad}{2\alpha}g(Y,\xi)\nabla_X\xi=-\tfrac{ad}{\alpha}A_\xi\big(\gamma_\xi(X,Y)\big),\\
&-\tfrac{a^2}{2\alpha}R(\nabla_Y\xi,\xi)X-\tfrac{a^2}{2\alpha}R(\nabla_X\xi,\xi)Y=-\tfrac{a^2}{\alpha}Hm_\xi(X,Y).
\end{aligned}$$

Thus,

$$\begin{aligned}
h\colon\quad &-\tfrac{a^2}{\alpha}Hm_\xi(X,Y)-\tfrac{ab}{\alpha}\rho_\xi(X,Y)+\tfrac{bd}{\alpha}\gamma_\xi(X,Y)-\tfrac{ad}{\alpha}A_\xi\big(\gamma_\xi(X,Y)\big)\\
&+\Big(\tfrac{ad+b^2}{(a+c+d)\alpha}\big(ag(Hm_\xi(X,Y),\xi)+bg(\rho_\xi(X,Y),\xi)\big)\\
&-\tfrac{bd}{\alpha}g(\gamma_\xi(X,Y),\xi)-\tfrac{d}{2(a+c+d)}\big(g(X,A_\xi Y)+g(Y,A_\xi X)\big)\Big)\xi.
\end{aligned}$$

Calculate the tangential component.

$$\begin{aligned}
t\colon\quad &\tfrac{b^2}{\alpha}R(X,\xi)Y-\tfrac{a(a+c)}{2\alpha}R(X,Y)\xi-\tfrac{(a+c)d}{\alpha}\gamma_\xi(X,Y)\\
&+\tfrac{1}{\alpha}\big[-b^2g(\rho_\xi(X,Y),\xi)+d(a+c)g\big(\gamma_\xi(X,Y),\xi\big)\big]\xi+\nabla_X\nabla_Y\xi\\
&+\tfrac{ab}{2\alpha}R(\nabla_Y\xi,\xi)X-\tfrac{bd}{2\alpha}g(X,\xi)\nabla_Y\xi-\tfrac{ab}{2\alpha}g\big(R(X,\xi)\nabla_Y\xi,\xi\big)\xi\\
&+\tfrac{ab}{2\alpha}R(\nabla_X\xi,\xi)Y-\tfrac{bd}{2\alpha}g(Y,\xi)\nabla_X\xi-\tfrac{ab}{2\alpha}g\big(R(\nabla_X\xi,\xi)Y,\xi\big)\xi\\
&-\nabla_{\nabla_XY}\xi.
\end{aligned}$$

Note that

$$\begin{aligned}&\tfrac{ab}{2\alpha}R(\nabla_Y\xi,\xi)X+\tfrac{ab}{2\alpha}R(\nabla_X\xi,\xi)Y=\\&\quad=\tfrac{ab}{2\alpha}\big(R(\xi,A_\xi Y)X+R(\xi,A_\xi X)Y\big)=\tfrac{ab}{\alpha}Hm_\xi(X,Y),\end{aligned}$$

$$\begin{aligned}&-\tfrac{bd}{2\alpha}g(X,\xi)\nabla_Y\xi-\tfrac{bd}{2\alpha}g(Y,\xi)\nabla_X\xi=\\&\quad=\tfrac{bd}{2\alpha}\big(g(X,\xi)A_\xi Y+g(Y,\xi)A_\xi X\big)=\tfrac{bd}{\alpha}A_\xi\big(\gamma_\xi(X,Y)\big),\end{aligned}$$

$$\begin{aligned}&-\tfrac{ab}{2\alpha}g\big(R(X,\xi)\nabla_Y\xi,\xi\big)\xi-\tfrac{ab}{2\alpha}g\big(R(\nabla_X\xi,\xi)Y,\xi\big)\xi=\\&\quad=-\tfrac{ab}{2\alpha}g\big(R(\xi,A_\xi Y)X+R(\xi,A_\xi X)Y,\xi\big)\xi=-\tfrac{ab}{\alpha}g\big(Hm_\xi(X,Y),\xi\big)\xi.\end{aligned}$$

Using the identity

$$R(X,\xi)Y+R(\xi,Y)X+R(Y,X)\xi=0,$$

we get that

$$R(X,\xi)Y=R(X,Y)\xi+R(Y,\xi)X$$

and

$$\begin{aligned}&\tfrac{b^2}{\alpha}R(X,\xi)Y-\tfrac{a(a+c)}{2\alpha}R(X,Y)\xi\\&\quad=\left(\tfrac{b^2}{\alpha}-\tfrac{a(a+c)}{2\alpha}\right)R(X,Y)\xi+\tfrac{b^2}{\alpha}R(Y,\xi)X\\&\quad=\left(\tfrac{b^2}{2\alpha}-\tfrac{1}{2}\right)R(X,Y)\xi+\tfrac{b^2}{\alpha}R(Y,\xi)X.\end{aligned}$$

Similarly, using

$$R(X,Y)\xi+R(Y,\xi)X+R(\xi,X)Y=0,$$

we obtain that

$$R(X,Y)\xi=R(X,\xi)Y-R(Y,\xi)X$$

and

$$\begin{aligned}&\left(\tfrac{b^2}{2\alpha}-\tfrac{1}{2}\right)R(X,Y)\xi+\tfrac{b^2}{\alpha}R(Y,\xi)X\\&\quad=\tfrac{b^2}{2\alpha}R(X,Y)\xi-\tfrac{1}{2}R(X,Y)\xi+\tfrac{b^2}{\alpha}R(Y,\xi)X\\&\quad=\tfrac{b^2}{2\alpha}R(X,\xi)Y-\tfrac{b^2}{2\alpha}R(Y,\xi)X-\tfrac{1}{2}R(X,Y)\xi+\tfrac{b^2}{\alpha}R(Y,\xi)X\\&\quad=\tfrac{b^2}{2\alpha}\big(R(X,\xi)Y+R(Y,\xi)X\big)-\tfrac{1}{2}R(X,Y)\xi\\&\quad=\tfrac{b^2}{\alpha}\rho_\xi(X,Y)-\tfrac{1}{2}R(X,Y)\xi.\end{aligned}$$

Finally, using the following relations

$$\begin{aligned}\nabla_X\nabla_Y\xi&=-\nabla_X(A_\xi Y),\\R(X,Y)\xi&=\nabla_X\nabla_Y\xi-\nabla_Y\nabla_X\xi-\nabla_{\nabla_XY}\xi+\nabla_{\nabla_YX}\xi\\&=\nabla_Y(A_\xi X)-\nabla_X(A_\xi Y)+A_\xi(\nabla_XY)-A_\xi(\nabla_YX),\end{aligned}$$

$$\nabla_X(A_\xi Y) = (\nabla_X A_\xi)Y + A_\xi(\nabla_X Y),$$

we get that

$$\begin{aligned}
&\tfrac{b^2}{\alpha}R(X,\xi)Y - \tfrac{a(a+c)}{2\alpha}R(X,Y)\xi + \nabla_X\nabla_Y\xi - \nabla_{\nabla_X Y}\xi \\
&\quad = \tfrac{b^2}{\alpha}\rho_\xi(X,Y) - \tfrac{1}{2}R(X,Y)\xi - \nabla_X(A_\xi Y) + A_\xi(\nabla_X Y) \\
&\quad = \tfrac{b^2}{\alpha}\rho_\xi(X,Y) - \tfrac{1}{2}\big(\nabla_Y(A_\xi X) + \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y) - A_\xi(\nabla_Y X)\big) \\
&\quad = \tfrac{b^2}{\alpha}\rho_\xi(X,Y) - \tfrac{1}{2}\big((\nabla_Y A_\xi)X + (\nabla_X A_\xi)Y\big) \\
&\quad = \tfrac{b^2}{\alpha}\rho_\xi(X,Y) - Hess_\xi(X,Y).
\end{aligned}$$

Thus

$$\begin{aligned}
t\colon\quad & - Hess_\xi(X,Y) + \tfrac{b^2}{\alpha}\rho_\xi(X,Y) - \tfrac{(a+c)d}{\alpha}\gamma_\xi(X,Y) \\
& + \tfrac{bd}{\alpha}A_\xi(\gamma_\xi(X,Y)) + \tfrac{ab}{\alpha}Hm_\xi(X,Y) + \\
& \Big(-\tfrac{b^2}{\alpha}g(\rho_\xi(X,Y),\xi) + \tfrac{d(a+c)}{\alpha}g\big(\gamma_\xi(X,Y),\xi\big) - \tfrac{ab}{\alpha}g\big(Hm_\xi(X,Y),\xi\big)\Big)\xi.
\end{aligned}$$

Therefore, using (2.3), we obtain (3.1). □

3.3. **Proof of the Theorem 3.1.** Let (M,g,η,ξ,ϕ) be a K-contact manifold, but not necessarily Sasakian. Using (2.9), (2.12) and (2.14), we have that

$$\begin{aligned}
Hess_\xi(X,Y) &= \tfrac{1}{2}\big(R(\xi,X)Y + R(\xi,Y)X\big), \\
Hm_\xi(X,Y) &= \tfrac{1}{2}\big(R(\xi,\phi X)Y + R(\xi,\phi Y)X\big), \\
\rho_\xi(X,Y) &= -Hess_\xi(X,Y), \\
\gamma_\xi(X,Y) &= \tfrac{1}{2}\big(\eta(X)Y + \eta(Y)X\big).
\end{aligned}$$

Then (2.8), (2.10) and (2.13) imply that

$$\begin{aligned}
\eta\big(Hm_\xi(X,Y)\big) &= 0, \\
\eta\big(Hess_\xi(X,Y)\big) &= g(X,Y) - \eta(X)\eta(Y), \\
\eta\big(\gamma_\xi(X,Y)\big) &= \eta(X)\eta(Y), \\
\eta\big(\phi(\gamma_\xi(X,Y))\big) &= 0.
\end{aligned}$$

Since $c = 1-2a$ and $d = a$, we can rewrite B_ξ as follows:

$$\begin{aligned}
B_\xi(X,Y) = \Big\{ & \tfrac{ab}{\alpha}\big(Hess_\xi(X,Y) + \gamma_\xi(X,Y)\big) \\
& - \tfrac{a^2}{\alpha}\big(Hm_\xi(X,Y) + \phi(\gamma_\xi(X,Y))\big) - \tfrac{a^2 b}{\alpha}g(X,Y)\xi\Big\}^h \\
& + \Big\{-\tfrac{a(1-a)}{\alpha}\big(Hess_\xi(X,Y) + \gamma_\xi(X,Y)\big)
\end{aligned}$$

$$+ \tfrac{ab}{\alpha}(\phi\big(\gamma_\xi(X,Y)\big) + Hm_\xi(X,Y))\Big\}^t.$$

Denote by V the vector filed

$$\begin{aligned} V = {} & \tfrac{ab}{\alpha}(Hess_\xi(X,Y) + \gamma_\xi(X,Y)) - \tfrac{a^2}{\alpha}(Hm_\xi(XkY) + \phi(\gamma_\xi(X,Y))) \\ & - \tfrac{a^2 b}{\alpha} g(X,Y)\xi. \end{aligned}$$

Then, the horizontal component of V is

$$\begin{aligned} V^h = {} & \Big\{ \tfrac{ab}{\alpha}\big(Hess_\xi(X,Y) + \gamma_\xi(X,Y)\big) - \tfrac{a^2}{\alpha}\big(Hm_\xi(X,Y) + \phi(\gamma_\xi(X,Y))\big)\Big\}^h - \\ & - \tfrac{a^2 b}{\alpha} g(X,Y)\xi^h. \end{aligned}$$

Denote further by W the vector field

$$W = -\tfrac{a(1-a)}{\alpha}\big(Hess_\xi(X,Y) + \gamma_\xi(X,Y)\big) + \tfrac{ab}{\alpha}\big(\phi(\gamma_\xi(X,Y)) + Hm_\xi(X,Y)\big).$$

Using (2.1), (2.2), $c = 1 - 2a$ and $d = a$, we can rewrite W^t as follows:

$$W^t = W^v + \eta(W)(b\xi^h - \xi^v). \tag{3.4}$$

Note that

$$\eta(W) = -\tfrac{a(1-a)}{\alpha} g(X,Y).$$

Therefore,

$$\begin{aligned} W^t = {} & -\tfrac{a(1-a)b}{\alpha} g(X,Y)\xi^h \\ & + \Big\{ -\tfrac{a(1-a)}{\alpha}(Hess_\xi(X,Y) + \gamma_\xi(X,Y)) \\ & \quad + \tfrac{ab}{\alpha}(\phi(\gamma_\xi(X,Y)) + Hm_\xi(X,Y)) + \tfrac{a(1-a)}{\alpha} g(X,Y)\xi \Big\}^v. \end{aligned}$$

Denote

$$\begin{aligned} \lambda_\xi(X,Y) &= Hess_\xi(X,Y) + \gamma_\xi(X,Y), \\ \mu_\xi(X,Y) &= Hm_\xi(X,Y) + \phi(\gamma_\xi(X,Y)). \end{aligned}$$

Then we can rewrite B_ξ as follows

$$\begin{aligned} B_\xi(X,Y) = {} & \Big\{ \tfrac{ab}{\alpha}\lambda_\xi(X,Y) - \tfrac{a^2}{\alpha}\mu_\xi(X,Y) - \tfrac{ab}{\alpha} g(X,Y)\xi \Big\}^h \\ & + \Big\{ -\tfrac{a(1-a)}{\alpha}\lambda_\xi(X,Y) + \tfrac{ab}{\alpha}\mu_\xi(X,Y) + \tfrac{a(1-a)}{\alpha} g(X,Y)\xi \Big\}^v. \end{aligned}$$

It follows that $B_\xi(X,Y) = 0$ if and only if

$$\begin{cases} \tfrac{ab}{\alpha}\lambda_\xi(X,Y) - \tfrac{a^2}{\alpha}\mu_\xi(X,Y) = \tfrac{ab}{\alpha} g(X,Y)\xi, \\ -\tfrac{a(1-a)}{\alpha}\lambda_\xi(X,Y) + \tfrac{ab}{\alpha}\mu_\xi(X,Y) = -\tfrac{a(1-a)}{\alpha} g(X,Y)\xi. \end{cases}$$

We get the system of linear equations with respect to unknowns $\lambda_\xi(X,Y)$ and $\mu_\xi(X,Y)$. Note that

$$\det\begin{pmatrix} \frac{ab}{\alpha} & -\frac{a^2}{\alpha} \\ -\frac{a(1-a)}{\alpha} & \frac{ab}{\alpha} \end{pmatrix} = -\frac{a^2}{\alpha} \neq 0.$$

Therefore, due to the Cramer's rule $\lambda_\xi(X,Y) = g(X,Y)\xi$, $\mu_\xi(X,Y) = 0$, that is

$$\begin{cases} Hess_\xi(X,Y) + \gamma_\xi(X,Y) = g(X,Y)\xi, \\ Hm_\xi(X,Y) + \phi(\gamma_\xi(X,Y)) = 0 \end{cases}$$

or, more detailed,

$$\begin{cases} R(\xi,X)Y + R(\xi,Y)X + \eta(X)Y + \eta(Y)X = 2g(X,Y)\xi, \\ R(\xi,\phi X)Y + R(\xi,\phi Y)X + \eta(X)\phi Y + \eta(Y)\phi X = 0. \end{cases}$$

Note that, using (2.14), we can rewrite the first equation as

$$(\nabla_X\phi)Y + (\nabla_Y\phi)X = 2g(X,Y)\xi - \eta(X)Y - \eta(Y)X.$$

This condition is exactly (2.16), that is (M,g,η,ξ,ϕ) is a nearly Sasakian manifold. Then by Theorem 2.6 it is Sasakian, which completes the proof.

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