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# Optimal codimension one gradient flows on closed surfaces

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**Abstract.** We consider codimension one gradient flows on closed surfaces with minimal number of singular points. There are two types of such flows: having saddle-node ($SN$) and having saddle connection ($SC$). We use chord diagrams of special types to classify such flows up to topological trajectory equivalence. Namely, up to a topological equivalence each $SN$-flow is determined by a certain chord diagrams with a marked arc, while each $SC$-flow is determined by a chord diagram with a distinguished T-subgraph. We list all such diagrams for flows on orientable surfaces of genus at most 2 and nonorientable surfaces of genus at most 3, and also find the corresponding diagrams of the inverse flows.

## 1. Introduction

In 1970, Palis and Smale [16] proved that the set of Morse-Smale gradient vector fields on a smooth manifold $M$ is open and dense in the space of all gradient fields $\mathrm{Grad}(M)$ with a $C^k$ topology ($k \geqslant 3$). Gradient vector fields of codimension 1 are the vector fields arising in typical 1-parameter families of gradient vector fields and they are not Morse-Smale fields. In 1983, Palis and Takens [17] proved that such vector fields are of the following two types: SN-fields, which contain one saddle-node singularity, and $SC$-fields, which contain one saddle connection, and except for those singularities, they have the same properties as Morse-Smale gradient fields. In particular, they are structurally stable in the set $\mathrm{Grad}(M)\setminus \mathrm{MS}(M)$.

Modern textbooks on bifurcation theory or qualitative theory of differential equations describe all possible local and global bifurcations of codimension 1 and local bifurcations of codimension 2. Gradient flows of codimension 1 on compact surfaces are described in [8, Theorem I, p.4794].

It is well known that a vector field on a closed manifold always generates a flow, while on compact manifolds with boundary, a vector field generates a flow only if it is tangent to the boundary. We will say that a smooth flow is *gradient*, (resp. *Morse-Smale*, *of codimension one*) if it is generated

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by a vector field having the corresponding property. On manifolds with boundary, Morse-Smale flows are described in [11], and flows of codimension one are discussed in [8]. Their structural stability in the corresponding spaces, as well as their openness and density in the corresponding spaces, are also proved.

Often, flows of special types (up to a topological equivalence) are clasified by certain graphs having special properties and endowed with additional information. That means that two such flows are topologically equivalent if and only if there is an isomorphism between the graphs which preserves the corresponding information. These invariants are rather effective and allow describe all topological types of flows with a given number of singular points.

For instance, Morse-Smale flows are classified by the so-called *separatrix diagram*, see [2, 12, 15, 18].

In the present paper we simplify this invariant for *Morse flows* (Morse-Smale flows without closed trajectories) which are topologically equivalent to gradient flows of Morse functions. Then the corresponding separatrix diagram will be a certain graph embedded into a surface. Its vertices are sources and its edges are stable manifolds of saddles.

Rotation system [3] is often used for defining a graph embedding into a surface. This is, in fact, a Peixoto invariant [18] for Morse flows. In the case of a one-source flow it is convenient to use chord diagrams to define Peixoto invariant, [7]. To define a chord diagram one can also use *codes* which are written down from a fixed enumeration of chords. Namely, a code then consists of the indexes of chords whose endpoints meet when traversing the circle. The ambiguity in numbering the chords or a starting point complicates the comparison of the constructed invariants. Also inverse diagrams were constructed for inverse flows on a surface [7, Theorem 7.2, p. 20].

Besides that, complete topological invariants of flows with gradient dynamics were constructed in dimensions 2, [22] and 3, [1], and for collective dynamics in dimension 2, [19, 20].

The purpose of this paper is to construct a complete invariant for codimension one flows on closed surfaces which resembles a chord diagram for Morse flows. This invariant has a marked point in the diagram, which allows us to define clearly a number code of the flow.

In Section 2 we consider Morse-Smale flows on 2-manifolds, as well as optimal Morse flows and their complete topological invariants. We also define inverse diagrams for inverse flows.

In Section 3 we investigate gradient codimension one flows on closed 2-manifolds which can be divided into two types – $SN$-flows (saddle-node flows) and $SC$-flows (saddle connection flows). We construct the appropriate topological invariants for such flows.

All the proofs of the theorems and statements from Section 3 are given in Section 4.

In Section 5 we describe all the possible structures of optimal $SN$-flows and $SC$-flows with at most 4 saddles on both oriented and non-oriented surfaces.

## 2. Preliminaries

**Flows and vector fields on closed smooth manifolds.** Let $M$ be a closed smooth manifold.

**Definition 2.1.** A smooth mapping $g\colon M \times \mathbb{R} \to M$ is called a *smooth dynamical system*, or a *flow*, on the manifold $M$ if:

(1) $g(x, 0) = x$ for each $x \in M$;

(2) $g(x, t + s) = g(g(x, s), t)$ for all $t, s \in \mathbb{R}$, $x \in M$.

It follows from this definition that for each $t \in \mathbb{R}$ the mapping

$$G\colon M \to M, \quad G(x) = g(x, t), \quad x \in M,$$

is a diffeomorphism.

**Definition 2.2.** A *stable manifold* (denoted by $W^s(p)$) of a singular point $p$ of a flow $g$ is the following set

$$W^s(p) = \{x \in M : \lim_{t\to\infty} g(x, t) = p\}.$$

Besides, the set

$$W^u(p) = \{x \in M : \lim_{t\to-\infty} g(x, t) = p\}$$

is called an *unstable manifold* (denoted by $W^u(p)$) of $p$.

**Definition 2.3.** A point $p \in M$ is called *non-wandering* for a flow $g$ if for any $T \in \mathbb{R}$ and any neighbourhood $U$ of $x$ there exists $t > T$ such that $g(U, t) \cap U \neq \varnothing$. Otherwise the point is said to be *wandering*.

The set of non-wandering points of the vector field $X$ (flow $g$) is denoted by $\Omega(g)$. Besides, an alpha-limit set of a point $x \in M$ (or of the trajectory it lies within) we denote with the symbol $\alpha(x)$, whereas an omega-limit of this point we denote with the symbol $\omega(x)$.

**Definition 2.4.** Suppose $M_1$ and $M_2$ are smooth closed manifolds. Two flows $g_i\colon M_i \times \mathbb{R} \to M$, $i = 1, 2$, are *topologically* or *trajectory equivalent*, if there exists a homeomorphism $h\colon M_1 \to M_2$ which maps each trajectory of the first flow onto a trajectory of the second one and preserves the natural orientation of these trajectories.

**Definition 2.5.** A *gradient flow* on $M$ is a flow which is topologically equivalent to a flow generated by a gradient vector field.

Note that gradient flows have no closed trajectories.

**Definition 2.6.** A vector field $X$ (flow $g$) on a compact manifold $M$ is a *Morse vector field* (*flow*), or a *gradient Morse-Smale vector field* (*flow*) if:

(i) it has a finite number of singular points and all of them are hyperbolic;

(ii) if $p$ and $q$ are singular points, then $W^s(p)$ and $W^u(q)$ have a transversal intersection;

(iii) the set of non-wandering points $\Omega(g)$ coincides with the union of all the singular points of $g$.

If the manifold has dimension 2, then condition (i) means that each singular point is a source, a hyperbolic saddle, or a sink, while condition (ii) means that the flow has no saddle connections (saddle connection is such a trajectory $\gamma$ for which $\alpha(\gamma)$ and $\omega(\gamma)$ are saddles).

It should be noted that for gradient flows on closed manifolds, condition (iii) from the definition of Morse flow follows from conditions (i) and (ii).

**Chord diagrams for optimal Morse flows on closed smooth surfaces.** Morse flows on connected surfaces with only one source can be studied by chord diagrams. But first we will explain a notion of *separatrix* which is crucial for understanding how chord diagrams are constructed.

Suppose that we have a smooth closed surface (2-manifold) $M$ and a gradient vector field $X\colon M \to TM$ of a smooth function $f\colon M \to \mathbb{R}$ for some Riemannian metric on $M$.

**Definition 2.7.** Let $p$ be a fixed point of a smooth vector field $X$ (flow $g$) on a closed smooth surface $M$. A closed subset $D$ of $M$ is called *a hyperbolic sector in the neighbourhood of $p$* if there is a homeomorphism $h\colon D \to S$, where

$$S = \{(x, y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\},$$

which maps intersections of trajectories of the flow $g$ on $M$ with $D$ into intersections of trajectories of the flow $g_1$ on $\mathbb{R}^2$, generated by the vector field $V = \{x, -y\}$ on $\mathbb{R}^2$, with $S$.

Speaking in the terms of the previous definition, we are ready to formulate a definition of separatrix.

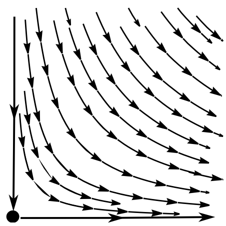


FIGURE 2.1. A hyperbolic sector in the neighbourhood of a fixed point

**Definition 2.8.** Let $g$ be a flow on a surface $M$ and $p$ its fixed point with a hyperbolic sector $D$ in its neighbourhood. A non-closed trajectory $\gamma$ of a flow $g$ is called a *separatrix* if it is contained either in the stable or in the unstable manifold of $p$ so that $\gamma \subset h^{-1}(S_0)$ where $S_0$ is either a set $\{(x, y) \in S \mid x = 0,\ y > 0\}$ or a set $\{(x, y) \in S \mid y = 0,\ x > 0\}$.

We can see an example of a hyperbolic sector on Figure 2.1. Furthermore, there are two separatrices here which lie along left and lower side of the square.

**Definition 2.9.** A *separatrix diagram* of a flow $g$ on a surface $M$ is this surface with embedded set of all the separatrices and the singular points of the flow $g$ on it.

We use the following notations:

(1) $\Omega^0$ is the set of sources, and the sources are $v_1, v_2, \ldots$;
(2) $\Omega^1$ is the set of saddles, and the saddles are $s_1, s_2, \ldots$;
(3) $\Omega^2$ is the set of sinks, and the sinks are $w_1, w_2, \ldots$

Thus, if $g$ is a Morse flow on a closed surface, then

$$\Omega(g) = \Omega^0 \cup \Omega^1 \cup \Omega^2.$$

If $\gamma$ is the separatrix of the Morse flow, then there exists $s_i \in \Omega^1$ such that $\gamma \subset W^s(s_i)$ (then $\omega(\gamma) = s_i$), or $\gamma \subset W^u(s_i)$ (then $\alpha(\gamma) = s_i$). If $x$ is a critical point of a Morse function and Morse index of $x$ is equal $k$, then $x \in \Omega^k$ for gradient vector field $X = \operatorname{grad} f$.

Now we are ready to explain the concept of chord diagrams.

**Definition 2.10.** Suppose that $n \in \{0\} \cup \mathbb{N}$. A configuration consisting of a unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$, $2n$ distinct points on it, and $n$ chords, which specify a partition of these points into pairs, are called *a chord diagram with n chords*, or *n-diagram*.

In fact, a chord diagram is a degree 3 topological graph with a fixed Hamiltonian cycle. Edges of a graph which belong to this cycle are called *arcs* whereas the other edges are called *chords*.

**Definition 2.11.** A chord diagram is *marked* if some of its chords are marked by the same symbol, say $X$, and others are not marked.

**Definition 2.12.** Two (marked) chord diagrams are said to be *equivalent* if there is an isomorphism between them (as of topological graphs) which sends circle to circle, unmarked (resp. marked) chords to unmarked (resp. marked) chords. If such an isomorphism additionally preserves the orientation of the circle, then we call such diagrams *isomorphic.*

**Definition 2.13.** A Morse flow on a surface is called *optimal* if it has the least number of singularities among all the Morse flows on this surface.

A Morse flow on a closed surface is optimal if and only if it contains only one sink and one source [7, Theorem 4.1, p.16].

Assume that $X$ is a vector field on a closed connected surface $M$ and the flow $g$ of $X$ has a unique source $v \in \Omega^0$. We will now associate a certain *marked chord diagram* to $X$.

First note that there exists a closed neighbourhood $U(v)$ of $v$ such that $\partial U(v)$ is a simple closed smooth curve, transversal to each trajectory.

Let $C$ be the union of all the separatrices which have $v$ as their alpha-limit set. Then the set $(C \backslash \overline{U(v)}) \cup \partial U(v)$ is a *chord diagram* of this Morse flow, and parts of separatrices which go into the same saddle form its *chords.*

If a stable manifold of a saddle *reverses* orientation, so that its thin enough neighbourhood is actually a Möbius strip, then we additionally mark the appropriate chord just putting a symbol $X$ on this chord. Otherwise, when a separatrix *preserves* orientation, we do not mark that chord.

The examples of chord diagrams are given in Figures 5.2 and 5.3.

Let's assume that we have a circle with $n$ chords so that the ends of these chords do not intersect. So we have a set $S = \{x_0, \ldots, x_{2n-1}\}$ of $2n$ points on circle. Assume that these points are numbered in counterclockwise direction.

Now we define three bijections on $S$:

(i) $\xi_+ : S \to S$, $\xi_+(x_i) = x_{i+1 (\mathrm{mod}\, 2n)}$, $i \in \{0, 1, \ldots, 2n-1\}$;

(ii) $\xi_- : S \to S$, $\xi_-(x_i) = x_{i-1 (\mathrm{mod}\, 2n)}$, $i \in \{0, 1, \ldots, 2n-1\}$;

(iii) $\pi : S \to S$ is a permutation which maps one end of a chord to the other end of it.

**Definition 2.14.** A cycle

$$q_0\ q_1\ \cdots\ q_{2m-2}\ q_{2m-1}\ q_{2m} = q_0,\ 1 \leqslant m \leqslant n,$$

in a chord diagram (as in a graph) is called a *chord cycle* if for $i = \overline{1, m}$:

(1) $q_{2i-1} = \pi(q_{2i-2})$;
(2) $q_{2i} = \xi_+(q_{2i-1})$ if the chord $q_{2i-2}\, q_{2i-1}$ is unmarked;
(3) $q_{2i} = \xi_-(q_{2i-1})$ if the chord $q_{2i-2}\, q_{2i-1}$ is marked.

What is important is that we may obtain more than one chord cycle in some cases. As soon as chord cycle covers all chords (twice) and arcs we have *a one-cycle diagram.*

**Codimension one gradient flows on closed smooth surfaces.** Suppose that we have a smooth closed surface (2-manifold) $M$.

Consider gradient flows on $M$ such that one of the following conditions in the Definition 2.6 fails for them:

(1) one singular point is non-hyperbolic – then we get a saddle-node point;
(2) existence of a saddle connection — a trajectory which goes out of one saddle and goes into another one so that it belongs to a stable manifold of one saddle and to an unstable manifold for another saddle; in this case we have a non-transversal intersection of stable and unstable manifolds of different saddles.

We will call such gradient flows on $M$ *codimension one gradient flows.*

It is known that every typical one-parameter gradient bifurcation of a flow on a closed surface is either a saddle-node bifurcation or a saddle connection bifurcation, and accordingly, each codimension one gradient flow is either a flow with one saddle–node point or a flow with one saddle connection ([17, Theorem, p.388]) so further speculations appear to be useful for both specialists in topology and dynamical systems.

Let's formulate more precise definitions of these flows.

**Definition 2.15.** A flow $g$ of a vector field $X$ on $M$ is called *$SN$-flow* if:

1) it has one singular saddle–node type point $sn$, and the other singular points are hyperbolic;
2) the set of non-wandering points $\Omega(g)$ is finite and coincides with the union of all the singular points of $g$;
3) the flow has no saddle connections, and if for some separatrix $\gamma$ an equality $\alpha(\gamma) = sn$ ($\omega(\gamma) = sn$) is true then $\omega(\gamma)$ is a sink ($\alpha(\gamma)$ is a source).

Thus, if $g$ is an $SN$-flow, then

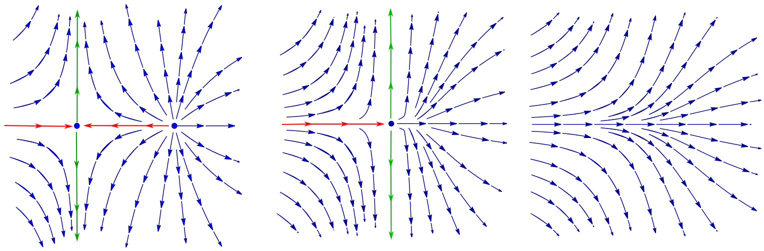
$$\Omega(g) = \Omega^1 \cup \Omega^2 \cup \Omega^3 \cup \{sn\}.$$

Saddle-node bifurcations come in two types: $SN_+$, where the node is a source, and $SN_-$, where the node is a sink.

We can observe the bifurcation $SN_+$ in Figure 2.2. Its family of vector fields can be described with the equation

$$V(x, y, a) = \{x^2 + a, y\},$$

where $x, y$ are local coordinates, $a$ is a real parameter.



1 2 3

FIGURE 2.2. Saddle-node bifurcation $SN_+$

When $a < 0$, we get the flow before the bifurcation (Figure 2.2.1); when $a > 0$, we get the flow after the bifurcation (Figure 2.2.3); when $a = 0$, we get the flow at the moment of the bifurcation (Figure 2.2.2) with $sn$.

The bifurcation $SN_-$ can be described with the equation

$$V(x, y, a) = \{-x^2 - a, -y\},$$

where $x, y$ are local coordinates, $a$ is a real parameter. Thus, $SN_-$ is obtained from $SN_+$ by reversing the direction of movement along each trajectory.

According to the type of the bifurcation, $SN$-flows come in two types – $SN_+$-flows and $SN_-$-flows.

**Definition 2.16.** A flow $g$ of a vector field $X$ on $M$ is called *$SC$-flow* if:

(1) it has a finite number of singular points and all of them are hyperbolic;

(2) the set of non-wandering points $\Omega(g)$ coincides with the union of all the singular points of $g$;

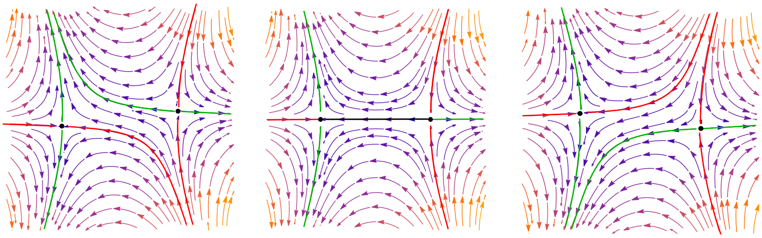
(3) this flow has a unique saddle connection.

An example of a gradient flow in the neighborhood of a saddle connection is given in Figure 2.3.2. The flow in this Figure can be described with the equation

$$V(x, y, a) = \{2xy - a, x^2 - y^2 + 1\},$$

where $x, y$ are local coordinates, and $a$ is a real parameter.

When $a < 0$, we get the flow before the bifurcation (Figure 2.2.1); when $a > 0$, we get the flow after the bifurcation (Figure 2.2.3); when $a = 0$, we get the flow at the moment of the bifurcation (Figure 2.2.2).



1 2 3

FIGURE 2.3. Saddle connection bifurcation

$SC$-flows belong to the class of $\Omega$-stable flows – gradient flows with at least one saddle connection. Topological classification of $\Omega$-stable flows by coloured graphs (like Oshemkov–Sharko graph) was obtained in [10, Theorem 5.3, p.11].

According to [17], $SN$- and $SC$-flows are gradient flows; furthermore, vector fields on some closed surface $M$ which generate these flows are structurally stable in the set $\mathrm{Grad}(M)\setminus\mathrm{MS}(M)$, where $\mathrm{Grad}(M)$ denote all possible gradient vector fields on $M$ and $\mathrm{MS}(M)$ denote all gradient Morse–Smale vector fields on $M$.

Our aim is to study $SN$- and $SC$-flows using their complete topological invariants – chord diagrams which had already been used for studying Morse flows in article [7].

## 3. Main results

All the proofs of the lemmas and the theorems given in this section can be read in Section 4.

**Saddle-node.** *Construction and properties of $C$-diagrams.*

**Definition 3.1.** An $SN$-flow on a closed connected surface $M$ is called *optimal* if there is no other $SN$-flow with a fewer number of singular points on $M$.

**Lemma 3.2.** *An $SN$-flow on a closed connected surface is optimal if and only if it has one source and one sink.*

**Definition 3.3.** A chord diagram is called *$C$-diagram* if it has a marked arc $C \subset S^1$ and the ends of $C$ do not coincide with ends of any chords in the diagram.

We will also need the following definition for the construction of $C$-diagrams.

**Definition 3.4.** A neighbourhood of a subset $S$ of the closed surface $M$ with a gradient flow on it is called *regular* if:

(1) it does not contain any other singular point of the flow apart from those ones which belong to the set $S$;

(2) its boundary is not intersected with any trajectory of the flow more than twice.

We will now construct a $C$-diagram for an optimal $SN_+$-flow on a closed connected surface. There is a unique separatrix $\gamma$, for which $\omega(\gamma) = sn$, where $sn$ is the saddle-node. According to the definition of $SN$-flow,

$$\alpha(\gamma) = v,\ v \in \Omega^0.$$

Let $L$ be the boundary of a closed regular neighbourhood $U$ of the union $v \cup \gamma \cup sn$ such that the flow in the points in $L$ is directed outward of $U$ and let $h\colon L \to S^1$ be some homeomorphism (see Figure 3.1).

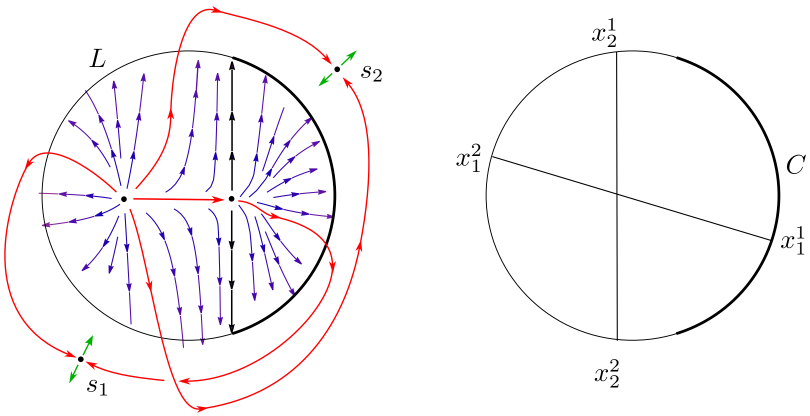



FIGURE 3.1. $SN$-flow on the torus and its $C$-diagram

Then we define the chord diagram similarly to optimal Morse flows: two points $x_i^1, x_i^2 \in S^1$ are connected with a chord if

$$h^{-1}(x_i^1) \cup h^{-1}(x_i^2) = W^s(s_i) \cap L$$

for some saddle $s_i \in \Omega^1$. Moreover, the arc

$$C = h(W^u(sn) \cap L) \subset S^1,$$

will be the marked arc from the Definition 3.3. This gives a $C$-diagram for $SN_+$-flow.

Let us colour the chords in the diagram above with two colors. All the chords are coloured with the first colour for an oriented surface. A chord is coloured with the second colour for an non-oriented surface if the union of a neighbourhood of the corresponding stable manifold and some neighborhood of a source is a Möbius strip. Otherwise the chord is coloured with the first colour.

An example of $C$-diagrams of optimal $SN_+$-flows on the torus $T^2$ is given in Figure 5.1.

A chord cycle for the given $C$-diagram is constructed in the same way as a chord cycle for the chord diagram of Morse flow.

A $C$-diagram with one cycle is called *one-cycle*. A $C$-diagram of an optimal flow is one-cycle which follows from Lemma 3.2.

A $C$-diagram for the optimal $SN_-$-flow $g$ is defined as the $C$-diagram of the flow that is inverse to $g$ and is an $SN_+$-flow.

**Theorem 3.5.** *Let $D$ be a one-cycle chord diagram and $C$ be a marked arc on $S^1$ whose ends are distinct from the ends of all chords of $D$. Then $D$ is a $C$-diagram of some optimal $SN$-flow on some closed connected surface.*

**Theorem 3.6.** *Two optimal $SN$-flows on closed connected surfaces are topologically equivalent if and only if they have the same type of saddle-node and there is an isomorphism between their chord diagrams preserving marked arcs.*

*$SN$-flow code.* Suppose we are given a $C$-diagram with $N$ chords of an $SN$-flow $g$ on a closed connected surface $M$.

Fix an orientation on the unit circle of the $C$-diagram in the counter-clockwise direction, and also denote $x$ the first end of the arc $C$. Let $S$ be a set $\{a_1, b_1, \dots, a_N, b_N\}$ of all ends of the chords in $C$-diagram and the second end of the arc $C$.

We begin a complete circumvention of the circle along the orientation from $x$ and thus write each point from $S$ in an ordered row $R$ in the order we meet them during a circumvention. Now, using this terminology, we formulate the next definition.

**Definition 3.7.** A ordered list of symbols of the form $SN_{\mathrm{Y}}\,\mathrm{XX}\ldots\mathrm{XX}$ will be called an *$SN$-flow code* if it has the following properties:
(1) $\mathrm{Y} = "-"$ if $g$ is an $SN_-$-flow, otherwise $\mathrm{Y} = "+"$;
(2) the part $\mathrm{XX}\ldots\mathrm{XX}$ of the code consists of one 0 and natural numbers $\{1, 2, \dots, N\}$ which are included in the code each twice;
(3) the first inclusions of every $i \in \{1, 2, \dots, N\}$ in the code form an increasing sequence;
(4) there is a bijection between the part of the code $\mathrm{XX}\ldots\mathrm{XX}$ as a multiset and the row $R$ preserving order so that 0 corresponds to the second end of the arc $C$, the same natural number corresponds to both of the ends of one chord;

(5) if the chord is marked then the first inclusion of the appropriate number is written in the code with a dash.

Two diagram codes are called *symmetric* if there is an axial symmetry for their diagrams (it corresponds the other bypass of the $S^1$ orientation). If we change the orientation of $S^1$ to the opposite one in such a chord diagram, then we get a symmetric diagram and a symmetric code.

A diagram code is called *self-symmetric* if a code, symmetric to it, equals to this code.

To every $SN_{\pm}$-flow one can naturally associate an $SN_{\pm}$-flow code.

**Examples 3.8.** • For $SN_{+}$-flow with $C$-diagram in Figure 3.1, the codes is $SN_{+}10212$. This code are self-symmetric.

• If $C$ is an upper half-circle in a diagram 4 in Figure 5.3 for $SN_{-}$-flow, then the code is following: $SN_{-}1'210323$. This code is also self-symmetric.

**Theorem 3.9.** *Two optimal $SN$-flows on closed connected surfaces are topologically equivalent if and only if they have either the same or symmetric codes.*

**Saddle connection.** *Construction and properties of $T$-diagrams.*

**Definition 3.10.** An $SC$-flow on a closed connected surface $M$ is called *optimal* if there is no other $SC$-flow with a fewer number of singular points on $M$.

**Lemma 3.11.** *An $SC$-flow on a closed connected surface of nonpositive Euler characteristics is optimal if and only if it has a unique source and a unique sink.*

By a *$T$-graph* we will mean a connected graph with four vertices where three ones of them have degree 1 but another one has degree 3. One of the three degree one vertices is marked and called *lower* vertex, while two others are called *side* vertices. Correspondent adjacent edges to them are called *lower* and *side* ones.

**Definition 3.12.** A graph is called a *$T$-diagram* if it is a chord diagram with an inscribed $T$-graph in it so that ends of the $T$-graph are on $S^1$ and are not ends of any other chords at once. One of the degree one vertices of $T$-graph is marked (lower vertex).

We will now construct a $T$-diagram of an optimal $SC$-flow on a closed connected surface. If a saddle is neither an $\alpha$-limit point nor an $\omega$-limit point of a saddle connection, we construct the chord for separatrices which go out of the source to this saddle as for optimal Morse flow.

Let $\alpha(\gamma) = x$ and $\omega(\gamma) = y$ for the saddle connection $\gamma$. Then we construct for $x$ a standard chord $L$ which represents $W^s(x)$. Let us denote a middle point of this chord with $z$, $z$ divides $L$ into $L_1$ and $L_2$. $W^s(y)$ represents a point $p$ in $S^1$.

We join $p$ with $z$ with a segment $N$. $T = L \cup N$ is a $T$-graph, $N$ is a lower edge, $L_1$ and $L_2$ are side edges. Besides, all the chords, $N$, $L_1$ and $L_2$ are coloured with two colours. We got a $T$-diagram of $SC$-flow.

Suppose that a simple regular curve $\mu$ is transversally intersected with oriented curves $\alpha$ and $\beta$, $x = \alpha \cap \mu \neq \beta \cap \mu = y$.

We say that orientations of the curves $\alpha$ and $\beta$ are coherent if there is a simply connected neighbourhood $U$ of the arc of $\mu$ between $x$ and $y$ such that $\alpha \subset \partial U$, $\beta \subset \partial U$ and the orientations of $\alpha$ and $\beta$ represent the same orientation $\partial U$.

A chord in the chord diagram is coloured with the first colour if an oriented path around the boundary of the neighbourhood of the source represents coherent orientations in neighbourhoods of the points of intersection with the appropriate separatrix. Otherwise the chord is coloured with the second colour.

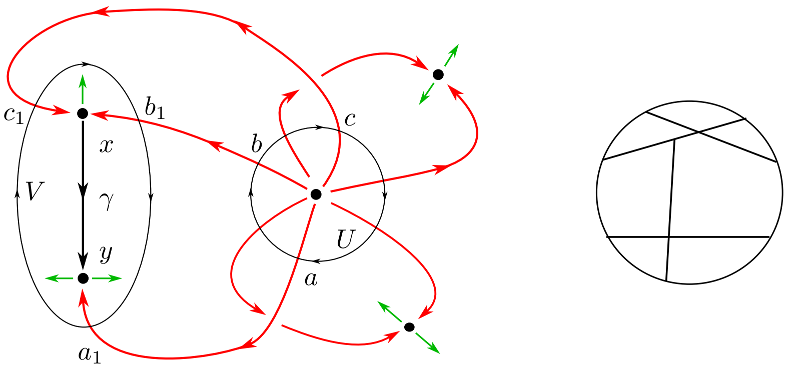



Figure 3.2. $SC$-flow on the double torus and its $T$-diagram

Let $\alpha(\gamma) = x$ and $\omega(\gamma) = y$ for the saddle connection $\gamma$, $U$ is a regular neighbourhood of the source, $V$ is a regular neighbourhood of $x \cup \gamma \cup y$. Let's denote

$$a = W^s(y) \cap \partial U, \qquad b \cup c = W^s(x) \cap \partial U,$$
$$a_1 = W^s(y) \cap \partial V, \qquad b_1 \cup c_1 = W^s(x) \cap \partial V,$$

$b$ and $b_1$ belong to the same separatrix. If the orientation of $\partial U$ is defined with the ordered triple $\{a, b, c\}$, the orientation of $\partial V$ is defined with the ordered triple $\{a_1, c_1, b_1\}$. Then we have two oriented transversal intersections with $\partial U$ and $\partial V$ for each of the three separatrices which go through the points $a, b, c$. If their orientations are coherent relatively to the separatrix, then we colour the appropriate edge with the first colour (leave unmarked), otherwise – with the second colour (mark it with X on the diagram).

Examples of such diagrams can be seen in Figure 3.4.

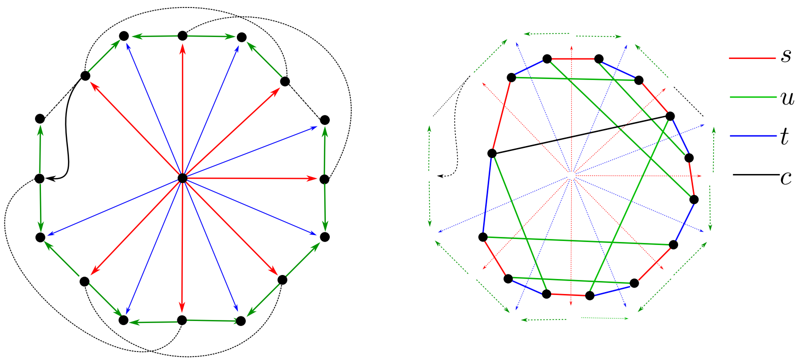



FIGURE 3.3. Separatrix diagram and multicolored graph of the $SC$-flow on the double torus

*Multicolored graphs.* Let's recall the construction of a multicolored graph ([15, p.1209], [19, p.53-54], [10, p.8-9]) in relation to the $SC$-flow. Let $\Omega_1$, $\Omega_2$ and $\Omega_3$ be the sets of all sources, saddles and sinks of a gradient $SC$-flow $g\colon M \to M$ with one saddle connection on a closed surface $M$. Suppose we have two sets

$$M_0 = M \backslash (\Omega_1 \cup W^s(\Omega_2) \cup W^u(\Omega_2) \cup \Omega_3), \quad M_1 = M \backslash (\Omega_1 \cup W^s(\Omega_2)).$$

A connected component of $M_0$ is called a *cell*.

We choose one trajectory $\Theta_J$ in every cell $J \subset M_0$ and call it a *t-curve*. Let

$$T = \bigcup_{J \subset M_0} \Theta_J, \; \overline{M} = M_0 \subset T.$$

Let us call a *c-curve* a separatrix connection on $M$. Any other separatrix on $M$ will be called an *s-curve* if it is a stable manifold of a saddle and a *u-curve* if it is an unstable manifold of a saddle. Also we will call a *polygonal region* a connecting component of $\overline{M}$.

We say that a four-colour graph $G(g)$ with edges of colours $u$, $s$, $c$, $t$ *bijectively corresponds to* $g$ if:

(1) its vertices bijectively correspond to polygonal regions of $g$;
(2) two vertices of $G(g)$ are incident to an edge of colour $u$, $s$, $c$ or $t$ if the polygonal regions corresponding to these vertices has a common $u$-, $s$-, $c$- or $t$-curve.

Let $\Gamma$ be a four-colour graph with colours $s$, $u$, $c$ and $t$, so as every its vertex is incident to exactly one edge of the colours $s$, $u$ and $t$. A four-colour graph $\Gamma$ is called *admissible* if it has one cycle with a $c$-edge which has four vertices.

The components of the boundary of $M_1$ with regular neighbourhoods of stable 1-manifolds of all saddles represent the *cycles* of the chord diagram.

A $T$-diagram with one cycle is called *one-cycle*. A $T$-diagram of an optimal flow is one-cycle which follows from Lemma 3.11.

**Lemma 3.13** ([10, Lemma 5.5]). *If we remove the $t$-edges from $G(g)$, we obtain a graph where for each component one of the next conditions is true:*

(1) *it is an su-cycle of length* 4*, corresponding to a saddle;*
(2) *it is a graph with* 7 *edges, six of which form an su-cycle, and the seventh is a c-edge that divides this su-cycle into equal parts with its endpoints.*

**Lemma 3.14** ([10, Lemma 5.5]). *If we remove the $u$-edges ($s$-edges) and the $c$-edge from $G(g)$, we obtain a graph where each component is an st-cycle (ut-cycle), corresponding to a source (sink).*

Thus, for the optimal $SC$-flow, the graph $G(g)$ has a unique $st$-cycle and a unique $ut$-cycle.

Next theorem states about a bijection between multicolored graphs and $T$-diagrams for $SC$-flows.

**Theorem 3.15.** *There exists a natural bijective mapping between the set of one-cycle $T$-diagrams and the set of graphs with the following properties:*
(1) *all edges of the graph are colored with* 4 *colors – s, t, u, c, with one edge colored c;*
(2) *the ends of the edge colored c have degree* 4*, while the other vertices have degree* 3*;*
(3) *for each vertex, the incident edges have different colors;*
(4) *the graph has exactly one st-cycle and one ut-cycle;*
(5) *there is one su-cycle of length* 6*, which is divided by c-edge into equal parts;*
(6) *the remaining su-cycles have a length of* 4*.*

A multicolored graph is a complete topological invariant and fully determines the flow [10, Lemma 5.6, p.12]. Hence, we obtain such corollaries.

**Corollary 3.16.** *If we have a one-cycle $T$-diagram, then there is an optimal $SC$-flow on some closed connected surface with this diagram.*

**Corollary 3.17.** *Two optimal SC-flows on closed connected surfaces are topologically equivalent if and only if their T-diagrams are isomorphic.*

From the definition of the $T$-diagram, the following statement emerges:

**Corollary 3.18.** *A T-diagram for optimal SC-flows on oriented closed connected surfaces and only on them has following properties:*

(1) *all the chords are coloured with the first colour;*

(2) *all the edges in T have the same colour.*

*SC-flow code.* Suppose we are given a $T$-diagram with $N$ chords of an $SC$-flow $g$ on a closed connected surface $M$.

Fix an orientation on the unit circle of the $T$-diagram in the counter-clockwise direction, and also denote $x_1$, $x_2$ and $x_3$ the lower, the upper left and the upper right end of the $T$-graph respectively. Let $S$ be a union of set $\{a_1, a_2, \dots, a_N, b_N\}$ of all ends of the chords in $C$-diagram and the set $\{x_1, x_2, x_3\}$.

We begin a complete circumvention of the circle along the orientation from $x_1$ and thus write each point from $S$ in an ordered row $R$ in the order we meet them during a circumvention. Now, using this terminology, we formulate next definition.

**Definition 3.19.** A certain ordered list of symbols $R'$ is called an *SC-flow code* if it has the form $SC$ XX...XX with next properties:

(1) the part XX...XX of the code consists of one 0 and natural numbers $\{1, 2, \dots, N+1\}$ which are included in the code each twice;
(2) all the first inclusions of natural numbers $\{2, \dots, N+1\}$ in the code form an increasing sequence;
(3) there is a bijection between the part of the code XX...XX as a multiset and the row $R$ preserving order so that 0 corresponds to the lower end of the $T$-graph, 1 corresponds to both of the upper ends of the $T$-graph, the same natural number greater than 1 corresponds to both of the ends of one chord;
(4) if the chord is marked then the first inclusion of the appropriate number is written in the code with a dash, and when some edge of the $T$-graph is marked then the appropriate number is included into the code with a dash.

We call diagram codes *symmetric* if there is an axial symmetry for their diagrams. If we change the orientation of $S^1$ to the opposite one in such a chord diagram, then we get a symmetric diagram and symmetric code.

A diagram code is called *self-symmetric* if a code, symmetric to it, equals to this code.

Again, to every $SC$-flow one can naturally associate an $SC$-flow code.

**Examples 3.20.** • For the left diagram in Fig.3.4, the code is $SC0'23'31'12$. The symmetric code is $SC0'211'3'32$.

• For the diagram 6 in Fig.5.6, the code is $SC01'212$ , the symmetric code is $SC02121'$.

**Theorem 3.21.** *Two optimal $SC$-flows on closed connected surfaces are topologically equivalent if and only if they have either the same or symmetric codes.*

**Inverse diagram and inverse code.** A chord diagram of an optimal $SN$-flow on a closed connected surface $M$ is inverse for itself, only the type of the singular point saddle-node is changed.

Let's analyze the construction of the inverse $T$-diagram for a $T$-diagram of an $SC$-flow on a closed connected surface $M$. We consider a closed regular neighborhood $S^1$ inside $D^2$ ($\partial D^2 = S^1$). It has a natural structure of $S^1 \times [0, \varepsilon]$ ($\varepsilon > 0$). We attach non-intersecting Smale 1-handles $h_i = D^1 \times D^1$ by embeddings

$$\phi_i \colon S^0 \times D^1 \to S^1 \times \{\varepsilon\}$$

such that

$$\phi_i(S^0 \times \{0\}) = (x_i^1 \cup x_i^2) \times \{\varepsilon\}, \quad (1 \leqslant i \leqslant n).$$

The handle $h_0 = D^1 \times D^1$ is attached by the embedding

$$\phi_0 \colon S^0 \times D^1 \to \partial(S^1 \times [0, \varepsilon] \cup_{\phi_1} h_1)$$

such that

$$\phi_0(S^0 \times \{0\}) = (\{x_0\} \times \{\varepsilon\}) \cup y$$

where $y \in \{0\} \times S^0 \subset h_1$. If the chords and edges in the $T$-graph are colored in the first color (there is not prime in the code), then for each 1-handle $h_i$, $1 \leqslant i \leqslant n$, we attach it so that $S^1 \times [0, \varepsilon] \cup_{\phi_i} h_i$ embeds into $D^2$, and for $h_0$, we require that $S^1 \times [0, \varepsilon] \cup_{\phi_1} h_1 \cup_{\phi_0} h_0$ embeds into $D^2$. The presence of a prime in the code leads to a change in the orientation of the attaching map to the opposite for the corresponding component. Let us denote

$$N = S^1 \times [0, \varepsilon] \cup_{\phi_1} h_1 \cup_{\phi_0} h_0 \cup_{\phi_2} h_2 \cdots \cup_{\phi_n} h_n.$$

Then $L = \partial N \backslash (S^1 \times \{0\})$ is a cycle on the chord diagram.

By $b_i$ we denote the cocore of $h_i$, that correspond to $\{0\} \times D^1$ in $h_i$. We fix $z_0 = b_1 \cap L$ and orientate $L$ in $z_0$ by vector, which is co-directional to vector $x_1^2 x_1^1$. For each handle $h_i$, $i = 0, 2, \ldots, n$, $b_i$ and $L$ have two points of intersection

$$b_0 \cap L = \partial b_0 = z_1^1 \cap z_1^2, \quad b_i \cap L = \partial b_i = z_i^1 \cap z_i^2, \quad i = 2, \ldots, n.$$

We choose the upper index in such a way, that cyclic order $z_0, z_i^1, z_i^2$ set the given orientation of $L$.

Let $g\colon L \to S^1$ is a homeomorphism, $\tilde{z}_i^k = g(z_i^k)$. We form inverse diagram by chords $\tilde{z}_i^1, \tilde{z}_i^2$, $i = 2, \ldots, n$ and $T$ with endpoints $\tilde{z}_0, \tilde{z}_1^1, \tilde{z}_1^2$ ($\tilde{z}_0$ is a low endpoint).

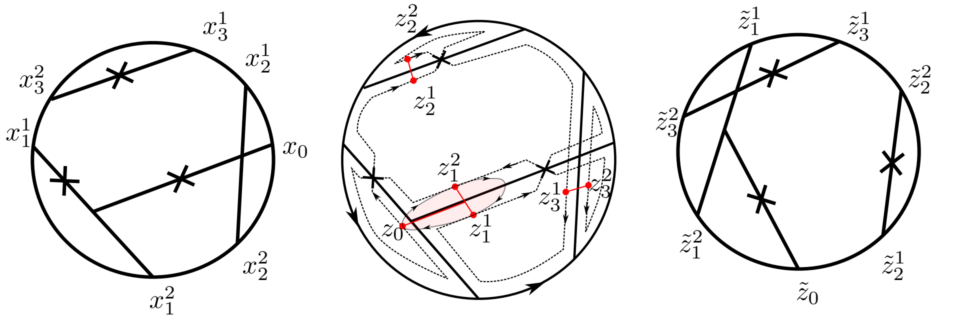



FIGURE 3.4. $T$-diagram (left), dotted cycle on the diagram (center), inverse diagram (right)

Let's describe how the orientation of chords and edges is defined in T. If the orientation L defines the co-directional vectors at points $z_i^1$ and $z_i^2$, then the chord $\tilde{z}_i^1\tilde{z}_i^2$ will be colored in the second color (with X), and if these vectors are oppositely directed, it will be colored in the first color (without X).

To determine the colors of the edges of the $T$-graph, consider the disk $U$, which touches the curve $L$ at points $z_0, z_1^1$ and $z_1^2$. We orient its boundary by establishing a cyclic order of the points $z_0, z_1^2, z_1^1$. Then, the edge ending at point $\tilde{z}_0(\tilde{z}_1^1, \tilde{z}_1^2)$ will be colored in the second color (with X) if the orientations $L$ and $\partial U$ coincide at this point. Otherwise, the edge will be colored in the first color (without X).

Note that when constructing the inverse $T$-diagram, the orientation of the cycle $L$ depends on the orientation of $S^1$ in the original chord diagram. Changing the orientation of $L$ will result in a diagram that is symmetric to the inverse one.

**Theorem 3.22.** *Let $g\colon M \times \mathbb{R} \to M$ be an SC-flow, and $h\colon M \times \mathbb{R} \to M$ be the flow given by*

$$h(x, t) = g(x, -t),$$

*so it is obtained by reversing all trajectories of $g$. Then the $T$-diagram of $h$ is inverse to the coreresponding $T$-diagram of $g$.*

## 4. Proofs of theorems

### Optimality criterion of $SN$-flows.

*Proof of Lemma 3.2.* Let $g\colon M \times \mathbb{R} \to M$ be an $SN_+$-flow on a closed connected surface $M$. We should show that it is optimal if and only if it has only one source and one sink. To do that we will establish the following three statements:

(1) an $SN_+$-flow on a surface contains at least one sink and one source;
(2) there exists an $SN_+$-flow on a given surface which contains only one sink and one source;
(3) if the flow has more than one sink or source then it is not optimal.

*Firstly.* Let $sn$ be a saddle-node point of $SN_+$-flow. According to the definition of $SN$-flow, if $\gamma$ is a separatrix and $\omega(\gamma) = sn$, then

$$\alpha(\gamma) = v_i \in \Omega^0.$$

If $\gamma$ is a separatrix and $\alpha(\gamma) = sn$, then

$$\omega(\gamma) = w_j \in \Omega^2.$$

So we have a source $v_i$ and a sink $w_j$ in the $SN$-flow.

*Secondly.* To construct a $SN$-flow with one source and one sink on $M$, we consider an optimal Morse flow $g$, existence of which was proved in [7]. We fix a regular point $x$. Then there is its neighbourhood $U$ in which $g$ is topologically equivalent to the flow in Figure 2.2.3. After deformation a vector field for $g$ in $U$ to the vector field in Figure 2.2.2, we obtain $SN_+$-flow with one source and one sink.

*Thirdly.* If a flow has more than one source or more than one sink, then according to the Poincaré–Hopf theorem a number of saddles is larger than the same number for an $SN$-flow with one source and one sink.

Hence a flow $g\colon M \times \mathbb{R} \to M$ with only one source and only one sink is an optimal one. $\square$

### Realization of $C$-diagram by SN-Flows.

*Proof of Theorem 3.5.* Suppose we have a one-cycle chord diagram with an marked arc C and chord endpoints $x_i^1, x_i^2$, $1 \leqslant i \leqslant n$, which do not intersect with the ends of the arc C. Our purpose is to prove that there is an optimal $SN_+$-flow on some surface with this chord diagram. Consider the flow generated by the vector field

$$X = \{x^2 + x^3, y\}$$

in the region

$$V = \{(x, y) \in \mathbb{R}^2 \,:\, (x + 0.5)^2 + y^2 \leqslant 1\}.$$

This flow can be seen in the Figure 3.1.

It has only two singular points: a source $v = (-1, 0)$ and a saddle-node $sn = (0, 0)$, for which

$$W^s(sn) = (-1, 0] \times \{0\}, \qquad W^u(sn) = \{(x, y) : x \geqslant 0\}.$$

For the given chord diagram, we consider a homeomorphism $h\colon S^1 \to \partial V$, for which

$$h(C) = \partial V \cap \{(x, y), x \geqslant 0\}.$$

Choose $\varepsilon > 0$ so that the $\varepsilon$-neighborhoods $B(h(x_i^1), \varepsilon)$ and $B(h(x_i^2), \varepsilon)$ for $1 \leqslant i \leqslant n$, do not intersect each other and do not intersect the line $x = 0$. Consider the standard flow in the neighborhood of the saddle generated by the vector field

$$Y_1 = \{-x, y\}$$

in the region

$$W = \{(x, y) \in \mathbb{R}^2 \, : \, -1 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1\}.$$

It is worth noticing that near the vertical sides of a square the flow is pointed inside the square, whereas the flow inside a circle is pointed outward.

For each pair of chord endpoints $x_i^1, x_i^2$, we consider homeomorphisms

$$g_i^{(j)}\colon \{(-1)^j\} \times [-1, 1] \to B(h(x_i^j), \varepsilon) \cap \partial V, \quad j = 1, 2,$$

so that $g_i^{(j)}((-1)^j, 0) = h(x_i^j)$. We glue the squares with the specified flows at one saddle point according to $g_i^{(1)}$ and $g_i^{(2)}$. This yield a surface with a boundary and a flow inside it directed outward at the boundary points.

From the condition of the one-cyclic nature of the diagram, it follows that the boundary of the constructed surface is connected, and therefore homeomorphic to a circle. We attach a 2-disk

$$B = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 \leqslant 1\}$$

to the surface with the specified flow generated by the vector field

$$Y_2 = \{-x, -y\}$$

on it. We smooth the resulting flow at the gluing points (using Gutierrez's smoothing theorem [4, Smoothing theorem, p.17]). As a result, we obtain a closed surface and an optimal $SN_+$-flow on it, whose $C$-diagram is isomorphic to the given one. $\square$

**C-Diagram criterion of topological equivalence.**

*Proof of Theorem 3.6.* Consider two optimal $SN$-flows $g_i\colon M_i \times \mathbb{R} \to M_i$, $i = 1, 2$, on closed connected surfaces $M_1$ and $M_2$. We should prove that they are topologically equivalent if and only if they have the same type of saddle-node and there is an isomorphism between their chord diagrams preserving marked arcs.

*Necessity.* A topological equivalence of flow maps the stable manifolds of one optimal flow into the stable ones of the other one. It defines a correspondence between the chords and the ends of marked arcs, hence, there is an isomorphism between the $C$-diagrams.

*Sufficiency.* Let $C$-diagram of the $SN_+$-flow $g$ be isomorphic to the $C$-diagram of the $SN_+$-flow $g'$. We fix the orientation of the circle $S^1$ in the $C$-diagram of the flow $g$, and we choose the orientation of $S^1$ in $C$-diagrams of $g'$ under the condition that the isomorphism preserves orientation. The endpoints of the chords (and saddle points) are numbered by traversing them according to the orientation of the circle, starting from the end of C where the movement is directed inward $C$. Thus, the first endpoint of the chord be $x_1^1$, the second $x_2^1$ if it is the end of another chord, or $x_1^2$ otherwise. We denote the separatrix corresponding to the endpoint $x_i^j$ as $\gamma_i^j$.

The separatrix between the source $v$ and the saddle-node $sn$ be denoted as $\gamma_0^1$. The separatrices emanating from the saddle $s_i$ be denoted as $\beta_i^1$ and $\beta_i^2$, such that the orientation generated by rotation in $s_i$,

$$\gamma_i^1 \to \beta_i^1 \to \gamma_i^2 \to \beta_i^2,$$

coincides on $\gamma_i^j$ with the orientation generated by the traversal of $S^1$. Under the isomorphism of $C$-diagrams, $x_i^j$ maps to $x_i^{j\prime}$.

Consider the bifurcation of the flow $g$ to the flow $g_-$ which turns picture 2 into picture 1 in Figure 2.2. It has two singular points instead of one saddle-node: the saddle $s_0$ and the source $v_1$. In this case, the new flow have an additional separatrix $\gamma_0^2$. Note that the inverse procedure of transitioning from the flow $g_-$ to the flow $g$ consists of collapsing $s_0 \cup \gamma_0^2 \cup v_1$ to a point. The same bifurcation of the flow $g'$ to the flow $g'_-$ takes place.

Since $g_-$ and $g'_-$ are Morse-Smale flows, it is sufficient to verify the equivalence of their Peixoto invariants (rotation systems for separatrix diagrams) to prove their equivalence. The graphs consisting of special points (as vertices) and separatrices (as edges) are isomorphic by construction. In the saddles, the choice of consistent orientations leads to identical cyclic orders of the separatrices within them. The order of the endpoints of the chords on the arc C corresponds to the cyclic order of the separatrices at the source $v_1$. The order of the endpoints of the chords outside C corresponds to the cyclic order of the separatrices at the source $v$. Finally, the uniqueness of writing the cycle for the $C$-diagram defines the cyclic order of the separatrices in the neighbourhood of the sink.

Thus, the isomorphism of $C$-diagrams generates an isomorphism of the graphs of separatrix diagrams and corresponding cyclic orders. Then, according to Peixoto's classification [18, Theorem 5.1, p.398], the flows $g_-$ and $g'_-$ are topologically equivalent. After collapsing the sets $s_0 \cup \gamma_0^2 \cup v_1$ to $sn$ for both flows, we obtain topological equivalence of $g$ to $g'$. □

**SN-code Criterion of Topological Equivalence.**

*Proof of Theorem 3.9.* We should show that two optimal $SN$-flows on closed connected surfaces are topologically equivalent if and only if they have either the same or symmetric codes.

*Necessity.* If we fix an orientation on $S^1$ in a $C$-diagram, then according to the procedure of constructing a circle, the code is defined clearly. So codes are either identical or symmetric for topologically equivalent flows.

*Sufficiency.* Each $C$-diagram is isomorphic to a diagram in which the vertices (ends of the chords) and the ends of the arc C are the vertices of a regular $(2n+2)$-gon. Here, $n$ is the number of chords. Let the starting point $p$ (the first end of the arc C) have coordinates $(1, 0)$. Each vertex (except $p$) corresponds to a number from the code. Two vertices are connected by a chord if they have the same number. Thus, if the codes are identical, the $C$-diagrams are the same, and the flows are topologically equivalent. If the codes are symmetric, then there is axial symmetry between their $C$-diagrams, which also defines the topological equivalence of the flows. The line of symmetry passes through the midpoint of arc C and the center of the circle. □

**Optimality Criterion of SC-Flows.** As a separatrix connection joins two different saddles, an $SC$-flow always contains at least 2 saddles. So we have

$$N = \chi + S \geqslant \chi + 2,$$

where $N$ is a general number of all the sources and sinks in a flow, $S$ is a general number of all saddles in a flow and $\chi$ is an Euler characteristics of the surface. Hence, the flow has either more than one source or more than one sink if $\chi$ is positive.

*Proof of Lemma 3.11.* Let $g\colon M \times \mathbb{R} \to M$ be an $SC$-flow on a closed connected surface $M$. We need to show that it is optimal if and only if it has only one source and one sink. The divided into the following three statements:

(1) an $SC$-flow on a surface contains at least one sink and one source;

(2) there exists an $SC$-flow on a given surface which contains only one sink and one source;

(3) if the flow has more than one sink or source then it is not optimal.

*Firstly.* Let $\gamma$ be a saddle connection in a $SN$-flow, $s_1 = \alpha(\gamma)$, $s_2 = \omega(\gamma)$. Suppose $W^s(s_1) = \gamma_1 \cup s_1 \cup \gamma_2$. Then

$$\alpha(\gamma_1) = v_1 \in \Omega^0.$$

So $\Omega^0 \neq \varnothing$. Analogously, $\Omega^2 \neq \varnothing$.

*Secondly.* To construct an $SC$-flow with one source and one sink on a closed surface $M$, we consider the optimal Morse flow on it. We select two saddles $s_1$ and $s_2$ and their stable separatrices $\gamma_1^1$, $\gamma_2^1$, which correspond to adjacent vertices on the chord diagram. Let $U$ be a connected neighborhood of the union $\gamma_1^1 \cup \gamma_2^1 \cup s_1 \cup s_2$, in which the flow is topologically equivalent to the flow shown in Figure 2.3.1. We replace it in this neighborhood with the flow in Figure 2.3.2. As a result, we obtain a flow with one source, one sink, and one separatrix connection.

*Thirdly.* If a flow has more than one source or more than one sink, then according to the Poincaré–Hopf theorem a number of saddles is larger than the same number for an optimal flow too. Hence a flows $g\colon M \times \mathbb{R} \to M$ with only one source and only one sink are an optimal ones. □

**T-Diagrams and Multicolored Graphs.** We demonstrate the existence of a bijection between $T$-diagrams and 4-coloured graphs of $\Omega$-stable flows with a single $c$-edge.

*Proof of Theorem 3.15.* First, we will show how to construct a graph with properties (1)–(6) from a $T$-diagram.

On $S^1$, we will choose regular neighborhoods of the ends of the chords and T. These will be the $s$-edges of the desired graph, and their boundaries will be the vertices. The complement in $S^1$ to the $s$-edges are the $t$-edges. For each chord, we will connect the ends of the corresponding $s$-edges with $u$-edges, forming a $su$-cycle of length 4. In this case, the $u$-edges are parallel if the chord has the first color and are crossed otherwise. Three $s$-edges corresponding to the $T$-graph will be included in a $su$-cycle of length 6.

Here, the $u$-edges cross or do not cross according to the colors of the edges in the $T$-graph. From the lower $s$-edge, two $u$-edges will emerge. Their ends, which are incident to the upper $s$-edges, will be connected by an $c$-edge. Thus, the $c$-edge divides the $su$-cycle of length 6 into equal parts. By construction, the $ut$-cycle corresponds to a cycle in the $T$-diagram. Therefore, the constructed graph has only one $ut$-cycle. This means that all 6 conditions are satisfied.

Now, we will construct a $T$-diagram based on the graph with properties (1)-(6).

To do this, we homeomorphically map the $st$-cycle onto $S^1$. We connect the midpoints of the $s$-edges with chords if they are part of a $su$-cycle of length 4. The remaining three $s$-edges are connected by a $T$-graph.

We fix an orientation on the $st$-cycle. If orientations can be assigned to the $su$-cycles of length 4 that are consistent with the given orientation, then the corresponding chords of the $T$-diagram be colored in the first color; otherwise, they be colored in the second color. A $c$-edge divides the $st$-cycle into two arcs. The orientation of the $c$-edge is aligned with the orientation of the arc that contains the lower $s$-edge.

On the other hand, the $c$-edge divides the $su$-cycle of length 6 into two parts. The part that contains the lower edge receives an orientation consistent with the $c$-edge and determines the orientation of the entire $su$-cycle of length 6. The colors of the edges of the $T$-graph are determined by the consistency of the orientations in the corresponding $s$-edges. Thus, the $T$-diagram is constructed. □

**T-Code Criterion of Topological Equivalence.** Here we study how the phenomenon of topological equivalence of $SC$-flows depends on similarity of the codes of their $T$-diagrams.

*Proof of Theorem 3.21. Necessity.* If we fix an orientation in the unit circle of a $T$-diagram, then according to the described procedure a code is defined clearly. We get a symmetric code for another orientation. So the codes are either identical or symmetric for topologically equivalent flows.

*Sufficiency.* We can renew the chord diagram definitely with the help of the code, if we have a regular $(2n+1)$-gon with its vertices in $S^1$ ($n$ is a quantity of mutually distinct numbers in the code apart from 0) with put numbers on its vertices according to the code (doing a counterclockwise motion along $S^1$). The pairs of the points with the same numbers are joined by the chords, whereas the center of the chord 1–1 is joined with 0 constructing T. The chords (the edges) having $'$ near their numbers are coloured with the second colour, all the others – with the first one.

Hence, if these codes are either the same or symmetric ones, we will get two isomorphic $T$-diagrams. Finally, according to Theorem 3.6, the appropriate flows are topologically equivalent. □

**T-Diagram of inverse $SC$-flow.**

*Proof of Theorem 3.22.* Let $g\colon M \times \mathbb{R} \to M$ be an $SC$-flow, and $h\colon M \times \mathbb{R} \to M$ be the flow obtained by reversing all trajectories of $g$. We need to prove that that the $T$-diagram of $h$ is inverse to the $T$-diagram of $g$.

The procedure for constructing the inverse T-diagram can be implemented by decomposing the surface $M$ into handles.

The 0-handle $h^0$ is the regular neighborhood of the source. The core of the 1-handles lie on stable manifolds of saddles and correspond to chords and edges of the $T$-graph (the $T$-neighborhood consists of two handles), while the cocore lie on unstable manifolds of saddles. In the inverse decomposition into handles for the inverse flow, the 0-handle will correspond to the sink of the original flow, and the 1-handles will correspond to the same saddles. The boundary of the 0-handle can be chosen as curve $L$. Then, the cocores of the original decomposition into handles will correspond to the chords, except for two of them that form the $T$-graph of the inverse diagram.

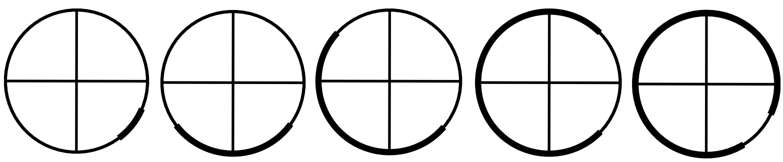
Thus, the arrangement of chords and the $T$-graph in the $T$-diagram of the inverse flow corresponds to the chords and the $T$-graph in the inverse diagram. The coloring of the edges of the $T$-graph and the chords corresponds to the coloring described during the construction of the $T$-diagram. The theorem has been proved. □

## 5. Examples

Since now we consider everywhere for the singularity "saddle-node" that the source is a node.

$SN$**-flows on oriented surfaces.** There is a unique optimal flow with a singularity "saddle-node" (source) on a sphere. $C$-diagram is an empty circle (without chords) with a marked arc in it.

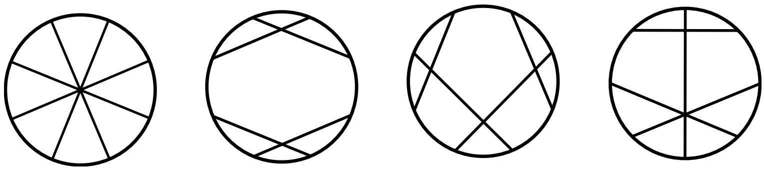
There is a unique chord diagram of an optimal Morse flow on a torus. We fix a standard orientation of $S^1$ – in anticlockwise direction. There is the only way of choice for the beginning of a $C$-arc to within circle symmetries. Then this point divides the arc it belongs to into two parts. Hence there are five options of arcs for the end of $C$-arc which can lie in it. So, there are five topologically non-equivalent $SN$-flows (SN-node is a source) on a torus. Their $C$-diagrams can be seen in Figure 5.1.



1 2 3 4 5

Figure 5.1. SN-diagrams on the torus

There are four chord diagrams of an optimal flow on an oriented surface of genus 2 (Figure 5.2). The beginning of a $C$-arc in the first diagram can be chosen in one way, whereas the end can be chosen in nine ways. So, there are nine different $C$-diagrams.



1 2 3 4

FIGURE 5.2. Chord diagrams on the genus 2 surface

We can choose the first end in the second diagram in three ways: right central arc, right upper arc and upper central arc. In the first case we have nine options for the second end, in the second case – seven options which were not considered before, in the third case – three options. Altogether we have nineteen opportunities for a placement of the $C$-arc in the second diagram.

We have five options for the first end of the $C$-arc in the third diagram, starting from the lower arc and doing counterclockwise circumvention. There be

$$9 + 8 + 6 + 4 + 2 = 29$$

opportunities for the placement of this arc in general, they define twenty-nine $C$-diagrams.

We have four options for the first end in the fourth diagram and

$$9 + 7 + 5 + 3 = 24$$

for the second end, so we have twenty-four $C$-diagrams in this case.

So, in general we have next statement: there are eighty-one $SN$-flows (source) on an oriented surface of genus 2.

**$SN$-flows on non-oriented surfaces.** Chord diagrams of optimal Morse flows on non-oriented surfaces of genuses 1, 2 and 3 are in Figure 5.3.

We have the first diagram on the projective plane (genus 1). The first end of the $C$-arc on it is defined clearly, whereas there are three options for the second one. So, there are three topologically unequivalent $SN$-flows (source) on the projective plane.

We have two diagrams (2 and 3 in Figure 5.3) on the Klein bottle (genus 2). There are two options for the first end of the $C$-arc in the diagram 2, 5+3=8 options for the second one. The first end of the $C$-arc is defined clearly for the diagram 3, but we have five opportunities for the second one.

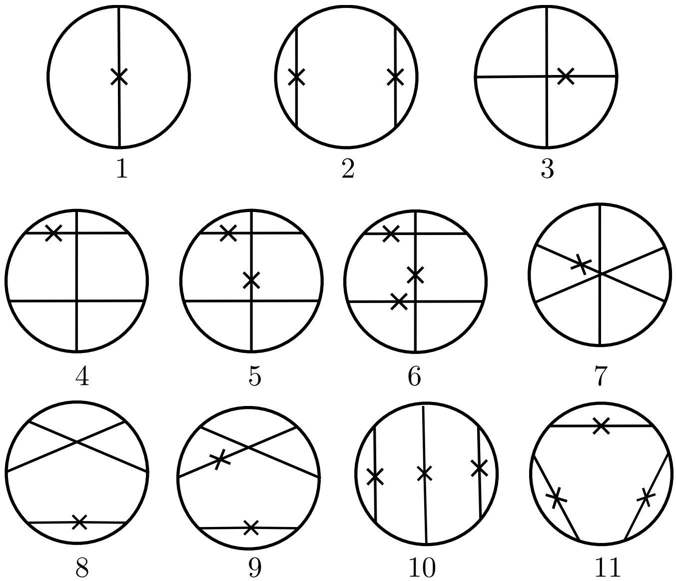



FIGURE 5.3. Non-oriented chord diagrams:
1 – genus 1, 2,3 – genus 2, 4-11 – genus 3.

Hence there are thirteen topologically unequivalent $SN$-flows (source) on the Klein bottle.

There are eight chord diagrams on a non-oriented surface of genus 3: diagrams 4–11 in Figure 5.3. Let's find options of placement of the $C$-diagram according to the previous scheme. We are getting next facts:

- for diagrams 4 and 5 number of options is equal to $7 + 5 + 3 = 15$ for each one,
- for diagrams 6 and 7 number of options is equal to $7 + 5 = 12$ for each one,
- for diagram 8 number of options is equal to $7 + 6 + 4 + 2 = 19$,
- for diagram 9 number of options is equal to $7 + 6 + 5 + 4 + 3 + 2 = 27$,
- for diagram 10 number of options is equal to $7 + 5 = 12$,
- for diagram 11 number of options is equal to $7 + 4 = 11$.

Altogether we have one hundred twenty-three flows on a non-oriented surface of genus 3.

$SC$**-flows on oriented surfaces.** Suppose we have a sphere and an optimal $SC$-flow on it. A saddle connection joins two saddles. Their Poincaré rotation index is equal to $-1$ for each one. Then, according to the Poincaré–Hopf theorem, general number of all the sources and sinks is equal to four. Three cases are possible:

(1) one source and three sinks;

(2) two sources and two sinks;

(3) three sources and one sink.

Everywhere we have a $T$-graph with no chords.

In the first case one unit circle corresponds to one source. All the edges of the $T$-graph have the first colour.

In the second case we have two circles and the vertices of the $T$-graph lie in them. Two subcases are possible:

(1) two side vertices lie in one circle, whereas the lower vertex does in the other one;

(2) side vertices lie in different circles.

In the third case we have three circles and each one of them has one vertex of the $T$-graph in it. So, general number of optimal $SC$-flows on a sphere is equal to four.

There is a unique optimal $SC$-flow on a torus, because there is a unique $T$-diagram without chords which corresponds a torus – all the edges are coloured with the second colour.

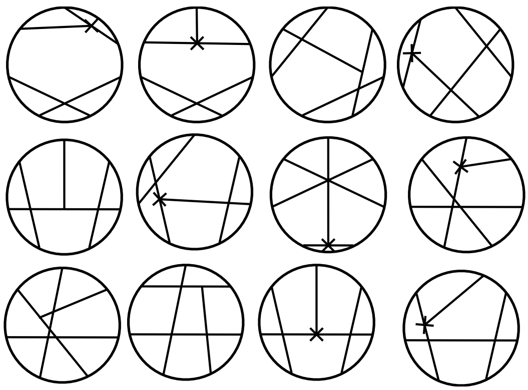
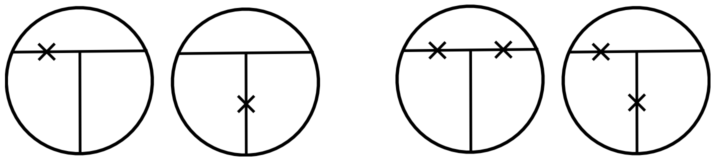


FIGURE 5.4. Twelve saddle connection bifurcations on the genus 2 surface

The $T$-diagram of an optimal $SC$-flow on an oriented surface of genus 2 has two chords and a $T$-graph. All such $T$-diagrams which are possible are showed in Figure 5.4. Here, a $T$-graph with a mark at the central vertex denotes a graph in which all three edges are marked.

**$SC$-flows on non-oriented surfaces.** If a flow with hyperbolic singular points on the projective plane has two saddles, then it has yet three points which are sources and sinks. We are analyzing flows with one source and two sinks. The $T$-diagram of such flows has no chords and has two cycles. We can see such diagrams in Figures 5.5.1 and 5.5.2.



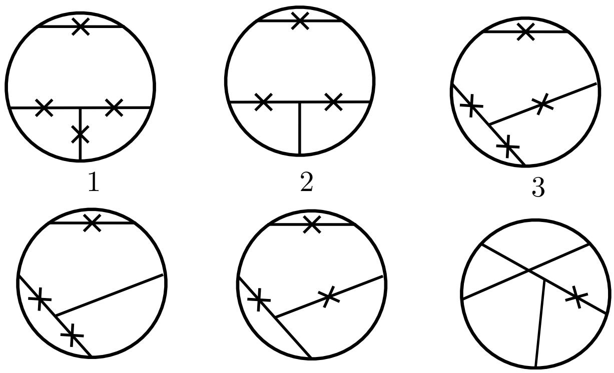
1 2 3 4

FIGURE 5.5. Non-oriented $T$-diagrams without chords

Flows with two sources and one sink we can get from these flows with reversing orientations of all trajectories. So we have four topologically unequivalent optimal $SC$-flows on the projective plane.

An optimal $SC$-flow on the Klein bottle has one source, one sink and two saddles. So its $T$-diagram has no chords and has one cycle. Two such diagrams which are possible we can see in Figures 5.5.3 and 5.5.4. Hence there are two topologically unequivalent $SC$-flows on the Klein bottle.

The $T$-diagrams of optimal $SC$-flows on a non-oriented surface of genus 3 can be seen in Figures 5.6 and 5.7.



1 2 3

4 5 6

FIGURE 5.6. Non-oriented self-inverse $T$-diagrams of type 3

First we coloured the edges of the $T$-graph with two colours for finding these diagrams. The case when all three edges have the first colour is impossible because one can't join three constructed cycles with one chords.

If only one edge is coloured with the second colour, then the chord has to join two constructed cycles which are an arc two edges of the first colour are attached to and one of the other two arcs. Colouring the chord can be arbitrary. So we have four diagrams for the side edge of the second colour: 6, 11, 16, 19. There are two diagrams for the lower edge – 10 and 15.

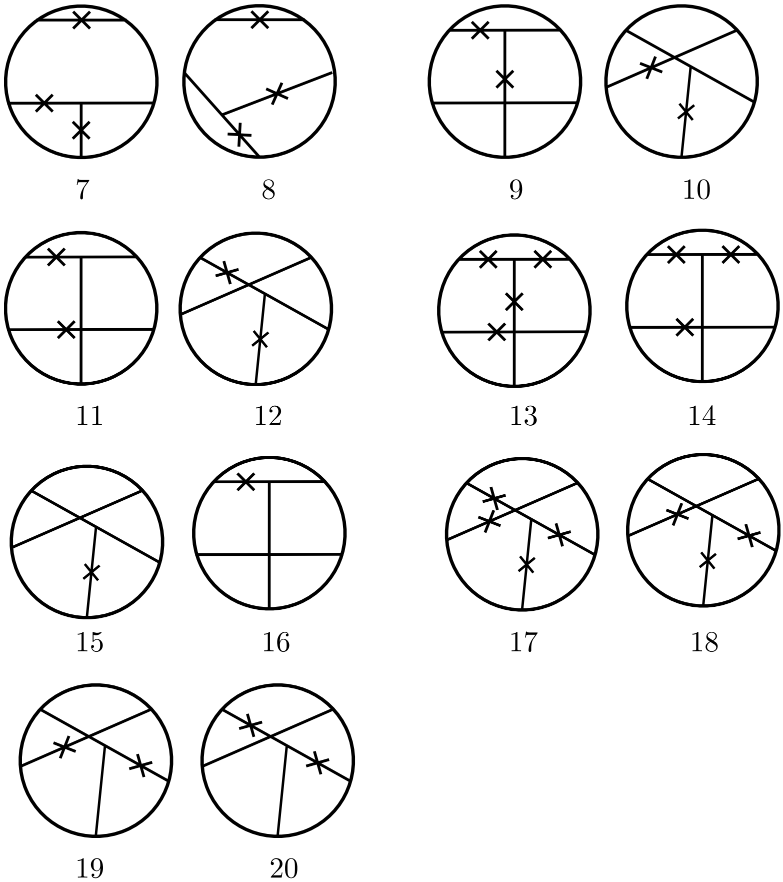



Figure 5.7. Pairs of mutually inverse non-oriented $T$-diagrams of type 3

If two or three edges are coloured with the second colour, then the chord can join two arbitrary points in $S^1$, but only one out of two colourings gives one cycle in the diagram in this case.

If the side edge has the first colour, then six options are possible for the chord: three options with a chord on one arc – 5, 7, 8, and three options with chords on different arcs – 9, 12, 18.

If the lower edge has the first colour, but the side ones have the second colour, then we have two options when the ends of the chord belong to the same arc – 2 and 4, and two options when they belong to different ones – 14 and 20.

If all the edges have the second colour, then the chord has it too. For the chord on one arc we have two options – 1 and 3, otherwise we also have two options – 13 and 17.

So, there are all possible twenty non-oriented diagrams with one edge in Figures 5.6 and 5.7.

We can use the method described in the previous chapter to verify which diagrams here are mutually inverse.

Hence there are twenty topologically unequivalent $SC$-flows on a non-oriented surface of genus 3, and there are six ones among them inverse to which are topologically equivalent to themselves and yet seven pairs of mutually inverse flows.

## 6. Conclusion

In this paper we construct complete topological invariants of codimension one gradient flows on closed surfaces. They were used for description of all the possible structures of flows on oriented surfaces of genus 0, 1, 2 and non-oriented surfaces of genus 1, 2, 3. Also we described an approach of finding the invariants of inverse flows according to the invariant of the initial flow.

The obtained resulted are similar to those ones about the structure of function deformations [5, 6, 9, 13, 14, 21, 23]. There the codimension zero functions are Morse functions of general position. Their structure is defined with Reeb graphs, their deformations are defined with deformations of Reeb graphs. Codimension one functions have a cubical speciality or two singular points in the same level. However, if the saddle-node bifurcation can be described as a gradient of a deformation with a cubical speciality, then the saddle connection is not associated with two singular points in the same function level.

We hope that the invariants developed in this paper pave the way for solving the following problems:

Problem 1. To construct a complete topological invariant for gradient flows of codimension 1 on spheres with holes.

Problem 2. To obtain a topological classification of all flows of codimension 1 on the sphere.

Problem 3. To construct a complete topological invariant for optimal gradient flows of codimension 2 on closed surfaces.

## References


[1] Stability of Morse-Smale vector fields on manifolds with boundary. *Topology*, 29(1):57–81, 1990. doi:10.1016/0040-9383(90)90025-F.

[2] G. Fleitas. Classification of gradient-like flows on dimensions two and three. *Bol. Soc. Brasil. Mat.*, 6(2):155–183, 1975. doi:10.1007/BF02584782.

[3] Jonathan L. Gross and Thomas W. Tucker. *Topological graph theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1987. A Wiley-Interscience Publication.

[4] Carlos Gutiérrez. Smoothing continuous flows on two-manifolds and recurrences. *Ergodic Theory Dynam. Systems*, 6(1):17–44, 1986. doi:10.1017/S0143385700003278.

[5] Bohdana Hladysh and Alexandr Prishlyak. Simple Morse functions on an oriented surface with boundary. *J. Math. Phys. Anal. Geom.*, 15(3):354–368, 2019. doi:10.15407/mag15.03.354.

[6] Olexandra Khohliyk and Sergiy Maksymenko. Diffeomorphisms preserving Morse-Bott functions. *Indag. Math. (N.S.)*, 31(2):185–203, 2020. doi:10.1016/j.indag.2019.12.004.

[7] Zlata Kibalko, Alexandr Prishlyak, and Roman Shchurko. Trajectory equivalence of optimal Morse flows on closed surfaces. *Proc. Int. Geom. Cent.*, 11(1):12–26, 2018. doi:10.15673/tmgc.v11i1.916.

[8] Vladislav Kibkalo and Tomoo Yokoyama. Topological characterizations of Morse-Smale flows on surfaces and generic non-Morse-Smale flows. *Discrete Contin. Dyn. Syst.*, 42(10):4787–4822, 2022. doi:10.3934/dcds.2022072.

[9] Anna Kravchenko and Sergiy Maksymenko. Automorphisms of Kronrod-Reeb graphs of Morse functions on 2-sphere. *Proc. Int. Geom. Cent.*, 11(4):72–79, 2018. doi:10.15673/tmgc.v11i4.1306.

[10] Vladislav Kruglov, Dmitry Malyshev, and Olga Pochinka. Topological classification of Ω-stable flows on surfaces by means of effectively distinguishable multigraphs. *Discrete Contin. Dyn. Syst.*, 38(9):4305–4327, 2018. doi:10.3934/dcds.2018188.

[11] R. Labarca and M.J. Pacifico. Stability of Morse-Smale vector fields on manifolds with boundary. *Topology*, 29(1):57–81, 1990.

[12] E. Leontovič and A. Maĭer. On a scheme determining the topological structure of the separation of trajectories. *Dokl. Akad. Nauk SSSR (N.S.)*, 103:557–560, 1955.

[13] Sergiy Maksymenko. Stabilizers and orbits of smooth functions. *Bull. Sci. Math.*, 130(4):279–311, 2006. doi:10.1016/j.bulsci.2005.11.001.

[14] Sergiy Maksymenko. Deformations of functions on surfaces by isotopic to the identity diffeomorphisms. *Topology Appl.*, 282:107312, 48, 2020. doi:10.1016/j.topol.2020.107312.

[15] A. A. Oshemkov and V. V. Sharko. On the classification of Morse-Smale flows on two-dimensional manifolds. *Mat. Sb.*, 189(8):93–140, 1998. doi:10.1070/SM1998v189n08ABEH000341.

[16] J. Palis and S. Smale. Structural stability theorems. In *Global Analysis (Proc. Sympos. Pure Math., Vols. XIV, XV, XVI, Berkeley, Calif., 1968)*, volume XIV-XVI of *Proc. Sympos. Pure Math.*, pages 223–231. Amer. Math. Soc., Providence, RI, 1970.

[17] J. Palis and F. Takens. Stability of parametrized families of gradient vector fields. *Ann. of Math. (2)*, 118(3):383–421, 1983. doi:10.2307/2006976.

[18] M.M. Peixoto. On the classication of flows of 2-manifolds. *Dynamical Systems (Proc. Symp. Univ. of Bahia, Salvador, Brasil, 1971)*, pages 389–419, 1973.

[19] Alexandr Prishlyak and Maria Loseva. Topology of optimal flows with collective dynamics on closed orientable surfaces. *Proc. Int. Geom. Cent.*, 13(2):50–67, 2020. doi:10.15673/tmgc.v13i2.1731.

[20] Alexandr Prishlyak, Andrii Prus, and Sophija Huraka. Flows with collective dynamics on a sphere. *Proc. Int. Geom. Cent.*, 14(1):60–79, 2021. doi:10.15673/tmgc.v14i1.1902.

[21] A.O. Prishlyak. Conjugacy of Morse functions on surfaces with values on a straight line and circle. *Ukrainian Mathematical Journal*, 52(10):1623–1627, 2000. doi:10.1023/A:1010461319703.

[22] A.O. Prishlyak and A.A. Prus. Three-color graph of the morse flow on a compact surface with boundary. *Journal of Mathematical Sciences*, 249(4):661–672, 2020. doi:10.1007/s10958-020-04964-1.

[23] Georges Reeb. Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique. *C. R. Acad. Sci. Paris*, 222:847–849, 1946.

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