

Applications of Interpolation theory to the regularity of some quasilinear PDEs

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Abstract. We present some regularity results on the gradient of the weak or entropic-renormalized solution u to the homogeneous Dirichlet problem for the quasilinear equations of the form

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x; u) = f,$$

where Ω is a bounded smooth domain of $\mathbb{R}^n$, V is a nonlinear potential and f belongs to non-standard spaces like Lorentz-Zygmund spaces.

The results, which have been exposed by the third author in a talk presented in AGMA 2024, the International Scientific Online Conference «Algebraic and geometric methods of analysis», May 27-30, 2024, Ukraine, constitute only a part of results proved in detail in a paper coauthored with I. Ahmed, A. Fiorenza, A. Gogatishvili, A. El Hamidi and J.-M. Rakotoson ([2]).

Moreover, we collect some well-known and new results of the identification of some interpolation spaces and we enrich some contents with details.

Анотація. В роботі наведено огляд деяких результатів про градієнти слабких або ентропічно ренормалізованих розв'язків u однорідної проблеми Діріхле квазілінійних рівнянь виду

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x; u) = f,$$

де Ω – це обмежена область з гладкою межею в $\mathbb{R}^n$, V – нелінійний потенціал, а f – належить до нестандартних просторів типу Лоренца-Загмунда.

Результати представлені третім автором в доповіді на міжнародній науковій онлайн конференції «Алгебраїчні та геометричні методи аналізу» (27-30 травня 2024, Україна) складають лише частину результатів доведених в спільній статті I. Ahmed, A. Fiorenza, A. Gogatishvili, A. El Hamidi and J.-M. Rakotoson ([2]).

В роботі також представлено деякі добре відомі та нові результати про розпізнавання певних інтерполяційних просторів.

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1. INTRODUCTION

Let Ω be a bounded smooth domain of $\mathbb{R}^n$, $n \geq 2$. We study the regularity on the gradient of the solutions to the quasilinear Dirichlet problem

$$-\Delta_p u + V(x; u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.1)$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian operator, $1 < p < \infty$, $V: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

(H1) the mapping $x \in \Omega \rightarrow V(x; \sigma)$ is in $L^\infty(\Omega)$ for every $\sigma \in \mathbb{R}$;

(H2) the mapping $\sigma \in \mathbb{R} \rightarrow V(x; \sigma)$ is continuous and non decreasing for almost every $x \in \Omega$ and $V(x; 0) = 0$.

The datum f of the equation will be assumed in non-standard spaces, such as Lorentz-Zygmund spaces or $G\Gamma$ spaces.

Most of our regularity results are based on applications of results on non-linear interpolation of α -Hölderian mappings between interpolation spaces with logarithm functors, which extend some results proved in [56, 57] by Luc Tartar, who first gave interpolation results on nonlinear Hölderian mappings (which include Lipschitz mappings) and applied them to PDEs. Other results concerning interpolation of Lipschitz operators and other applications of Interpolation theory, also in PDEs, were given in [14, 41, 42, 44].

In Section 2 we define the spaces involved and recall some properties. In Section 3 we define the interpolation spaces with logarithm function and we identify some interpolation spaces between couples of Lebesgue or Lorentz spaces, recovering spaces as Lorentz-Zygmund spaces or $G\Gamma$ -spaces. In Section 4 we give results concerning interpolation of Hölderian mappings to couple of spaces with a logarithm function and in Section 5 we study the action of these mappings on those couples. Finally in Sections 6, 7 and 8 we provide applications of these results to regularity on the gradient of the weak or entropic-renormalized solution u to quasilinear Dirichlet problem (1.1). We also show that the mapping $\mathcal{T}: \mathcal{T}f = \nabla u$ is locally or globally α -Hölderian under suitable values of α and appropriate assumptions on V . We will exhibit only some proofs to give an idea of the tools involved.

2. DEFINITIONS OF SOME FUNCTIONAL SPACES

Let Ω be a bounded open set of $\mathbb{R}^n$ with Lebesgue measure $|\Omega|$.

Here and in the sequel, if A_1 and A_2 are two quantities depending on some parameters, we write $A_1 \lesssim A_2$ if there exists $c > 0$ independent of the parameters such that $A_1 \leq cA_2$, and $A_1 \simeq A_2$ if and only if $A_1 \lesssim A_2$ and $A_2 \lesssim A_1$.

Moreover sometimes, to simplicity the notations, for a function space X on Ω we will only write X instead of $X(\Omega)$.

Definition 2.1 (Decreasing rearrangement). For a measurable function $f: \Omega \rightarrow \mathbb{R}$, for any $t \geq 0$, the *distribution function* of f is

$$D_f(t) = |\{x \in \Omega : |f(x)| \geq t\}|,$$

and the *decreasing rearrangement* of f is defined by

$$f_*(s) = \inf\{t : D_f(t) \leq s\}, \quad \forall s \in (0, |\Omega|),$$

(with the convention $\inf \emptyset = +\infty$).

The *maximal function* f_{**} of f is defined by

$$f_{**}(s) = \frac{1}{s} \int_0^s f_*(t) dt.$$

Definition 2.2 (Lorentz spaces, e.g. [7]). Let $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. The *Lorentz space* $L^{p,q}(\Omega)$ is defined as the set of all measurable functions f on Ω for which the quantity

$$\|f\|_{p,q} = \|f\|_{L^{p,q}(\Omega)} = \|t^{\frac{1}{p}-\frac{1}{q}} f_*(t)\|_{L^q(0,|\Omega|)}$$

is finite, where $\|\cdot\|_{L^q(0,|\Omega|)}$ is the norm in the Lebesgue space L^q .

For $1 < p < +\infty$, the following equivalence holds,

$$\|f\|_{p,q} \simeq \|t^{\frac{1}{p}-\frac{1}{q}} f_{**}(t)\|_{L^q(0,|\Omega|)} = \begin{cases} \left(\int_0^{|\Omega|} [t^{\frac{1}{p}} f_{**}(t)]^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 1 \leq q < +\infty, \\ \sup_{0 < t < |\Omega|} t^{\frac{1}{p}} f_{**}(t) & \text{if } q = +\infty \end{cases}$$

up to multiplicative constants.

In particular,

$$L^{p,p}(\Omega) = L^p(\Omega).$$

The inclusions among Lorentz spaces are given by

$$\begin{aligned} L^{p,q}(\Omega) &\subset L^{p,r}(\Omega), & \text{if } 1 < p < +\infty, \quad 1 \leq q < r \leq +\infty; \\ L^{p,q}(\Omega) &\subset L^{r,s}(\Omega), & \text{if } 1 < r < p < +\infty, \quad 1 \leq q, s \leq +\infty. \end{aligned}$$

In particular, for $1 \leq q < p < r \leq +\infty$, we have

$$L^r(\Omega) \subset L^{p,q}(\Omega) \subset L^p(\Omega) \subset L^{p,r}(\Omega) \subset L^{p,\infty}(\Omega) \subset L^q(\Omega).$$

Definition 2.3 (Lorentz-Zygmund spaces, e.g. [7]). Assume now, for simplicity, $|\Omega| = 1$, $0 < p, q \leq +\infty$, $\lambda \in \mathbb{R}$. The *Lorentz-Zygmund space* $L^{p,q}(\log L)^\lambda(\Omega)$ consists of all Lebesgue measurable functions f on Ω such

that

$$\|f\|_{L^{p,q}(\log L)^\lambda(\Omega)} = \begin{cases} \left(\int_0^1 \left[t^{\frac{1}{p}} (1 - \log t)^\lambda f_*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}}, & \text{if } 0 < q < +\infty \\ \sup_{0 < t < 1} t^{\frac{1}{p}} (1 - \log t)^\lambda f_*(t), & \text{if } q = +\infty \end{cases}$$

is finite.

These spaces reduce to the Lorentz spaces for $\lambda = 0$ and to the Zygmund spaces for $p = q$, $0 < p < +\infty$:

$$\begin{aligned} L^{p,q}(\log L)^0(\Omega) &= L^{p,q}(\Omega), & 0 < p, q \leq +\infty, (\lambda = 0) \\ L^{p,p}(\log L)^\lambda(\Omega) &= L^p(\log L)^\lambda(\Omega), & 0 < p < +\infty, (\lambda \in \mathbb{R}). \end{aligned}$$

When $p = 1$ and $\lambda = 1$ the Zygmund space $L^1(\log L)^1(\Omega)$ is also denoted simply by $L(\log L)(\Omega)$.

Inclusion relations. (see [6, pp.31-33]) For $0 < p \leq +\infty$, $0 < q, s \leq +\infty$, $\lambda_1, \lambda_2 \in \mathbb{R}$, let

- $L^{p,q}(\log L)^{\lambda_1} \subset L^{r,s}(\log L)^{\lambda_2}$, $0 < r < p \leq +\infty$;
- $L^{p,q}(\log L)^{\lambda_1} \subset L^{p,s}(\log L)^{\lambda_2}$, whenever
either $q \leq s$, $\lambda_1 \geq \lambda_2$ or $q > s$, $\lambda_1 + \frac{1}{q} > \lambda_2 + \frac{1}{s}$;
- $L^{\infty,q}(\log L)^{\lambda_1} \subset L^{\infty,s}(\log L)^{\lambda_2}$, $0 < q < s \leq \infty$, $\lambda_1 + \frac{1}{q} = \lambda_2 + \frac{1}{s}$.

In particular, for $0 < q < p < r < \infty$ and for any $\varepsilon > 0$, as a consequence of the previous results, the following chain of inclusions holds (see [49, p.192]:

$$\begin{aligned} L^p(\log L)^{\frac{1}{q} - \frac{1}{p} + \varepsilon} &\subset L^{p,q} \subset L^p \subset L^{p,q}(\log L)^{\frac{1}{p} - \frac{1}{q} - \varepsilon}, \\ L^{p,r}(\log L)^{\frac{1}{p} - \frac{1}{r} + \varepsilon} &\subset L^p \subset L^{p,r} \subset L^p(\log L)^{\frac{1}{r} - \frac{1}{p} - \varepsilon}. \end{aligned}$$

All the previous inclusions are the sharpest possible in the sense that we can nowhere permit that $\varepsilon = 0$.

Definition 2.4 (Grand and small Lebesgue spaces, see [16, 25, 26, 36] and references therein). Suppose $|\Omega| = 1$. Let $1 < p < \infty$ and $\alpha > 0$. The *grand Lebesgue space* $L^{(p),\alpha}(\Omega)$ is the set of all measurable functions such that the norm

$$\|f\|_{L^{(p),\alpha}(\Omega)} = \|f\|_{p,\alpha} = \sup_{0 < \varepsilon < p-1} \left(\varepsilon^\alpha \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

is finite. For $\alpha = 1$ this space was defined by C. Sbordone and T. Iwaniec in [38] and it is denoted by $L^{(p),1} = L^{(p)}$.

The *small Lebesgue space* $L^{(p'),\alpha}(\Omega)$, where $p' = \frac{p}{p-1}$, was introduced by A. Fiorenza in [24] as the associate space to the grand Lebesgue space

$L^{(p),\alpha}(\Omega)$, that is $\|\cdot\|_{L^{(p),\alpha}(\Omega)} = \|\cdot\|_{(p',\alpha)}$ is the smallest functional defined on the measurable functions such that a kind of Hölder type inequality holds:

$$\int_{\Omega} f(x) g(x) dx \leq \|f\|_{L^{(p),\alpha}(\Omega)} \|g\|_{L^{(p',\alpha)}(\Omega)}.$$

We have therefore $(L^{(p),\alpha})' = L^{(p',\alpha)}$. The equivalence with the following quasi-norms hold (see [16, 28]):

$$\begin{aligned} \|f\|_{L^{(p),\alpha}(\Omega)} &\simeq \sup_{0 < t < 1} (1 - \log t)^{-\frac{\alpha}{p}} \left(\int_t^1 f_*^p(\sigma) d\sigma \right)^{\frac{1}{p}}, \\ \|g\|_{L^{(p',\alpha)}(\Omega)} &\simeq \int_0^1 (1 - \log t)^{\frac{\alpha}{p}-1} \left(\int_0^t f_*^{p'}(\sigma) d\sigma \right)^{\frac{1}{p'}} \frac{dt}{t}. \end{aligned}$$

The space $L^{(p,1)}$ is denoted by $L^{(p)}$.

These spaces and their generalizations have an important role in applications to PDEs, in Calculus of Variations, and also in Probability and Statistics (see, e.g. [31–33] and references therein).

Definition 2.5 (Generalized Gamma spaces with double weights, [3, 26]). Suppose $|\Omega| = 1$. Let also w_1, w_2 be two weights on $(0, 1)$, $m \in [1, +\infty]$, $1 \leq p < +\infty$. Assume further that the following conditions hold:

$$(c1) \quad \exists k > 0 : w_2(2t) \leq K w_2(t), \quad \forall t \in (0, 1/2);$$

$$(c2) \quad \int_0^t w_2(\sigma) d\sigma \in L^{\frac{m}{p}}(0, 1; w_1).$$

The the *generalized Gamma space*

$$G\Gamma(p, m; w_1, w_2)(\Omega) = G\Gamma(p, m; w_1, w_2)$$

with double weights is the set of all measurable functions $f: \Omega \rightarrow \mathbb{R}$ such that

$$\left(\int_0^t f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} \in L^m(0, 1; w_1),$$

endowed with the quasi-norm

$$\|f\|_{G\Gamma(p, m; w_1, w_2)} = \left[\int_0^1 w_1(t) \left(\int_0^t f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}},$$

with the usual change when $m = \infty$.

If $w_2 = 1$ we denote $G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1)$.

In [1] we defined the $G\Gamma(p, m; w_1, w_2)$ spaces with a slight modification.

The scale of $G\Gamma(p, m; w_1, w_2)$ spaces is very general and covers many well-known scales of spaces. In next examples we collect some particular cases.

Examples.

1) If w is an integrable weight on $(0, 1)$, then $G\Gamma(p, m; w) = L^p$ (see [29, Remark 1, p.798]).

2) Let $1 \leq p, q < \infty$, w_1 be an integrable weight on $(0, 1)$, and let $w_2(t) = t^{\frac{q}{p}-1}$. Then

$$G\Gamma(q, m; w_1, w_2) = L^{p,q}.$$

In fact, the conditions (c1) and (c2) are satisfied. The first is obvious, while the second derives from

$$\int_0^1 w_1(t) \left(\int_0^t \sigma^{\frac{q}{p}-1} d\sigma \right)^{\frac{m}{q}} dt \simeq \int_0^1 t^{\frac{m}{p}} w_1(t) dt \leq \int_0^1 w_1(t) dt < \infty.$$

Now, if $f \in L^{p,q}$, we have

$$\begin{aligned} \|f\|_{G\Gamma(q,m;w_1,w_2)} &= \left[\int_0^1 w_1(t) \left(\int_0^t f_*^q(\sigma) \sigma^{\frac{q}{p}-1} d\sigma \right)^{\frac{m}{q}} dt \right]^{\frac{1}{m}} \\ &\leq \|f\|_{L^{p,q}} \left(\int_0^1 w_1(t) dt \right)^{\frac{1}{m}} < \infty, \end{aligned}$$

since w_1 is integrable on $(0, 1)$. Therefore $f \in G\Gamma(q, m; w_1, w_2)$.

For the reverse inclusion $G\Gamma(q, m; w_1, w_2) \subseteq L^{p,q}$ see [3, Proposition 2.1].

3) If $m = p$, $w(t) = t^{-1}(1 - \log t)^{\theta p-2}$, and $\theta > \frac{1}{p}$, we have

$$G\Gamma(p, p; t^{-1}(1 - \log t)^{\theta p-2}) = L^p(\log L)^{\theta - \frac{1}{p}},$$

(see [26, Proposition 6.2]), which can be written also in this form

$$G\Gamma(p, p; t^{-1}(1 - \log t)^{\theta p-1}) = L^p(\log L)^\theta,$$

(see [19, Lemma 3.5]).

4) $G\Gamma(p, 1; t^{-1}(1 - \log t)^{\frac{\theta}{p'}-1}) = L^{(p,\theta)}$, $p' = \frac{p}{p-1}$, (see [27, p. 813]).

5) Suppose $m = 1$, $p > 1$, $\gamma, \beta \in \mathbb{R}$, $\gamma > -1$, $\gamma + \frac{\beta}{p} + 1 > 0$,

$$\theta = p'(\gamma + \frac{\beta}{p} + 1), \quad w_1(t) = t^{-1}(1 - \log t)^\gamma, \quad w_2(t) = (1 - \log t)^\beta,$$

then

$$G\Gamma(p, 1; w_1, w_2) = L^{(p,\theta)},$$

(see [3, Corollary 2.7, p. 10]).

6) If $m, p \in [1, \infty)$, $\gamma, \beta \in \mathbb{R}$, $\gamma > -1$,

$$\gamma + \beta \frac{m}{p} + 1 < 0, \quad w_1(t) = t^{-1}(1 - \log t)^\gamma, \quad w_2(t) = (1 - \log t)^\beta,$$

then

$$\|f\|_{\text{GI}(p,m;w_1,w_2)}^m \simeq \int_0^1 (1 - \log t)^{\gamma + \beta \frac{m}{p} + 1} \left(\int_t^1 f_*^p(x) dx \right)^{\frac{m}{p}} \frac{dt}{t}$$

(see [3, Lemma 2.4, p. 9]).

Here we collect some properties of inclusion among the spaces defined above:

$$\begin{aligned} L^{(p)} \subsetneq L^p \subsetneq L^{p,\infty} \subsetneq L^{(p)} \subsetneq L^{p,\infty}(\log L)^{-\frac{1}{p}}, \quad (\text{see [28, p. 669]}), \\ \bigcup_{\varepsilon > 0} L^{p+\varepsilon} \subsetneq \bigcup_{\beta > 1} L^p(\log L)^{\frac{\beta\theta}{p'-1}} \subsetneq L^{(p,\theta)}(\Omega) \subsetneq L^p(\log L)^{\frac{\theta}{p'-1}} \subsetneq L^p(\Omega), \\ (\text{see [25, p. 26]}), \\ L^p \subsetneq L^p(\log L)^{-\theta} \subsetneq L^{(p),\theta} \subsetneq \bigcap_{\alpha > 1} L^p(\log L)^{-\alpha\theta} \subsetneq \bigcap_{0 < \varepsilon < p-1} L^{p-\varepsilon}, \\ (\text{see [25, p.24]}). \end{aligned}$$

3. INTERPOLATION SPACES WITH LOGARITHM FUNCTION

3.1. Preliminaries. Let $(X_0, \|\cdot\|_0)$ and $(X_1, \|\cdot\|_1)$ be two normed spaces continuously embedded in the same Hausdorff topological vector space, that is, (X_0, X_1) is a *compatible couple*.

For $f \in X_0 + X_1$, and $t > 0$ the *Peetre K -functional* is defined by

$$K(f, t) \doteq K(f, t; X_0, X_1) = \inf_{f=f_0+f_1} \left(\|f_0\|_{X_0} + t\|f_1\|_{X_1} \right),$$

where the infimum extends over all decompositions $f = f_0 + f_1$ of f with $f_0 \in X_0$ and $f_1 \in X_1$.

For $0 \leq \theta \leq 1$, $1 \leq q \leq +\infty$, and $\alpha \in \mathbb{R}$, the *logarithmic interpolation space* $(X_0, X_1)_{\theta,q;\alpha}$ is the set of all functions $f \in X_0 + X_1$ such that

$$\|f\|_{\theta,q;\alpha} = \|t^{-\theta-\frac{1}{q}}(1 - \log t)^\alpha K(f, t)\|_{L^q(0,1)} < \infty.$$

For the limiting cases it only makes sense to consider $\theta = 0$ and $\alpha \geq -1/q$ or $\theta = 1$ and $\alpha < -1/q$ or $\theta = 1, q = \infty$ and $\alpha = 0$.

We refer, for example, to [8, 20, 35].

The spaces $(X_0, X_1)_{0,q;\alpha}$ and $(X_0, X_1)_{1,q;\alpha}$ produce spaces which are “very close” to X_0 and to X_1 respectively.

If $X_1 \subset X_0$ and $\alpha = 0$, the space $(X_0, X_1)_{\theta,q;0}$ reduces to the classical real interpolation space $(X_0, X_1)_{\theta,q}$ defined by J. Peetre.

We recall some properties (see [8, p.46]).

- (1) $(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}$
- (2) $(X_0, X_1)_{\theta,q} \subset (X_0, X_1)_{\theta,r}$ if $1 \leq q \leq r \leq +\infty$.

(3) If $X_1 \subset X_0$ and $\theta_0 < \theta_1$, then $(X_0, X_1)_{\theta_1, q} \subset (X_0, X_1)_{\theta_0, q}$.

Theorem 3.2 (Associate space of an interpolation space). *Let X_0, X_1 be two Banach function spaces, such that $X_1 \subset X_0$. Let also $1 \leq q < +\infty$, $\alpha \in \mathbb{R}$, and $0 < \theta < 1$. Then*

$$[(X_0, X_1)_{\theta, q; \alpha}]' = (X_1', X_0')_{1-\theta, q'; -\alpha}, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

where X_i' is the associate space of X_i , $i = 0, 1$.

3.3. Identifications of some interpolation spaces. Here we collect some well known and new results concerning the interpolation spaces between Lebesgue, Lorentz, Lorentz-Zygmund, small Lebesgue and $G\Gamma$ spaces.

Let $0 < \theta < 1$, $1 \leq q \leq +\infty$. We have (see, e.g. [8, Theorem 5.2.1, Theorem 5.3.1], [58, Theorem 2, p.134], [7, Theorem 1.9, p.300])

$$L^q = (L^{p_0}, L^{p_1})_{\theta, q}, \quad 1 \leq p_0 < p_1 \leq \infty, \quad \frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

In particular,

$$\begin{aligned} L^q &= (L^1, L^\infty)_{1-\frac{1}{q}, q}, & 1 < q < \infty \\ L^q &= (L^p, L^\infty)_{\theta, q}, & 1 \leq p < q < \infty, \quad \theta = 1 - \frac{p}{q}. \end{aligned}$$

For $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,

$$L^{p, q} = (L^{p_0}, L^{p_1})_{\theta, q} = (L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q}; \quad (3.1)$$

if $p_0 = p_1 = p$,

$$L^{p, q} = (L^{p, q_0}, L^{p, q_1})_{\theta, q}, \quad \text{with } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \quad (3.2)$$

Theorem 3.4 (Interpolation with logarithm function between Lorentz spaces, [2, p. 908-909]). *Let $1 \leq p_0 < p_1 \leq \infty$, $1 \leq q_0, q_1 \leq \infty$, $1 \leq q < \infty$, $\alpha \in \mathbb{R}$. Let also either*

- $0 < \theta < 1$ and $\alpha \in \mathbb{R}$, or
- $\theta = 0$ and $\alpha \geq -\frac{1}{q}$, or
- $\theta = 1$ and $\alpha < -\frac{1}{q}$.

Then the following identifications hold.

(1) *Case $0 < \theta < 1$ and $\alpha \in \mathbb{R}$. Then*

$$\|f\|_{(L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q; \alpha}} \simeq \left(\int_0^1 \left[t^{\frac{1-\theta}{p_0} + \frac{\theta}{p_1}} f_*(t) (1 - \log t)^\alpha \right]^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

which implies

$$(L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q; \alpha} = L^{p, q} (\log L)^\alpha, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \quad (3.3)$$

As particular cases, for $1 \leq r < p < m \leq \infty$, $1 \leq q < \infty$, and $\alpha \in \mathbb{R}$, we have

$$L^{p,q}(\log L)^\alpha = (L^{r,\infty}, L^m)_{\theta,q;\alpha}, \quad \frac{1}{p} = \frac{1-\theta}{r} + \frac{\theta}{m}, \quad (3.4)$$

$$L^{p,q}(\log L)^\alpha = (L^1, L^m)_{\theta,q;\alpha}, \quad \theta = m' \left(1 - \frac{1}{p}\right), \quad m' = \frac{m}{m-1} \quad (3.5)$$

$$L^{p,q}(\log L)^\alpha = (L^{r,\infty}, L^\infty)_{\theta,q;\alpha} = (L^r, L^\infty)_{\theta,q;\alpha}, \quad \theta = 1 - \frac{r}{p}. \quad (3.6)$$

(2) Limiting case $\theta = 0$, with $\alpha \geq -\frac{1}{q}$ and $1 \leq q_0 < \infty$:

$$\|f\|_{(L^{p_0,q_0}, L^{p_1,q_1})_{0,q;\alpha}} \simeq \left(\int_0^1 \left[\left(\int_0^t s^{\frac{q_0}{p_0}-1} f_*^{q_0}(s) ds \right)^{\frac{1}{q_0}} (1 - \log t)^\alpha \right]^q \frac{dt}{t} \right)^{\frac{1}{q}},$$

which yields

$$(L^{p_0,q_0}, L^{p_1,q_1})_{0,q;\alpha} = G\Gamma(q_0, q; w_1, w_2), \quad (3.7)$$

where $w_1(t) = t^{-1}(1 - \log t)^{\alpha q}$, $w_2(t) = t^{\frac{q_0}{p_0}-1}$, $t \in (0, 1)$.

As particular case, for $1 < p \leq \infty$, $p_0 = q_0 = 1$, and $p_1 = q_1 = p$, we have that

$$(L^1, L^p)_{0,q;\alpha} = G\Gamma(1, q; w_1). \quad (3.8)$$

If $1 < q_0 = p_0 < p_1 < \infty$, $1 \leq q_1 \leq \infty$,

$$q = 1, \quad \alpha > -1, \quad \alpha_0 = p'_0(\alpha + 1), \quad \frac{1}{p_0} + \frac{1}{p'_0} = 1,$$

we have the small Lebesgue spaces (see also [4, Theorem 4.5]):

$$(L^{p_0}, L^{p_1,q_1})_{0,1;\alpha} = L^{(p_0,\alpha_0)}(\Omega), \quad (3.9)$$

since

$$\begin{aligned} \|f\|_{(L^{p_0}, L^{p_1,q_1})_{0,1;\alpha}} &\simeq \int_0^1 \left(\int_0^t f_*^{p_0}(s) ds \right)^{\frac{1}{p_0}} (1 - \log t)^{\frac{\alpha_0}{p'_0}-1} \frac{dt}{t} \\ &\simeq \|f\|_{L^{(p_0,\alpha_0)}(\Omega)}. \end{aligned}$$

In the case $q = 1$, $p_1 = q_1$, and $p_0 = q_0$, we recover (see [26, Propositions 4.1 and 4.2, p. 437])

$$(L^{p_0}, L^{p_1})_{0,1;\frac{\alpha}{p'_0}-1} = L^{(p_0,\alpha)}(\Omega), \quad 1 < p_0 < p_1 \leq \infty.$$

(3) Limiting case $\theta = 0$, with $\alpha \geq -\frac{1}{q}$ and $q_0 = \infty$:

$$\|f\|_{(L^{p_0,\infty}, L^{p_1,q_1})_{0,q;\alpha}} \simeq \left(\int_0^1 \left(\operatorname{ess\,sup}_{0 < s < t} s^{\frac{1}{p_0}} f_*(s) \right)^q (1 - \log t)^{\alpha q} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Therefore, we have

$$(L^{p_0, \infty}, L^{p_1, q_1})_{0, q; \alpha} = G\Gamma(\infty, q; w_1, w_2), \quad (3.10)$$

where $w_1(t) = t^{-1}(1 - \log t)^{\alpha q}$, $w_2(t) = t^{\frac{1}{p_0}}$, $t \in (0, 1)$.

(4) *Limiting case $\theta = 1$, with $\alpha < -\frac{1}{q}$ and $1 \leq q_1 < \infty$.*

$$\|f\|_{(L^{p_0, q_0}, L^{p_1, q_1})_{1, q; \alpha}} \simeq \left(\int_0^1 \left(\int_t^1 s^{\frac{q_1}{p_1}-1} f_*^{q_1}(s) ds \right)^{\frac{q}{q_1}} (1 - \log t)^{\alpha q} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

If, in particular, $q_1 = p_1 < \infty$, $\alpha = \frac{\gamma}{q} + \frac{\beta}{p_1}$ with $\gamma > -1$, $\beta \in \mathbb{R}$, $\gamma + \beta \frac{q}{p_1} + 1 < 0$, so that $\alpha < -1/q$, then we have

$$\|f\|_{(L^{p_0, q_0}, L^{p_1})_{1, q; \alpha}} \simeq \left(\int_0^1 \left(\int_t^1 f_*^{p_1}(s) ds \right)^{\frac{q}{p_1}} (1 - \log t)^{\gamma + \beta \frac{q}{p_1}} \frac{dt}{t} \right)^{\frac{1}{q}}$$

and, from [3, Lemma 2.4, p. 9],

$$(L^{p_0, q_0}, L^{p_1})_{1, q; \alpha} = G\Gamma(p_1, q; w_1, w_2). \quad (3.11)$$

with $w_1(t) = t^{-1}(1 - \log t)^\gamma$, $w_2(t) = (1 - \log t)^\beta$.

(5) *Limiting case $\theta = 1$, with $\alpha < -\frac{1}{q}$ and $q_1 = +\infty$:*

$$\|f\|_{(L^{p_0, q_0}, L^{p_1, \infty})_{1, q; \alpha}} \simeq \left(\int_0^1 \left(\operatorname{ess\,sup}_{0 < s < t} s^{\frac{1}{p_1}} f_*(s) \right)^q (1 - \log t)^{\alpha q} \frac{dt}{t} \right)^{\frac{1}{q}}.$$

Therefore we have

$$(L^{p_0, q_0}, L^{p_1, \infty})_{1, q; \alpha} = G\Gamma(\infty, q; w_1, w_2), \quad (3.12)$$

where $w_1(t) = t^{-1}(1 - \log t)^{\alpha q}$, $w_2(t) = t^{\frac{1}{p_1}}$, $t \in (0, 1)$.

The following result can be found in [35, Lemma 5.5, p. 100], in the more general setting of the Lorentz-Karamata spaces, of which the Lorentz-Zygmund spaces are a special case.

Theorem 3.5 (Interpolation with logarithm function between Lorentz-Zygmund spaces). *Let $1 \leq p_0 < p_1 \leq +\infty$, $1 \leq q_0, q_1 \leq \infty$, $1 \leq q < \infty$, $0 < \theta < 1$, $\alpha_0, \alpha_1, \alpha \in \mathbb{R}$. Then*

$$(L^{p_0, q_0}(\log L)^{\alpha_0}, L^{p_1, q_1}(\log L)^{\alpha_1})_{\theta, q; \alpha} = L^{p_\theta, q}(\log L)^{\alpha_\theta} \quad (3.13)$$

with $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha_\theta = (1 - \theta)\alpha_0 + \theta\alpha_1 + \alpha$.

In particular, for $\alpha_0 = \alpha_1 = 0$, we recover (3.3) in case (1):

$$(L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q; \alpha} = L^{p_\theta, q}(\log L)^\alpha. \quad (3.14)$$

Proof. In [35], for $\Omega \subset \mathbb{R}^n$ bounded with $|\Omega| = 1$, the Lorentz-Karamata spaces are defined through

$$\|f\|_{L_{p, q; b}(\Omega)} := \|t^{\frac{1}{p} - \frac{1}{q}} b(t) f_*(t)\|_{L^q(0, 1)}, \quad (3.15)$$

where $b(t)$ is a slowly varying function on $(0, 1)$. It has been proved that

$$(L_{p_0, q_0; b_0}, L_{p_1, q_1; b_1})_{\theta, q; b} = L_{p_\theta, q; b_\theta^\#} \quad (3.16)$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and

$$b_\theta^\#(t) = b_0^{1-\theta}(t) b_1^\theta(t) b \left(t^{\frac{1}{p_0} - \frac{1}{p_1}} \frac{b_0(t)}{b_1(t)} \right), \quad \text{for all } t \in (0, 1).$$

Choosing

$$b_0(t) = (1 - \log t)^{\alpha_0}, \quad b_1(t) = (1 - \log t)^{\alpha_1}, \quad b(t) = (1 - \log t)^\alpha,$$

we have that

$$\begin{aligned} L_{p_0, q_0; b_0}(\Omega) &= L^{p_0, q_0}(\log L)^{\alpha_0}(\Omega), \\ L_{p_1, q_1; b_1}(\Omega) &= L^{p_1, q_1}(\log L)^{\alpha_1}(\Omega). \end{aligned}$$

Therefore, the left hand side of (3.16) becomes

$$(L^{p_0, q_0}(\log L)^{\alpha_0}, L^{p_1, q_1}(\log L)^{\alpha_1})_{\theta, q; \alpha}.$$

Now, to identify the space $L_{p_\theta, q; b_\theta^\#}$, we determine $b_\theta^\#$:

$$b_\theta^\#(t) = (1 - \log t)^{\alpha_0(1-\theta) + \alpha_1\theta} [1 - \log(t^{\frac{1}{p_0} - \frac{1}{p_1}} (1 - \log t)^{\alpha_0 - \alpha_1})]^\alpha.$$

It is easy to see that

$$\lim_{t \rightarrow 0^+} \frac{1 - \log(t^{\frac{1}{p_0} - \frac{1}{p_1}} (1 - \log t)^{\alpha_0 - \alpha_1})}{1 - \log t} = \frac{1}{p_0} - \frac{1}{p_1},$$

since

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{1 - \left(\frac{1}{p_0} - \frac{1}{p_1} \right) \log t}{1 - \log t} &= \frac{1}{p_0} - \frac{1}{p_1}, \\ \lim_{t \rightarrow 0^+} \frac{(\alpha_0 - \alpha_1) \log(1 - \log t)}{1 - \log t} &= 0. \end{aligned}$$

Therefore

$$b_\theta^\#(t) \simeq (1 - \log t)^{\alpha_0(1-\theta) + \alpha_1\theta} (1 - \log t)^\alpha = (1 - \log t)^{\alpha_0(1-\theta) + \alpha_1\theta + \alpha},$$

hence

$$L_{p_\theta, q; b_\theta^\#} = L^{p_\theta, q}(\log L)^{\alpha_0(1-\theta) + \alpha_1\theta + \alpha} = L^{p_\theta, q}(\log L)^{\alpha_\theta}. \quad \square$$

In the limiting case $p_0 = p_1$ and $q_0 = q_1$, the following description of interpolation spaces between Lorentz–Zygmund spaces is given in [1, Eq. (7.2), p. 21], where $G\Gamma$ spaces are defined in slightly different way, but that here we rewrite according our notations.

Let $0 < p, q, r < \infty$, $0 < \theta < 1$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Let also

$$w_1(t) = t^{-1}(1 - \ln t)^{\theta r(\beta - \alpha) - 1}, \quad w_2(t) = t^{q/p - 1}(1 - \ln t)^{\alpha q}, \quad 0 < t < 1.$$

Then

$$\left(L^{p, q}(\log L)^\alpha, L^{p, q}(\log L)^\beta \right)_{\theta, r} = G\Gamma(q, r; w_1, w_2).$$

4. INTERPOLATION OF HÖLDERIAN MAPPINGS

Let $0 < \alpha \leq 1$ and X, Y two normed spaces. A mapping

$$\mathcal{T}: X \rightarrow Y$$

is *globally α -Hölderian*, with Hölder constant c , if

$$\exists c > 0 : \|\mathcal{T}a - \mathcal{T}b\|_Y \leq c\|a - b\|_X^\alpha, \quad \forall a, b \in X.$$

In the sequel we will consider four normed spaces $X_1 \subset X_0$, $Y_1 \subset Y_0$ and

$$\mathcal{T}: X_i \rightarrow Y_i, \quad i = 0, 1,$$

a nonlinear mapping satisfying, for $0 < \alpha \leq 1$ and $\beta > 0$, the following conditions:

$$(1) \quad \|\mathcal{T}a - \mathcal{T}b\|_{Y_0} \leq f(\|a\|_{X_0}, \|b\|_{X_0})\|a - b\|_{X_0}^\alpha, \quad \forall a, b \in X_0,$$

$$(2) \quad \|\mathcal{T}a\|_{Y_1} \leq g(\|a\|_{X_0})\|a\|_{X_1}^\beta, \quad \forall a \in X_1,$$

where g is a continuous increasing function on $\mathbb{R}_+$, f is continuous on $\mathbb{R}_+^2$ and increasing for any $\sigma \in \mathbb{R}_+$.

We put

$$G(\sigma) = \max\{g(2\sigma); f(\sigma, 2\sigma)\}, \quad \sigma \in \mathbb{R}_+. \quad (4.1)$$

In the next theorems we extend some Tartar's results contained in [56, 57].

Theorem 4.1. *Let $\lambda \in \mathbb{R}$ and $0 < \alpha \leq 1$. Let also $X_1 \subset X_0$, $Y_1 \subset Y_0$ be normed spaces and assume that X_1 is dense in X_0 . If*

$$\mathcal{T}: X_i \rightarrow Y_i, \quad \text{is globally } \alpha\text{-Hölderian, } i = 0, 1,$$

then, for $0 \leq \theta \leq 1$ and $1 \leq p \leq +\infty$,

$$\mathcal{T}: (X_0, X_1)_{\theta, p; \lambda} \rightarrow (Y_0, Y_1)_{\theta, \frac{p}{\alpha}; \lambda\alpha} \quad \text{is } \alpha\text{-Hölderian,}$$

$$\text{i.e. } \|\mathcal{T}a - \mathcal{T}b\|_{\theta, \frac{p}{\alpha}; \lambda\alpha} \leq c\|a - b\|_{\theta, p; \lambda}^\alpha, \quad \forall a, b \in (X_0, X_1)_{\theta, p; \lambda}.$$

Theorem 4.2. *Let $\lambda \in \mathbb{R}$, $0 < \alpha \leq 1$ and $\beta > 0$, with $\alpha \leq \beta$. Let also $X_1 \subset X_0$, $Y_1 \subset Y_0$ be normed spaces and*

$$\mathcal{T}: X_i \rightarrow Y_i, \quad i = 0, 1,$$

a mapping satisfying the assumptions (1) and (2). Then, for $0 \leq \theta \leq 1$ and $1 \leq p \leq +\infty$,

$$\mathcal{T} \text{ maps } (X_0, X_1)_{\theta, p; \lambda} \text{ into } (Y_0, Y_1)_{\theta \frac{\alpha}{\beta}, \frac{p}{\alpha}; \lambda \alpha}$$

and

$$\|\mathcal{T}a\|_{\theta \frac{\alpha}{\beta}, \frac{p}{\alpha}; \lambda \alpha} \lesssim [(1 + \|a\|_{X_0}^{\beta-\alpha}) G(\|a\|_{X_0})] \|a\|_{\theta, p; \lambda}^\alpha.$$

5. ESTIMATES OF THE K -FUNCTIONAL RELATED TO THE MAPPING $\mathcal{T}$

Here we study the action of the nonlinear mappings $\mathcal{T}$ on the K -functional, defined in Section 3 (see [2, Section 2.1]).

Theorem 5.1. *Let $0 < \alpha \leq 1$, $\beta > 0$. Let also $X_1 \subset X_0$, $Y_1 \subset Y_0$, be four normed spaces and*

$$\mathcal{T}: X_i \rightarrow Y_i, \quad i = 0, 1,$$

be a nonlinear mapping satisfying the assumptions (1) and (2) from Section 4 and G be the function defined in (4.1). Then, $\forall a \in X_0$, $\forall t > 0$, we have

$$K(\mathcal{T}a, t^\beta, Y_0, Y_1) = K(\mathcal{T}a, t^\beta) \leq G(\|a\|_{X_0}) ([K(a, t)]^\beta + [K(a, t)]^\alpha).$$

Moreover, if $\beta \geq \alpha$, then

$$K(\mathcal{T}a, t^\beta) \leq G(\|a\|_{X_0}) (1 + \|a\|_{X_0}^{\beta-\alpha}) [K(a, t)]^\alpha.$$

As a particular case we have the following consequences.

Corollary 5.2. *Let $0 < \alpha \leq 1$, $\beta > 0$. Let also $X_1 \subset X_0$, $Y_1 \subset Y_0$, be four normed spaces. Assume that the mapping $\mathcal{T}: X_0 \rightarrow Y_0$ is globally α -Hölderian with constant M_0 , i.e.*

$$\|\mathcal{T}a - \mathcal{T}b\|_{Y_0} \leq M_0 \|a - b\|_{X_0}^\alpha, \quad \forall a, b \in X_0,$$

and $\mathcal{T}$ maps X_1 into Y_1 , in the sense that there exist $M_1 > 0$ and $\beta > 0$ such that

$$\|\mathcal{T}a\|_{Y_1} \leq M_1 \|a\|_{X_1}^\beta.$$

Then

$$K(\mathcal{T}a, t^\beta) \leq \max(M_0; M_1) ([K(a, t)]^\beta + [K(a, t)]^\alpha), \quad \forall a \in X_0, \quad \forall t > 0.$$

Moreover, if $\beta \geq \alpha$, then

$$K(\mathcal{T}a, t^\beta) \leq \max(M_0; M_1) (1 + \|a\|_{X_0}^{\beta-\alpha}) [K(a, t)]^\alpha.$$

Corollary 5.3. *Let $0 < \alpha \leq 1$. Let also $X_1 \subset X_0$, $Y_1 \subset Y_0$, be four normed spaces. Assume that the mapping*

$$\mathcal{T}: X_i \rightarrow Y_i, \quad i = 0, 1,$$

is globally α -Hölderian with constant M_i , for $i = 0, 1$, i.e.

$$\|\mathcal{T}a - \mathcal{T}b\|_{Y_i} \leq M_i \|a - b\|_{X_i}^\alpha, \quad \forall a, b \in X_i.$$

Then, for all $a \in X_0$, $b \in X_1$, and $t > 0$, we have

$$K(\mathcal{T}a - \mathcal{T}b, t^\alpha) \leq 2 \max(M_0; M_1) [K(a - b, t)]^\alpha.$$

Furthermore, if X_1 is dense in X_0 , then the above inequality holds also for all $b \in X_0$.

6. APPLICATIONS TO THE REGULARITY OF THE SOLUTION OF A p -LAPLACIAN EQUATION

The Marcinkiewicz interpolation theorems for linear operators acting on Lebesgue spaces turned out to be a powerful tool for studying regularity of solutions for linear PDEs in L^p -spaces. The K -method introduced by J. Peetre ([46, 48]) allowed to extend the study of regularity of solutions of linear equations on spaces different from L^p -spaces. The main difficulty to apply Peetre's definition is the identification of the interpolation spaces between two normed spaces embedded in a same topological space. In [3, 25–27] we did such a study with applications to linear PDEs using non-standard spaces as grand or small Lebesgue spaces and GT -spaces.

In [57] L. Tartar gave interpolation results on nonlinear Hölderian mappings (which include Lipschitz mappings) with applications to semi-linear PDEs.

Other results concerning interpolation of Lipschitz operators and other applications of Interpolation theory, also in PDEs, can be found, e.g. in [14, 41–44, 47].

6.1. The quasilinear p -Laplacian equation. In the sequel we will consider Ω a bounded smooth domain of $\mathbb{R}^n$, $n \geq 2$, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Recall that $W^{1,p}(\Omega)$ is the classical Sobolev space of all real-valued functions $f \in L^p(\Omega)$ whose first-order weak (or distributional) partial derivatives on Ω belong to $L^p(\Omega)$, normed by

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

The space $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$; roughly speaking $W_0^{1,p}(\Omega)$ is a subspace of $W^{1,p}(\Omega)$ consisting of functions which vanish on the boundary of Ω . Its dual is $W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))'$.

We provide applications of the results described in Sections 3, 4 and 5 to regularity on the gradient of the weak or entropic-renormalized solution u to the quasilinear equation of the form

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + V(x; u) = f, \quad u = 0 \quad \text{on } \partial\Omega, \quad (6.1)$$

associated to the Dirichlet homogeneous condition on the boundary, where $1 < p < \infty$, V is a nonlinear potential and f belongs to non-standard spaces such as Lorentz-Zygmund spaces.

More precisely, we assume that $V : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying:

- (H1) the mapping $x \in \Omega \rightarrow V(x; \sigma)$ is in $L^\infty(\Omega)$ for every $\sigma \in \mathbb{R}$;
- (H2) the mapping $\sigma \in \mathbb{R} \rightarrow V(x; \sigma)$ is continuous and non decreasing for almost every $x \in \Omega$ and $V(x; 0) = 0$.

The p -laplacian is *strong coercive*, that is satisfies the following inequality, which can be found in [17, Lemma 4.10, p.264].

For $p \geq 2$, there exists a constant $\alpha_p > 0$ such that, for all $\xi, \xi' \in \mathbb{R}^n$,

$$(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi - \xi')_{\mathbb{R}^n} \geq \alpha_p |\xi - \xi'|^p, \quad (6.2)$$

where the symbol $(\cdot, \cdot)_{\mathbb{R}^n}$ in the left-hand side denotes the inner product in $\mathbb{R}^n$ and $|\cdot|$ is the associated norm.

The next Proposition follows from Leray–Lions’ method for monotone operators (see [40]) or the usual fixed point theorem of Leray–Schauder’s type (see [34]).

Proposition 6.2 (Weak solution). *Let*

$$f \in L^1(\Omega) \cap W^{-1,p'}(\Omega), \quad 1 < p < +\infty, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

*Then there exists a unique **weak solution** of the Dirichlet problem (6.1), i.e. a unique $u \in W_0^{1,p}(\Omega)$ such that*

$$\begin{aligned} \int_{\Omega} \varphi(x) V(x; u) dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx &= \int_{\Omega} f \varphi dx, \\ \forall \varphi &\in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \end{aligned} \quad (6.3)$$

Remark 6.3. We observe that if $p > n$, then $L^1(\Omega) \subset W^{-1,p'}(\Omega)$. In this case it is well known that, if the datum f of (6.1) is in $L^1(\Omega)$, there is existence and uniqueness of the weak solution, which is bounded (thanks to the Sobolev embedding). The existence follows from the classical results on operators acting between Sobolev spaces in duality (see, e.g. [40, p.107], [11, 15, 39]).

If $1 < p < n$ then $L^{(p^*)'}(\Omega) \subset W^{-1,p'}(\Omega)$, where $p^* = \frac{np}{n-p}$, and also $L^{(p^*)'}(\Omega) \subset L^1(\Omega)$. Therefore, if the datum f of (6.1) is in $L^{(p^*)'}(\Omega)$, we have $f \in L^1(\Omega) \cap W^{-1,p'}(\Omega)$, hence Proposition 6.2 can be applied.

If $p = n$ then $L^{q'}(\Omega) \subset W^{-1,p'}(\Omega)$ for any $q \in [1, \infty)$, where $q' = \frac{q}{q-1}$. Therefore, if $f \in L^{q'}(\Omega)$, for the same above reasons, we can apply Proposition 6.2.

Remark 6.4. In the case $p = n$ and f in $L^1(\Omega)$ the Iwaniec-Sbordone's method guarantees existence and uniqueness of the weak solution of (6.1); see, e.g. [36], where the authors use the Hodge decomposition in a smart way in a variety of questions (see also [21, 45]); see [30] for the particular case $p = n = 2$).

For the above Remark, the meaningful case is

$$p < n,$$

even if all our next results remain still true for $p \geq n$.

We define a nonlinear mapping

$$\begin{aligned} \mathcal{T}: L^1(\Omega) \cap W^{-1,p'}(\Omega) &\longrightarrow [L^p(\Omega)]^n \\ f &\longmapsto \mathcal{T}f = \nabla u. \end{aligned} \tag{6.4}$$

We intend to extend the mapping $\mathcal{T}$ over all $L^1(\Omega)$.

If $p < n$ and f is only in $L^1(\Omega)$, the formulation by equation (6.3) cannot ensure the uniqueness of the solution.

This case attracted the interest of several researchers, who tried to find a satisfying notion of solution in order to get both existence and uniqueness of the solution (see, for instance, [5, 9, 10, 12, 18, 50, 52]).

Here we focus our attention to the so-called entropic-renormalized solutions, considered by Jean Michel Rakotoson in [51, 52] (see also [5]), which are defined as follows.

Definition 6.5 (Entropic-renormalized solution). Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. For all $k > 0$, we consider the truncation operator $T_k: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$T_k(\sigma) = \{|k + \sigma| - |k - \sigma|\}/2, \tag{6.5}$$

and we define $\mathbb{S}_0^{1,p}$ as the set of all measurable functions $v: \Omega \rightarrow \mathbb{R}$ satisfying:

- 1) $\tan^{-1}(v) \in W_0^{1,1}(\Omega)$;
- 2) $\forall k > 0, T_k(v) \in W_0^{1,p}(\Omega)$;
- 3) $\sup_{k>0} k^{-\frac{1}{p}} \|\nabla T_k(v)\|_{L^p(\Omega)} < \infty$.

A function u defined on Ω is an *entropic-renormalized solution* of the Dirichlet problem

$$-\Delta_p u + V(x; u) = f \in L^1(\Omega), \quad u = 0 \quad \text{on } \partial\Omega \quad (6.6)$$

if

- (1) $u \in \mathbb{S}_0^{1,p}(\Omega)$, $V(\cdot, u) \in L^1(\Omega)$.
- (2) $\forall \eta \in W^{1,r}(\Omega)$, $r > n$, $\forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and all $B \in W^{1,\infty}(\mathbb{R})$ with $B(0) = 0$, $B'(\sigma) = 0$ for all σ such $|\sigma| \geq \sigma_0 > 0$, one has:

$$\begin{aligned} \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (\eta B(u - \varphi)) dx + \int_{\Omega} V(x; u) \eta B(u - \varphi) dx = \\ = \int_{\Omega} f \eta B(u - \varphi) dx. \end{aligned} \quad (6.7)$$

Remark 6.6. If $f \in L^{p'}(\Omega)$ then the formulation (6.7) is equivalent to the formulation (6.3), i.e. a weak solution is an entropic-renormalized solution (see [51] or [45] for the case $p = n$). In [5, 9, 12, 50, 53] existence and uniqueness of an entropic-renormalized solution have been proved.

Theorem 6.7. Let $f \in L^1(\Omega)$ and $V: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Caratheodory function satisfying the assumptions (H1) and (H2). Then there exists a unique entropic-renormalized solution of the equation (6.6).

Moreover, let (u_j) be a sequence of the solutions of (6.6) with respective data f_j , $j \in \mathbb{N}$. If the sequence (f_j) converges to f in $L^1(\Omega)$, then the sequence $(\nabla u_j(x))$ converges to $\nabla u(x)$ almost everywhere in Ω (up to subsequences still denoted by $\{u_j\}$). When $p > 2 - \frac{1}{n}$, the solution $u \in W_0^{1,1}(\Omega)$.

The next Proposition gathers well known results (see [13, 14, 23, 54, 55]). We shall need the following additional growth assumption on V .

(H3) There exist a constant $c > 0$ and $m_1 \in [p - 1, \bar{m}_1)$, where

$$\bar{m}_1 = \begin{cases} (p-1) \left(1 + \frac{1}{n-p}\right) & \text{if } p < n \\ < +\infty & \text{if } p \geq n, \end{cases}$$

such that

$$|V(x, \sigma)| \leq c |\sigma|^{m_1}, \quad \forall \sigma \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Proposition 6.8. Let u be the solution of the Dirichlet problem (6.6), with $f \in L^{p'}(\Omega)$. Let us assume that V satisfies (H1) and (H2).

- If $f \in L^{\frac{n}{p}, \frac{1}{p-1}}(\Omega)$ and $p \leq n$, then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq c \|f\|_{L^{\frac{n}{p}, \frac{1}{p-1}}(\Omega)}^{\frac{1}{p-1}}. \quad (6.8)$$

- If $f \in L^{1, \frac{1}{p-1}}(\Omega)$ and $p > n$, then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty(\Omega)} \leq c \|f\|_{L^{1, \frac{1}{p-1}}(\Omega)}^{\frac{1}{p-1}}. \quad (6.9)$$

- If V satisfies also the growth assumption (H3) and $f \in L^{n,1}(\Omega)$, $n \geq 3$, then

$$\|\nabla u\|_{L^\infty(\Omega)} \leq c \left(1 + \|f\|_{L^{1, \frac{1}{p-1}}(\Omega)}^{\frac{m_1+1-p}{p-1}}\right) \|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}}. \quad (6.10)$$

All the constants denoted by c depend only on p , Ω , and V .

Remark 6.9. We observe that, $p > n$ implies $L^{p'} \subset L^{1, \frac{1}{p-1}}$.

For $p \geq 2$ implies $L^{n,1} \subset L^{p'}$ since, if $p > n$, we have $p' < n' \leq n$; if $2 \leq p \leq n$, then $\frac{p}{p-1} \leq p \leq n$, hence $p' \leq n$.

The following Lemma provides estimates for the potential V .

Lemma 6.10. Let u be a weak solution of (6.1):

$$-\Delta_p u + V(x; u) = f, \quad u = 0 \quad \text{on } \partial\Omega.$$

Assume that V satisfies (H1), (H2), (H3). Let m_1 the number defined in condition (H3). Let also $m_3 = \frac{n}{n-p}(p-1)$ if $1 < p < n$ or $m_3 \in [nm_1, +\infty)$ if $p \geq n$.

- If $u \in L^\infty(\Omega)$, then

$$\|V(\cdot; u(\cdot))\|_{L^{n,1}}^{\frac{1}{p-1}} \lesssim \|u\|_\infty \cdot \|u\|_{L^{m_3,1}}^{\frac{m_1+1-p}{p-1}}. \quad (6.11)$$

- If $f \in L^{n,1}(\Omega)$, then

$$\|V(\cdot; u(\cdot))\|_{L^{n,1}}^{\frac{1}{p-1}} \lesssim \|f\|_{L^{n,1}}^{\frac{1}{p-1}} \cdot \|f\|_{L^1}^{\frac{m_1+1-p}{p-1}}. \quad (6.12)$$

We point out that the proof of (6.10) in Proposition 6.8 is a consequence of Cianchi-Maz'ya estimate and (6.12) in Lemma 6.10. In fact, if u is a weak solution of (6.1) for $f \in L^{n,1}(\Omega)$, then

$$-\Delta_p u = f - V(\cdot; u) \in L^{n,1}.$$

By Cianchi-Maz'ya result (see [14, Theorem 4.4]) and our estimate (6.12), we have

$$\|\nabla u\|_{L^\infty} \lesssim \|f - V(\cdot; u)\|_{L^{n,1}}^{\frac{1}{p-1}} \lesssim \|f\|_{L^{n,1}}^{\frac{1}{p-1}} + \|V(\cdot; u)\|_{L^{n,1}}^{\frac{1}{p-1}}$$

$$\begin{aligned}
&\lesssim \|f\|_{L^{n,1}}^{\frac{1}{p-1}} + \|f\|_{L^{n,1}}^{\frac{1}{p-1}} \cdot \|f\|_{L^1}^{\frac{m_1+1-p}{p-1}} \\
&\lesssim \left(1 + \|f\|_{L^1}^{\frac{m_1+1-p}{p-1}}\right) \|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}}.
\end{aligned}$$

7. THE HÖLDERIAN MAPPINGS FOR THE CASE $p \geq 2$

In this Section we apply our previous results to Hölderian mappings in order to obtain regularity results on the gradient of the solution of the equation (6.6):

$$-\operatorname{div}(|\nabla u|^{p-2} \nabla u) + V(x; u) = f \in L^1, \quad u = 0 \text{ on } \partial\Omega.$$

Recall again that, even if the next results remain valid in the case $p \geq n$, we will consider only the meaningful case

$$p < n.$$

For the case $p \geq n$, the number $p^* = \frac{np}{n-p}$ appearing in the case $p < n$, should be replaced by any finite number.

Recall also that

$$p' = \frac{p}{p-1}, \quad n' = \frac{n}{n-1}, \quad (p^*)' = \frac{np}{np - n + p}.$$

Theorem 7.1. *Let $2 \leq p < n$, $f_1, f_2 \in L^1(\Omega)$ and u_1, u_2 be the corresponding entropic-renormalized solution of (6.6). Then*

- (1) $\int_{\Omega} |\nabla T_k(u_1 - u_2)|^p dx \leq c \int_{\Omega} |f_1 - f_2| dx,$
- (2) $u_1, u_2 \in W_0^{1,1}(\Omega)$ and

$$\|\nabla(u_1 - u_2)\|_{L^{n'(p-1),\infty}(\Omega)} \leq c \|f_1 - f_2\|_{L^1(\Omega)}^{\frac{1}{p-1}},$$

where $n' = \frac{n}{n-1}$ and c is a constant depending only on p, Ω, V .

As a consequence we have

Corollary 7.2 (of Theorem 7.1). *Assume (H1) and (H2) and let u be the unique entropic-renormalized solution of the Dirichlet problem (6.6). Let $\mathcal{T}$ be the mapping $f \mapsto \mathcal{T}f$, with*

$$\mathcal{T}f = \nabla u.$$

For $2 \leq p < n$ there exists an extension of the mapping

$$\mathcal{T}: L^1(\Omega) \rightarrow [L^{n'(p-1),\infty}(\Omega)]^n,$$

which is $\frac{1}{p-1}$ -Hölderian, i.e. there exists $c(p, \Omega) > 0$ such that

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{L^{n'(p-1),\infty}} \leq c(p, \Omega) \|f_1 - f_2\|_{L^1(\Omega)}^{\frac{1}{p-1}},$$

where $n' = \frac{n}{n-1}$.

Remark 7.3. If $2 \leq p < n$ then $L^p(\Omega) \subset L^{n'(p-1),\infty}(\Omega)$, since $n' < p'$ and hence $n'(p-1) < p$.

Theorem 7.4. Assume (H1) and (H2) and let u be the unique entropic-renormalized solution of the Dirichlet problem (6.6). Let $\mathcal{T}$ be the mapping $f \mapsto \mathcal{T}f$, with

$$\mathcal{T}f = \nabla u.$$

For $2 \leq p < n$ the previous mapping in Corollary 7.2 is also $\frac{1}{p-1}$ -Hölderian from $L^{(p^*)}'$ into $[L^p(\Omega)]^n$, i.e.

$$\mathcal{T}: L^{(p^*)}' \rightarrow [L^p(\Omega)]^n,$$

which means that there exists $c_p > 0$ such that

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{L^p} \leq c_p \|f_1 - f_2\|_{L^{(p^*)}'}^{\frac{1}{p-1}},$$

where

$$p' = \frac{p}{p-1}, \quad p^* = \frac{np}{n-p}, \quad (p^*)' = \frac{np}{np-n+p}.$$

Proof. The proof is based on the strong coercivity of the p -laplacian and the Poincaré-Sobolev inequality. In fact, for $p \geq 2$, we recall that, from (6.2), there exists a constant $\alpha_p > 0$ such that for all $\xi \in \mathbb{R}^n$ and $\xi' \in \mathbb{R}^n$, we have that

$$\left(|\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi - \xi' \right)_{\mathbb{R}^n} \geq \alpha_p |\xi - \xi'|^p.$$

Therefore, for two data f_1 and f_2 in $L^{p'}(\Omega)$, dropping the non negative term, we have

$$\begin{aligned} c_p \int_{\Omega} |\mathcal{T}f_1 - \mathcal{T}f_2|^p dx &\leq \int_{\Omega} \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla(u_1 - u_2) \right) dx \\ &= - \int_{\Omega} (u_1 - u_2) \cdot \operatorname{div} \left(|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) dx \\ &= \int_{\Omega} \left[-\operatorname{div} (|\nabla u_1|^{p-2} \nabla u_1) \cdot (u_1 - u_2) + \right. \\ &\quad \left. + \operatorname{div} (|\nabla u_2|^{p-2} \nabla u_2) \cdot (u_1 - u_2) \right] dx \\ &= \int_{\Omega} \left[(f_1 - V(x, u_1)) \cdot (u_1 - u_2) + (-f_2 + V(x, u_2)) \cdot (u_1 - u_2) \right] dx \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} [(f_1 - f_2)(u_1 - u_2) + (V(x, u_2) - V(x, u_1)) \cdot (u_1 - u_2)] dx \\
&\leq \int_{\Omega} (f_1 - f_2)(u_1 - u_2) dx,
\end{aligned}$$

where in the last inequality we have used the monotonicity of V with respect to the second variable, which implies

$$(V(x, u_2) - V(x, u_1)) \cdot (u_1 - u_2) \leq 0.$$

By Poincaré–Sobolev inequality we have

$$\int_{\Omega} (f_1 - f_2)(u_1 - u_2) dx \leq c_{1p} \|f_1 - f_2\|_{L(p^*)'} \|\mathcal{T}f_1 - \mathcal{T}f_2\|_{L^p}$$

so that

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{L^p(\Omega)} \leq c_{2p} \|f_1 - f_2\|_{L(p^*)'}^{\frac{1}{p-1}}. \quad \square$$

Theorem 7.5. Assume (H1) and (H2) and let u be the unique entropic-renormalized solution of the Dirichlet problem (6.6). Let also

$$2 \leq p < n, \quad r \in [1, +\infty], \quad 1 \leq k \leq (p^*)', \quad \theta = p^* \left(1 - \frac{1}{k}\right).$$

(1) If $0 < \theta < 1$, i.e. $1 < k < (p^*)'$, then

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{L^{k^*(p-1), r(p-1)}} \leq c \|f_1 - f_2\|_{L^{k, r}}^{\frac{1}{p-1}},$$

for f_1, f_2 in $L^{k, r}(\Omega)$, with $k^* = \frac{nk}{n-k}$.

In particular, if $f \in L^{k, r}(\Omega)$, then $\nabla u \in [L^{k^*(p-1), r(p-1)}(\Omega)]^n$.

(2) If $\theta = 0$, i.e. $k = 1$, then

$$\mathcal{T}: G\Gamma(1, r; t^{-1}) \rightarrow G\Gamma(\infty, r(p-1); t^{-1}, t^{\frac{1}{n'(p-1)}})$$

is $\frac{1}{p-1}$ -Hölderian.

(3) If $\theta = 1$, i.e. $k = (p^*)'$, then

$$\mathcal{T}: G\Gamma((p^*)', r; v, w) \rightarrow G\Gamma(p, r(p-1); v_1, w_1)$$

is $\frac{1}{p-1}$ -Hölderian, where

$$v = t^{-1}(1 - \log t)^\gamma, \quad w = (1 - \log t)^\beta, \quad \gamma > -1,$$

$$\beta \in \mathbb{R}, \quad \gamma + \beta \frac{r}{(p^*)'} + 1 < 0,$$

$$v_1 = t^{-1}(1 - \log t)^{\gamma_1}, \quad w_1 = (1 - \log t)^{\beta_1}, \quad \gamma_1 > -1,$$

$$\beta_1 \in \mathbb{R}, \quad \gamma_1 + \frac{\beta_1 r(p-1)}{p} + 1 < 0$$

are such that

$$\frac{\gamma}{r} + \frac{\beta}{(p^*)'} = \frac{\gamma_1}{r} + \frac{\beta_1(p-1)}{p}. \quad (7.1)$$

Proof. Note that

$$\mathcal{T}: L^1(\Omega) \rightarrow [L^{n'(p-1),\infty}(\Omega)]^n \quad \text{is } \frac{1}{p-1}\text{-H\"olderian (by Corollary 7.2)}$$

$$\mathcal{T}: L^{(p^*)'}(\Omega) \rightarrow [L^p(\Omega)]^n \quad \text{is } \frac{1}{p-1}\text{-H\"olderian (by Theorem 7.4).}$$

It is known from (3.1) that, for $0 < \theta < 1$,

$$L^{k,r}(\Omega) = (L^1, L^{(p^*)'})_{\theta,r}, \quad \theta = p^*(1 - \frac{1}{k})$$

since

$$\frac{1}{k} = 1 - \theta + \frac{\theta}{(p^*)'} \Rightarrow \frac{1}{k} = 1 - \theta + \theta \left(1 - \frac{1}{p^*}\right) \Rightarrow 1 - \frac{1}{k} = \frac{\theta}{p^*}$$

and $0 < \theta < 1 \Rightarrow 1 < k < (p^*)'$.

Moreover, again from (3.1), we have

$$L^{k^*(p-1),r(p-1)} = \left(L^{n'(p-1),\infty}, L^p \right)_{\theta,r(p-1)},$$

in fact

$$\frac{1}{k^*(p-1)} = \frac{1-\theta}{n'(p-1)} + \frac{\theta}{p},$$

with $k^* = \frac{nk}{n-k}$, since $k < (p^*)' = \frac{np}{np-n+p} < n$.

It is easy to see that

$$\begin{aligned} \frac{1-\theta}{n'(p-1)} + \frac{\theta}{p} &= \frac{n-1}{n(p-1)} + \theta \left(\frac{1}{p} - \frac{n-1}{n(p-1)} \right) \\ &= \frac{n-1}{n(p-1)} + \theta \left(\frac{np-n-np+p}{np(p-1)} \right) \\ &= \frac{n-1}{n(p-1)} - \frac{\theta(n-p)}{np(p-1)} = \frac{np-p-\left(1-\frac{1}{k}\right)\frac{np}{n-p}(n-p)}{np(p-1)} \\ &= \frac{n-k}{nk} \cdot \frac{1}{p-1} = \frac{1}{k^*(p-1)}. \end{aligned}$$

Then, for f_1, f_2 in $L^{k,r}(\Omega)$, from Theorem 4.1 with $\lambda = 0$ and $\alpha = \frac{1}{p-1}$, $L^{(p^*)'} \subset L^1$, $L^p \subset L^{n'(p-1),\infty}$, we have

$$\mathcal{T}: (L^1, L^{(p^*)'})_{\theta,r} \rightarrow \left[(L^{n'(p-1),\infty}, L^p)_{\theta,r(p-1)} \right]^n$$

is also $\frac{1}{p-1}$ -H\"olderian, which implies

$$\mathcal{T}: L^{k,r}(\Omega) \rightarrow [L^{k^*(p-1),r(p-1)}(\Omega)]^n$$

and

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{\theta,r(p-1)} \lesssim \|f_1 - f_2\|_{\theta,r}^{\frac{1}{p-1}},$$

which yields

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{L^{k^*(p-1),r(p-1)}(\Omega)} \lesssim \|f_1 - f_2\|_{L^{k,r}(\Omega)}^{\frac{1}{p-1}},$$

since the bounded functions are dense in $L^{k,r}(\Omega)$.

For $\theta = 0$ we have from Theorem 4.1, with $\lambda = 0$ and $\alpha = \frac{1}{p-1}$,

$$\mathcal{T}: (L^1, L^{(p^*)}')_{0,r} \rightarrow \left[(L^{n'(p-1),\infty}, L^p)_{0,r(p-1)} \right]^n$$

and the assertion follows by (3.8) and (3.10), since

$$\begin{aligned} (L^1, L^{(p^*)}')_{0,r} &= G\Gamma(1, r; t^{-1}), \\ \left(L^{n'(p-1),\infty}, L^p \right)_{0,r(p-1)} &= G\Gamma(\infty, r(p-1); t^{-1}, t^{\frac{1}{n'(p-1)}}). \end{aligned}$$

For $\theta = 1$ we have from Theorem 4.1, with $\lambda < -\frac{1}{r}$ and $\alpha = \frac{1}{p-1}$, that

$$\mathcal{T}: (L^1, L^{(p^*)}')_{1,r;\lambda} \rightarrow \left[(L^{n'(p-1),\infty}, L^p)_{1,r(p-1);\frac{\lambda}{p-1}} \right]^n$$

and the assertion follows by (3.11), since

$$\begin{aligned} (L^1, L^{(p^*)}')_{1,r;\lambda} &= G\Gamma((p^*)', r; t^{-1}(1 - \log t)^\gamma, (1 - \log t)^\beta), \\ \left(L^{n'(p-1),\infty}, L^p \right)_{1,r(p-1);\frac{\lambda}{p-1}} &= G\Gamma(p, r(p-1); t^{-1}(1 - \log t)^{\gamma_1}, (1 - \log t)^{\beta_1}), \end{aligned}$$

where

$$\lambda = \frac{\gamma}{r} + \frac{\beta}{(p^*)'}, \quad \gamma > -1, \quad \beta \in \mathbb{R}, \quad \gamma + \beta \frac{r}{(p^*)'} + 1 < 0,$$

so that

$$\begin{aligned} \lambda < -1/r, \quad \frac{\lambda}{p-1} &= \frac{\gamma_1}{r(p-1)} + \frac{\beta_1}{p}, \quad \gamma_1 > -1, \\ \beta_1 &\in \mathbb{R}, \quad \gamma_1 + \beta_1 \frac{r(p-1)}{p} + 1 < 0, \end{aligned}$$

whence $\frac{\lambda}{p-1} < -\frac{1}{r(p-1)}$, i.e. $\lambda < -1/r$, with the condition (7.1):

$$\frac{\gamma}{r} + \frac{\beta}{(p^*)'} = \frac{\gamma_1}{r} + \frac{\beta_1(p-1)}{p}, \quad \text{with} \quad (p^*)' = \frac{np}{np-n+p}. \quad \square$$

Remark 7.6. In the case $k = r \in (1, (p^*)')$ we improve previous known results, in fact the usual estimate was only obtained in $[L^{r^*(p-1)}(\Omega)]^n$ (see [13]) and $L^{r^*(p-1),r(p-1)}(\Omega) \subset L^{r^*(p-1)}(\Omega)$.

Remark 7.7. If $p = \frac{2n}{n+1}$, we have $p < n$ and $(p^*)' = \frac{np}{np-n+p} = p$. Therefore, in this particular case, the condition (7.1) is certainly satisfied if $\gamma = \gamma_1$ and $\beta = \beta_1(p-1)$.

The identification of interpolation spaces between couples of Lebesgue or Lorentz spaces, recovering spaces such as Lorentz–Zygmund spaces or GF spaces, permits us to obtain precise regularity of the gradient of an entropic-renormalized solution.

Theorem 7.8. *Let $2 \leq p < n$. Assume (H1) and (H2) and let u be the unique entropic-renormalized solution of the Dirichlet problem (6.6). Let also $\mathcal{T}$ the mapping $f \mapsto \mathcal{T}f$, with*

$$\mathcal{T}f = \nabla u.$$

(1) *Let $0 < \theta < 1$, $1 \leq q < \infty$, $\lambda \in \mathbb{R}$, $\frac{1}{p_\theta} = \frac{(1-\theta)}{n'(p-1)} + \frac{\theta}{p}$, $n' = \frac{n}{n-1}$. Then*

$$\mathcal{T}: L_{p^{*-}\theta}^{p^*,q}(\log L)^\lambda \rightarrow [L^{p_\theta,q(p-1)}(\log L)^{\frac{\lambda}{p-1}}]^n$$

is $\frac{1}{p-1}$ -Hölderian, where $p^ = \frac{np}{n-p}$.*

(2) *Let $\theta = 0$ and $\lambda \geq -\frac{1}{p}$. Then*

$$\mathcal{T}: GF(1, p; t^{-1}(1 - \log t)^{\lambda p}) \rightarrow (L^{n'(p-1),\infty}(\Omega), L^p(\Omega))_{0,p(p-1);\frac{\lambda}{p-1}}$$

is $\frac{1}{p-1}$ -Hölderian. Moreover, observing that

$$\begin{aligned} (L^{n'(p-1),\infty}; L^p)_{0,p(p-1);\frac{\lambda}{p-1}} &= \\ &= GF(\infty, p(p-1); t^{-1}(1 - \log t)^{\lambda p}, t^{\frac{1}{n'(p-1)}}), \end{aligned}$$

an equivalent norm of ∇u , for σ such that $\frac{1}{\sigma} = \frac{p'}{pn'} - \frac{1}{p}$, is given by

$$\begin{aligned} \|\nabla u\|_{(L^{n'(p-1),\infty}(\Omega), L^p(\Omega))_{0,p(p-1);\frac{\lambda}{p-1}}} &\approx \\ &\approx \left[\int_0^1 \left(\sup_{0 < s < t^\sigma} s^{\frac{p'}{pn'}} |\nabla u|_*(s) (1 - \log t)^{\frac{\lambda}{p-1}} \right)^{p(p-1)} \frac{dt}{t} \right]^{\frac{1}{p(p-1)}}. \end{aligned}$$

(3) *Let $\theta = 1$ and $\lambda < -\frac{1}{p}$. Then*

$$\mathcal{T}: (L^1(\Omega), L^{(p^*)'}(\Omega))_{1,p;\lambda} \rightarrow (L^{n'(p-1),\infty}(\Omega), L^p(\Omega))_{1,p(p-1);\frac{\lambda}{p-1}}$$

is $\frac{1}{p-1}$ -Hölderian and we have

$$\left[\int_0^1 \left(\left(\int_t^1 |\nabla u|_*(s)^p ds \right)^{\frac{1}{p}} (1 - \log t)^{\frac{\lambda}{p-1}} \right)^{p(p-1)} \frac{dt}{t} \right]^{\frac{1}{p(p-1)}} \leq$$

$$\leq c \left[\int_0^1 \left(\left(\int_t^1 f_*(s)^{(p^*)'} ds \right)^{\frac{1}{(p^*)'}} (1 - \log t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}$$

Proof. Let $0 < \theta < 1$. Then

$$\begin{aligned} \mathcal{T}: L^1(\Omega) &\longrightarrow [L^{n'(p-1),\infty}(\Omega)]^n && \text{is } \frac{1}{p-1}\text{-H\"olderian (by Corollary 7.2),} \\ \mathcal{T}: L^{(p^*)'}(\Omega) &\longrightarrow [L^p(\Omega)]^n && \text{is } \frac{1}{p-1}\text{-H\"olderian (by Theorem 7.4)} \end{aligned}$$

Since smooth functions are dense in the Lorentz-Zygmund spaces $L^{p,q}(\log L)^\lambda$, $1 \leq p, q < \infty$, it follows that

$$\mathcal{T}: (L^1(\Omega), L^{(p^*)'}(\Omega))_{\theta,q;\lambda} \rightarrow (L^{n'(p-1),\infty}(\Omega), L^p(\Omega))_{\theta,\frac{q}{\alpha};\lambda\alpha}$$

is $\alpha = \frac{1}{p-1}$ -H\"olderian.

Moreover, we identify

$$\begin{aligned} (L^1(\Omega), L^{(p^*)'}(\Omega))_{\theta,q;\lambda} &= L^{\frac{p^*}{p^*-\theta},q}(\log L)^\lambda = L^{\frac{np}{np-\theta(n-p)},q}(\log L)^\lambda \quad (\text{by (3.5)}), \\ (L^{n'(p-1),\infty}; L^p)_{\theta,\frac{q}{\alpha};\lambda\alpha} &= L^{p\theta,\frac{q}{\alpha}}(\log L)^{\lambda\alpha}, \quad \frac{1}{p\theta} = \frac{1-\theta}{n'(p-1)} + \frac{\theta}{p} \quad (\text{by (3.4)}). \end{aligned}$$

Then, by Theorem 4.1 with $\alpha = \frac{1}{p-1}$, we get that

$$\mathcal{T}: L^{\frac{p^*}{p^*-\theta},q}(\log L)^\lambda \rightarrow \left[L^{p\theta,q(p-1)}(\log L)^{\frac{\lambda}{p-1}} \right]^n \quad \text{is } \frac{1}{p-1}\text{-H\"olderian.}$$

In the case $\theta = 0$, $\lambda \geq -\frac{1}{p}$, we get from (3.8) that

$$(L^1(\Omega), L^{(p^*)'}(\Omega))_{0,p;\lambda} = G\Gamma(1, p; t^{-1}(1 - \log t)^{\lambda p})$$

and $L^{(p^*)'}$ is dense therein. By (3.10),

$$(L^{n'(p-1),\infty}; L^p)_{0,p(p-1);\frac{\lambda}{p-1}} = G\Gamma(\infty, p(p-1); t^{-1}(1 - \log t)^{\lambda p}, t^{\frac{1}{n'(p-1)}}).$$

Therefore, by Theorem 4.1, with $\alpha = \frac{1}{p-1}$, the mapping

$$\mathcal{T}: (L^1(\Omega), L^{(p^*)'}(\Omega))_{0,p;\lambda} \rightarrow (L^{n'(p-1),\infty}; L^p)_{0,p(p-1);\frac{\lambda}{p-1}}$$

is $\frac{1}{p-1}$ -H\"olderian. Due to the identification of the above interpolation spaces, we have

$$\mathcal{T}: G\Gamma(1, p; t^{-1}(1 - \log t)^{\lambda p}) \rightarrow G\Gamma(\infty, p(p-1); t^{-1}(1 - \log t)^{\lambda p}, t^{\frac{1}{n'(p-1)}})$$

and the assertion follows.

The same argument holds for $\theta = 1$. □

Remark 7.9. Observe that, in the case $0 < \theta < 1$, for $\lambda = 0$, we recover the conclusion of Theorem 7.5, in fact $k = \frac{p^*}{p^* - \theta}$ and $\frac{1}{k^*(p-1)} = \frac{1}{p_\theta}$, so in this case, for $q = r$, we have

$$L^{\frac{p^*}{p^* - \theta}, q}(\log L)^\lambda = L^{k, r}, \quad L^{p_\theta, q(p-1)}(\log L)^{\frac{\lambda}{p-1}} = L^{k^*(p-1), r(p-1)},$$

while, for $\lambda \neq 0$, we have

$$L^{\frac{p^*}{p^* - \theta}, q}(\log L)^\lambda \subset L^{k, r}, \quad L^{p_\theta, q(p-1)}(\log L)^{\frac{\lambda}{p-1}} \subset L^{k^*(p-1), r(p-1)} \quad \text{if } \lambda > 0,$$

$$L^{k, r} \subset L^{\frac{p^*}{p^* - \theta}, q}(\log L)^\lambda, \quad L^{k^*(p-1), r(p-1)} \subset L^{p_\theta, q(p-1)}(\log L)^{\frac{\lambda}{p-1}} \quad \text{if } \lambda < 0,$$

(see inclusion relations in Section 2).

Remark 7.10. We also observe that, by (3.11),

$$(L^1, L^{(p^*)'})_{1, p; \lambda} = G\Gamma((p^*)', p; t^{-1}(1 - \log t)^{\gamma_1}, (1 - \log t)^{\beta_1}),$$

$$(L^{n'(p-1), \infty}, L^p)_{1, p(p-1); \frac{\lambda}{p-1}} = G\Gamma(p, p(p-1); t^{-1}(1 - \log t)^{\gamma_2}, (1 - \log t)^{\beta_2}),$$

where

$$\lambda = \frac{\gamma_1}{p} + \frac{\beta_1}{(p^*)'}, \quad \gamma_1 > -1, \quad \beta_1 \in \mathbb{R}, \quad \gamma_1 + \beta_1 \frac{p}{(p^*)'} + 1 < 0, \quad \lambda < -\frac{1}{p},$$

$$\lambda = \frac{\gamma_2}{p} + \frac{\beta_2(p-1)}{p}, \quad \gamma_2 > -1, \quad \beta_2 \in \mathbb{R}, \quad \gamma_2 + \beta_2(p-1) + 1 < 0, \quad \lambda < -\frac{1}{p},$$

$$\gamma_1 - \gamma_2 + (\beta_1 - \beta_2)(p-1) + \beta_1 \frac{p}{n} = 0.$$

The last condition follows by the equality

$$\frac{\gamma_1}{p} + \frac{\beta_1}{(p^*)'} = \frac{\gamma_2}{p} + \frac{\beta_2(p-1)}{p}, \quad \text{with } (p^*)' = \frac{np}{np-n+p}.$$

To obtain the boundedness of the solution in a more general situation, stated in next Theorem, we need to assume the growth condition (H3).

Theorem 7.11. *Let $2 \leq p < n$. Assume that (H1), (H2), and (H3) hold, and let u the entropic-renormalized solution of the Dirichlet problem (6.6). Let also $\mathcal{T}$ be the mapping $f \mapsto \mathcal{T}f$, with*

$$\mathcal{T}f = \nabla u,$$

$1 < q < +\infty$, $\lambda \in \mathbb{R}$, and $0 \leq \theta < 1$.

- If $f \in L^{\frac{n'}{n' - \theta}, q}(\log L)^\lambda$, $0 < \theta < 1$, then

$$\nabla u \in L^{\frac{n(p-1)}{(1-\theta)(n-1)}, q(p-1)}(\log L)^{\frac{\lambda}{p-1}}.$$

- If $f \in G\Gamma(1, q; t^{-1}(1 - \log t)^{\lambda q})$, then

$$\nabla u \in G\Gamma(\infty, q(p-1); t^{-1}(1 - \log t)^{\lambda q}, t^{\frac{1}{n'(p-1)}}).$$

Proof. Note that

$$\begin{aligned}\mathcal{T}: L^1(\Omega) &\rightarrow [L^{n'(p-1),\infty}(\Omega)]^n \quad \text{is } \frac{1}{p-1}\text{-H\"olderian (by Corollary 7.2),} \\ \mathcal{T}: L^{n,1}(\Omega) &\rightarrow [L^\infty(\Omega)]^n, \quad \text{is bounded (by (6.10) in Proposition 6.8).}\end{aligned}$$

Using Theorem 4.1, that is the fact that

$$\mathcal{T}: (X_0, X_1)_{\theta, q; \lambda} \rightarrow (Y_0, Y_1)_{\theta, \frac{\alpha}{\beta}, \frac{q}{\alpha}; \lambda \alpha}$$

is bounded, for $\beta = \alpha = \frac{1}{p-1}$, we have that

$$\mathcal{T}: (L^1(\Omega), L^{n,1}(\Omega))_{\theta, q; \lambda} \rightarrow (L^{n'(p-1),\infty}(\Omega), L^\infty(\Omega))_{\theta, q(p-1); \frac{\lambda}{p-1}}$$

is bounded.

If $0 < \theta < 1$, we get from (3.3) and (3.6) respectively, that

$$\begin{aligned}(L^1, L^{n,1})_{\theta, q; \lambda} &= L^{\frac{n'}{n'-\theta}, q}(\log L)^\lambda \\ (L^{n'(p-1),\infty}, L^\infty)_{\theta, q(p-1); \frac{\lambda}{p-1}} &= L^{\frac{n'(p-1)}{1-\theta}, q(p-1)}(\log L)^{\frac{\lambda}{p-1}}.\end{aligned}$$

If $\theta = 0$, then it follows respectively from (3.7) and (3.10) that

$$\begin{aligned}(L^1(\Omega), L^{n,1}(\Omega))_{0, q; \lambda} &= G\Gamma(1, q; t^{-1}(1 - \log t)^{\lambda q}), \\ (L^{n'(p-1),\infty}, L^p)_{0, q(p-1); \frac{\lambda}{p-1}} &= G\Gamma(\infty, q(p-1); t^{-1}(1 - \log t)^{\lambda q}, t^{\frac{1}{n'(p-1)}}).\square\end{aligned}$$

Remark 7.12. Recall that, as pointed out at the beginning of Section 7, all the previous results are also true for $p \geq n$, and in this case the symbol p^* must be considered as any finite number. Hence, if in (1) of Theorem 7.8 we assume $p \geq n$ and choose $p^* = n'$, then comparing the result with the first one in Theorem 7.11 for $p \geq n$, we see that the datum f is in the same space $L^{\frac{n'}{n'-\theta}, q}(\log L)^\lambda$, while the gradient of the solution u in Theorem 7.11 belongs to a smaller space, since

$$L^{\frac{n(p-1)}{(1-\theta)(n-1)}, q(p-1)}(\log L)^{\frac{\lambda}{p-1}} \subset L^{p\theta, q(p-1)}(\log L)^{\frac{\lambda}{p-1}}$$

with

$$\frac{1}{p\theta} = \frac{(1-\theta)}{n'(p-1)} + \frac{\theta}{p}.$$

Moreover, since

$$\frac{n(p-1)}{(1-\theta)(n-1)} \geq \frac{n}{1-\theta} > n$$

for $p \geq n$ and $0 < \theta < 1$, we obtain that $\nabla u \in L^r$, $r > n$. Then, by Sobolev theorem, the solution u is bounded.

8. THE HÖLDERIAN MAPPINGS FOR THE CASE $1 < p < 2$

Some of results for $2 \leq p < n$ remain true in the case $1 < p < 2$. The fundamental changes concern the Hölder properties than can exist but are not sharp as for the case $p \geq 2$, and the Hölder constant appearing depends on the data.

Theorem 8.1 (Local Lipschitz contraction). *Let*

$$1 < p < 2, \quad n \geq 2, \quad p^* = \frac{np}{n-p}, \quad (p^*)' = \frac{np}{np+p-n}.$$

Suppose also that $f_1, f_2 \in L^{(p^)'}(\Omega)$ and V satisfy (H1) and (H2). Further, let $u_i, i = 1, 2$, be the **weak solution** of*

$$-\Delta_p u_i + V(x; u_i) = f_i, \quad u_i = 0 \text{ on } \partial\Omega.$$

Then

$$\|\nabla u_1\|_{L^p} \leq c \|f_1\|_{L^{(p^*)}'}^{\frac{1}{p-1}}, \quad \|\nabla u_2\|_{L^p} \leq c \|f_2\|_{L^{(p^*)}'}^{\frac{1}{p-1}},$$

$$\|\nabla(u_1 - u_2)\|_{L^p} \leq c \left(\|\nabla u_1\|_{L^p} + \|\nabla u_2\|_{L^p} \right)^{2-p} \|f_1 - f_2\|_{L^{(p^*)}'}.$$

Here the constant c depends only on p and Ω .

Corollary 8.2 (of Theorem 8.1). *Under the same assumptions as in Theorem 8.1, there exists a constant c depending only on p and Ω such that*

$$\|\nabla(u_1 - u_2)\|_{L^p} \leq c \left(\|f_1\|_{L^{(p^*)}'}^{\frac{1}{p-1}} + \|f_2\|_{L^{(p^*)}'}^{\frac{1}{p-1}} \right)^{2-p} \|f_1 - f_2\|_{L^{(p^*)}'}. \quad (8.1)$$

In particular,

$$\mathcal{T}: L^{(p^*)}'(\Omega) \rightarrow [L^p(\Omega)]^n \quad \text{is locally Lipschitz.}$$

Theorem 8.3. *Let $1 < p < 2, r \in [1, +\infty]$ and k be such that $(p^*)' < k < n$. Assume also that (H1), (H2), and (H3) hold. Let u be the **entropic-renormalized solution** of the equation*

$$-\Delta_p u + V(x; u) = f, \quad u = 0 \text{ on } \partial\Omega.$$

Then, the nonlinear mapping $\mathcal{T}$ is bounded from $L^{k,r}(\Omega)$ into $L^{k_1,r}(\Omega)$, with

$$k_1 = \frac{p}{1-\theta(p-1)}, \quad \theta = \frac{\frac{1}{(p^*)'} - \frac{1}{k}}{\frac{1}{(p^*)'} - \frac{1}{n}} \in (0, 1).$$

Proof. From Corollary 8.2 we have that

$$\mathcal{T}: L^{(p^*)}'(\Omega) \rightarrow [L^p(\Omega)]^n \quad \text{is locally Lipschitz}$$

and (8.1) holds. By Proposition 6.8 we also have that

$$\mathcal{T}: L^{n,1}(\Omega) \rightarrow [L^\infty(\Omega)]^n \text{ is bounded.}$$

Further, from Theorem 4.2 with $\lambda = 0$, $\alpha = 1$ and $\beta = \frac{1}{p-1}$ we get that

$$\mathcal{T} \text{ maps } (X_0, X_1)_{\theta, r; \lambda} \text{ into } (Y_0, Y_1)_{\theta \frac{\alpha}{\beta}, \frac{r}{\alpha}; \lambda \alpha},$$

i.e.

$$\mathcal{T}: (L^{(p^*)'}(\Omega), L^{n,1}(\Omega))_{\theta, r} \rightarrow (L^p(\Omega), L^\infty(\Omega))_{\theta(p-1), r}$$

is a locally bounded mapping, with $\theta \in (0, 1)$. Now the assertion follows from (3.1), since

$$\begin{aligned} (L^{(p^*)'}(\Omega), L^{n,1}(\Omega))_{\theta, r} &= L^{k, r}(\Omega), & \frac{1}{k} &= \frac{1-\theta}{(p^*)'} + \frac{\theta}{n}, \\ (L^p(\Omega), L^\infty(\Omega))_{\theta(p-1), r} &= L^{k_1, r}(\Omega), & \frac{1}{k_1} &= \frac{1-\theta(p-1)}{p}. \end{aligned} \quad \square$$

For other results concerning equations with data in Lorentz spaces, see, e.g. [22, 37].

9. CONCLUSION

We conclude by highlighting that in [2] also applications to the anisotropic equation and variable exponents version of the p -Laplacian are given. We refer the reader to the paper [2] for the results.

Anisotropic equation.

$$\begin{cases} -\Delta_{\vec{p}} u + V(x; u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\Delta_{\vec{p}} u = - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right),$$

$\vec{p} = (p_1, \dots, p_n)$, $1 < p_i < +\infty$, $\vec{p}' = (p'_1, \dots, p'_n)$, p'_i is the conjugate of p_i .

Variable exponents version of the p -Laplacian.

$$\begin{cases} -\Delta_{p(\cdot)} u + V(x; u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{p(\cdot)} u = \operatorname{div} (|\nabla u|^{p(x)-2} \nabla u)$.

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