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Split curvature

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Abstract. We consider spaces with a special kind of Riemannian tensor. It is proved that they are semisymmetric spaces. These spaces are divided into three types and we investigate nontrivial geodesic mappings for each type. In particular, it is proved that if these spaces admit nontrivial geodesic mappings, then they have a constant scalar curvature.

Анотація. Розглядаються простори зі спеціальним видом тензора Рімана. Доведено, що вони є півсиметричними просторами. Вказані простори розділені на три типи і для кожного з цих типів в роботі вивчені нетривіальні геодезичні відображення. Зокрема, доведено, що якщо вказані простори допускають нетривіальні геодезичні відображення, то вони мають сталу скалярну кривину.

The study of generalized spaces often requires modeling that preserves certain properties of the object being modeled. When studying dynamic processes, it is crucial to preserve the trajectories of motion, namely, geodesic lines. This, in turn, leads to the need of investigation of geodesic mappings of generalized spaces.

Today the theory of geodesic mappings is a well developed area of mathematical research, yet the study of geodesic mappings remains an topical area as far as new possibilities for studying and special classes of spaces appear.

The specialization of investigated spaces follows from internal needs arising in the process of developing the theory of geodesic mappings itself and the needs of its application in the relativity theory and mechanics.

In this paper, we consider geodesic mappings of special pseudo-Riemannian spaces, namely, spaces of split curvature. We study them locally, in a tensor form, in the class of sufficiently differentiable functions.

Keywords: Riemannian tensor, geodesic mappings, scalar curvature, semisymmetric spaces

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1. SPACES OF SPLIT CURVATURE

Let $V_n (n > 2)$ be a pseudo-Riemannian space with a metric tensor g_{ij}, $\Gamma^h_{ij}(x)$ be its Christoffel symbols, and R^h_{ijk} be the Riemannian tensor such that

$$R^h_{ijk} = \partial_j \Gamma^h_{ik}(x) + \Gamma^\alpha_{ik}(x)\Gamma^h_{j\alpha}(x) - \partial_k \Gamma^h_{ij}(x) - \Gamma^\alpha_{ij}(x)\Gamma^h_{k\alpha}(x).$$

The curvature tensor R_{hijk} is defined by the formula

$$R_{hijk} = g_{\alpha h} R^\alpha_{ijk}.$$

If the curvature tensor of a pseudo-Riemannian space V_n, different from the space of constant curvature, is represented as follows:

$$R_{hijk} = S_{hi} S_{jk}, \tag{1.1}$$

where S_{hi} is some skew-symmetric tensor, then the space V_n is said to be a space of *split curvature.*

A pseudo-Riemannian space V_n is said to be *semisymmetric* if its Riemannian tensor satisfies the following conditions

$$R^h_{ijk,[lm]} = 0. \tag{1.2}$$

Here, a comma "," denotes the sign of the covariant derivative in V_n, and brackets [ ] denote the alternation, [20, 22].

Given the Ricci identity, the equation (1.2) can be rewritten as

$$R_{\alpha ijk} R^\alpha_{hlm} + R_{h\alpha jk} R^\alpha_{ilm} + R_{hi\alpha k} R^\alpha_{jlm} + R_{hij\alpha} R^\alpha_{klm} = 0. \tag{1.3}$$

By substituting the conditions from (1.3) into the equation (1.1), we see that the following theorem holds:

Theorem 1.1. *Spaces of split curvature are semisymmetric spaces.*

Using Ricci tensor, which is calculated as

$$R_{ij} = R^\alpha_{ij\alpha},$$

one can define Ricci semisymmetric spaces in which

$$R_{ij,[lm]} = 0.$$

As a consequence of Theorem 1.1 we have the following:

Corollary 1.2. *Spaces of split curvature belong to the Ricci-semisymmetric spaces.*

Multiplying equation (1.1) by the tensor g^{hk}, being inverse of the metric tensor, and wrapping the resulting equation by the indices h and k, we obtain that

$$R_{ij} = g^{\alpha\beta} S_{\alpha i} S_{j\beta}.$$

Multiply further (1.1) by $g^{h\alpha}S_{\alpha l}$ and wrapping by the index h, we get

$$S_{\alpha l}R^{\alpha}_{ijk} = -R_{li}S_{jk}.$$

Since Ricci tensor is symmetric, we have the following alternatives

$$S_{\alpha l}R^{\alpha}_{ijk} - S_{\alpha i}R^{\alpha}_{ijk} = 0 \qquad \text{or} \qquad S_{\alpha l}R^{\alpha}_{ijk} + S_{i\alpha}R^{\alpha}_{ijk} = 0,$$

that is $S_{il,[jk]} = 0$.

Note that if A_{ij} is a tensor in the space V_n such that $A_{ij} \neq cg_{ij}$ and $A_{ij,[lm]} = 0$, then such spaces are called *A-semisymmetric.* Hence, following statement holds.

Theorem 1.3. *Each space of split curvature is S-semisymmetric, i.e. its tensor S satisfies the identity $S_{il,[jk]} = 0$.*

2. TYPES OF SPLIT CURVATURE SPACES

Since the result of looping over the covariant indices of the Riemannian tensor is identically zero, we obtain from (1.1) the following equality

$$S_{hi}S_{jk} + S_{hj}S_{ki} + S_{hk}S_{ij} = 0. \tag{2.1}$$

Choose non-zero vectors and $\overset{1}{\tau}{}^{i}$, $\overset{2}{\tau}{}^{i}$ such that $S_{\alpha\beta}\,\overset{1}{\tau}{}^{\alpha}\,\overset{2}{\tau}{}^{\beta} = 1$. Dividing (2.1) by $\overset{1}{\tau}{}^{j}\cdot\overset{2}{\tau}{}^{k}$, and rolling up by indices j, k, we obtain

$$S_{hi} = a_h b_i - a_i b_h,$$

where $a_i = S_{i\alpha}\,\overset{1}{\tau}{}^{\alpha}$, $b_i = S_{i\alpha}\,\overset{2}{\tau}{}^{\alpha}$ are non-zero vectors.

Then the equation (1.1) will be written as follows:

$$R_{hijk} = (a_h b_i - a_i b_h)(a_j b_k - a_k b_j), \tag{2.2}$$

$$R_{ij} = a_\alpha b^\alpha a_i b_j + a_\alpha b^\alpha a_j b_i - b_\alpha b^\alpha a_i a_j - a_\alpha a^\alpha b_i b_j, \tag{2.3}$$

where $a^i = a_\alpha g^{\alpha i}$, $b^i = b_\alpha g^{\alpha i}$ and g^{ij} are elements of the matrix inverse to the metric g_{ij}.

The scalar curvature R satisfies the following equation

$$R = R_{\alpha\beta}g^{\alpha\beta} = 2(a_\alpha b^\alpha a_\beta b^\beta - a_\alpha a^\alpha b_\beta b^\beta). \tag{2.4}$$

Rewrite (2.2) as follows

$$R_{hijk} = a_h b_i a_j b_k - a_h a_k b_i b_j - a_i b_h a_j b_k + a_i b_h a_k b_j. \tag{2.5}$$

Multiplying (2.5) by the vector a_m and cycling it up by indices m, k and l we obtain:

$$a_m R_{ijkl} + a_k R_{ijlm} + a_l R_{ijmk} = 0. \tag{2.6}$$

Doing the same procedure for the vector b_m we also get:

$$b_m R_{ijkl} + b_k R_{ijlm} + b_l R_{ijmk} = 0. \tag{2.7}$$

Pseudo-Riemannian spaces in which there exists a vector field a_m satisfying conditions (2.6) are called *weakly recurrent*, [6,7]. Thus the following theorem holds:

Theorem 2.1. *Each space of split curvature is weakly recurrent.*

Consider further spaces of split curvature for which the vectors a_i and b_i are not isotropic. We will prove the following theorem:

Theorem 2.2. *If vectors a_i and b_i are not isotropic in the space of split curvature, then the following condition holds:*

$$\frac{R}{2}R_{ijkl} = R_{li}R_{kj} - R_{lj}R_{ki}. \tag{2.8}$$

Proof. It follows from (2.6) that

$$a_\alpha R^\alpha_{jkl} = a_l R_{jk} - a_k R_{jl}. \tag{2.9}$$

Multiplying (2.6) by the vector a^m and wrapping by the index m we obtain:

$$a_\alpha a^\alpha R_{ijkl} + a^\alpha R_{\alpha lji} a_k + a^\alpha R_{\alpha kij} a_k = 0. \tag{2.10}$$

Substituting further (2.9) into (2.10) we get:

$$a_\alpha a^\alpha R_{ijkl} = a_k a_j R_{li} - a_k a_i R_{lj} + a_l a_i R_{kj} - a_l a_j R_{ki}. \tag{2.11}$$

Performing ismilar operations for the vector b^m we obtain that

$$b_\alpha b^\alpha R_{ijkl} = b_k b_j R_{li} - b_k b_i R_{lj} + b_l b_i R_{kj} - b_l b_j R_{ki}. \tag{2.12}$$

Multiply (2.6) by the vector b^m and fold

$$a_\alpha b^\alpha R_{ijkl} = a_k b_j R_{li} - a_k b_i R_{lj} + a_l b_i R_{kj} - a_l b_j R_{ki}. \tag{2.13}$$

From equation (2.7) we obtain

$$a_\alpha b^\alpha R_{ijkl} = b_k a_j R_{li} - b_k a_i R_{lj} + b_l a_i R_{kj} - b_l a_j R_{ki}.$$

Let us perform the following transformations. Taking into account (2.11), (2.12), (2.13), multiply equation (2.10) by $b_\beta b^\beta, a_\beta a^\beta, a_\beta b^\beta, a_\beta b^\beta$. This will give the following identities:

$$\begin{aligned} a_\alpha a^\alpha b_\beta b^\beta R_{ijkl} = {} & a_k a_j b_\beta b^\beta R_{li} - a_k a_i b_\beta b^\beta R_{lj} + \\ & + a_l a_i b_\beta b^\beta R_{kj} - a_l a_j b_\beta b^\beta R_{ki} \end{aligned} \tag{2.14}$$

$$\begin{aligned} a_\beta a^\beta b_\alpha b^\alpha R_{ijkl} = {} & b_k b_j a_\beta a^\beta R_{li} - b_k b_i a_\beta a^\beta R_{lj} + \\ & + b_l b_i a_\beta a^\beta R_{kj} - b_l b_j a_\beta a^\beta R_{ki}. \end{aligned} \tag{2.15}$$

$$\begin{aligned} a_\alpha b^\alpha a_\beta b^\beta R_{ijkl} = {} & a_k b_j a_\beta b^\beta R_{li} - a_k b_i a_\beta b^\beta R_{lj} + \\ & + a_l b_i a_\beta b^\beta R_{kj} - a_l b_j a_\beta b^\beta R_{ki}. \end{aligned} \tag{2.16}$$

$$a_\alpha b^\alpha a_\beta b^\beta R_{ijkl} = b_k a_j a_\beta b^\beta R_{li} - b_k a_i a_\beta b^\beta R_{lj} + \\ + b_l a_i a_\beta b^\beta R_{kj} - b_l a_j a_\beta b^\beta R_{ki}. \qquad (2.17)$$

Further, adding equations (2.16) and (2.17), and subtracting (2.14) and (2.15) from the resulting sum, we get

$$\begin{aligned} &R_{ijkl}(a_\alpha b^\alpha a_\beta b^\beta + a_\alpha b^\alpha a_\beta b^\beta - a_\alpha a^\alpha b_\beta b^\beta - a_\beta a^\beta b_\alpha b^\alpha) = \\ &= R_{li}(a_k b_j a_\beta b^\beta + b_k a_j a_\beta b^\beta - a_k a_j b_\beta b^\beta - b_k b_j a_\beta a^\beta) + \\ &+ R_{lj}(a_k a_i b_\beta b^\beta + b_k b_i a_\beta a^\beta - a_k b_i a_\beta b^\beta - b_k a_i a_\beta b^\beta) + \\ &+ R_{kj}(a_l b_i a_\beta b^\beta + b_l a_i a_\beta b^\beta - a_l a_i b_\beta b^\beta - b_l b_i a_\beta a^\beta) + \\ &+ R_{ki}(a_l a_j b_\beta b^\beta + b_l b_j a_\beta a^\beta - a_l b_j a_\beta b^\beta - b_l a_j a_\beta b^\beta). \qquad (2.18) \end{aligned}$$

Taking into account equations (2.4), (2.5) and the reducing similar terms, we will see that the equation (2.18) will be reduced to (2.8). Theorem is completed. □

Note that is theorem is also valid in the case when vectors a_i and b_i are orthogonal. Moreover, if at least one of the vectors a_i or b_i is non-isotropic, then the scalar curvature R is non-zero.

The formulas (2.2), (2.3) and (2.4) allow us to divide the set of pseudo-Riemannian spaces of split curvature into the following three non-overlapping classes:

(A) $a_\alpha a^\alpha \neq 0;\ b_\alpha b^\alpha \neq 0;$
(B) $a_\alpha a^\alpha \neq 0;\ b_\alpha b^\alpha = 0;$
(C) $a_\alpha a^\alpha = b_\alpha b^\alpha = 0.$

For spaces of type (B), the Ricci tensor satisfies the condition

$$R_{ij} = -a_\alpha a^\alpha b_i b_j, \qquad R = 0. \qquad (2.19)$$

Given (2.19), the equation (2.5) can be expressed as

$$R_{hijk} = \frac{1}{(a_\alpha a^\alpha)}(a_h a_k R_{ij} - a_h a_j R_{ik} + a_i a_j R_{hk} - a_i a_k R_{hj}). \qquad (2.20)$$

The third (C) type includes Ricci flat spaces, i.e. the spaces in which

$$R_{ij} = 0. \qquad (2.21)$$

Thus, the following theorem holds:

Theorem 2.3. *Pseudo-Riemannian spaces V_n of split curvature can be divided into three classes, in which conditions* (2.2) *are satisfied, namely* (2.8) *holds for type* (A), (2.20) *for type* (B), *and* (2.21) *for type* (C).

3. EQUIDISTANT SPACES

A pseudo-Riemannian space V_n with a metric tensor g_{ij} is called *equidistant* if there exists a vector field $\psi_i \neq 0$ satisfying the following itentity:

$$\psi_{i,j} = \tau g_{ij}, \tag{3.1}$$

where τ is some invariant. It is also called the equidistant space of the *main class* if $\tau \neq 0$, and of *special class* if $\tau = 0$.

A vector field satisfying equations (3.1) was called a *circular field* by K. Yano [2, 23]. Following N. S. Sinyukov [22], will call it an *equidistant vector field.*

The integrability conditions for the basic equations (3.1) can be formulated as follows:

$$\psi_\alpha R^\alpha_{ijk} = g_{ij}\tau_{,k} - g_{ik}\tau_{,j}. \tag{3.2}$$

These conditions imply that

$$\tau_{,i} = \frac{1}{n-1}\psi_\alpha R^\alpha_i,$$

here $R^h_i = g^{\alpha h}R_{\alpha i}$.

From the integrability conditions (3.1), it is also easy to see that

$$\tau_{,k} = B\psi_k, \tag{3.3}$$

where B is some invariant.

Given (3.3), one can rewrite (3.2) as follows:

$$\psi_\alpha R^\alpha_{ijk} = B(g_{ij}\psi_k - g_{ik}\psi_j). \tag{3.4}$$

Thus

$$\psi_\alpha R^\alpha_i = (n-1)B\psi_i. \tag{3.5}$$

Now we consider equidistant spaces of split curvature.

Multiply (3.4) by a_m and cycle it up by the indices j, k and m. Since conditions (2.6) and (2.7) hold for the spaces of split curvature, we obtain the following expression for the vector a_m:

$$B(a_m g_{ij}\psi_k + a_j g_{ik}\psi_m + a_k g_{im}\psi_j - a_m g_{ik}\psi_j - a_k g_{ij}\psi_m - a_j g_{im}\psi_k) = 0.$$

Folding the latter, we get that

$$B(a_m\psi_k - a_k\psi_m) = 0,$$

Similar identity holds for the vector b:

$$B(b_m\psi_k - b_k\psi_m) = 0.$$

If $B \neq 0$, then we get a contradiction of the requirement that V_n is a non-flat pseudo-Riemannian space. This implies that following theorem:

Theorem 3.1. *In equidistant pseudo-Riemannian spaces of split curvature, the invariant B is zero.*

Thus if $B = 0$, (3.4) and (3.5) reduce to the following identities:

$$\psi_\alpha R^\alpha_{ijk} = 0, \qquad \psi_\alpha R^\alpha_i = 0, \tag{3.6}$$

while equation (3.3) becomes $\tau_{,k} = 0$. Taking into account (3.6), (2.6) and (2.7), we see that the vectors ψ_i, a_i, b_i are mutually orthogonal, i.e.

$$\psi_\alpha a^\alpha = 0, \qquad \psi_\alpha b^\alpha = 0. \tag{3.7}$$

Differentiating (3.7), we see that

$$\tau a_i + \psi_\alpha a^\alpha_{,i} = 0, \qquad \tau b_i + \psi_\alpha b^\alpha_{,i} = 0.$$

In next section we will consider geodesic mappings.

4. GEODESIC MAPPINGS OF PSEUDO-RIEMANNIAN SPACES

Definition 4.1. A bijection between the points of pseudo-Riemannian spaces V_n with metric tensor g_{ij} and $\bar{V}_n$ with metric tensor $\bar{g}_{ij}$ is called a *geodesic mapping* if every geodesic line V_n turns into a geodesic line $\bar{V}_n$.

Two pseudo-Riemannian spaces V_n and $\bar{V}_n$ are said to be in a *geodesic correspondence* or *belong to the same geodesic class* if there exists a geodesic mapping between them.

The deformation tensor of Christoffel symbols under geodesic mappings is given by

$$P^h_{ij} = \delta^h_{(i}\varphi_{j)},$$

where φ_i is some gradient vector, by a necessity, and the brackets (ij) denote symmetrization by indices i and j.

This condition is necessary and sufficient for the existence of a geodesic mappings between two pseudo-Riemannian spaces V_n and $\bar{V}_n$.

It follows from the definition of the deformation tensor that:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \varphi_i\delta^h_j + \varphi_j\delta^h_i. \tag{4.1}$$

In other words, taking into account the covariant constancy of a metric tensor, we get

$$\bar{g}_{ij},_k = 2\varphi_k\bar{g}_{ij} + \varphi_i\bar{g}_{jk} + \varphi_j\bar{g}_{ik}. \tag{4.2}$$

Equations (4.1) and (4.2) are equivalent, and they are the necessary and sufficient conditions for the pseudo-Riemannian space V_n and $\bar{V}_n$ to be in a geodesic correspondence.

Note that for a geodesic mapping the following equations hold:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \varphi_{ij}\delta^h_k - \varphi_{ik}\delta^h_j,$$
$$\bar{R}_{ij} = R_{ij} + (n-1)\varphi_{ij},$$

where $\varphi_{ij} = \varphi_{i,\,j} - \varphi_i \varphi_j$.

A geodesic mapping distinct from homothety is called *nontrivial.*

A pseudo-Riemannian space V_n admits a nontrivial geodesic mapping if and only if there exists a solution of the following system of differential equations with respect to the tensor $a_{ij} = a_{ji} \neq c g_{ij}$ and the vector $\lambda_i = \lambda_{,i} \neq 0$:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{4.3}$$

This system is said to be the *linear form* of the basic equations.

Integrability conditions for these equations are given by the following identities:

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha j} R^{\alpha}_{ikl} = \lambda_{li} g_{jk} + \lambda_{lj} g_{ik} - \lambda_{ki} g_{jl} - \lambda_{kj} g_{il}. \tag{4.4}$$

Note that (4.4) imply that

$$n \lambda_{i,j} = \mu g_{ij} + a_{\alpha i} R^{\alpha}_{j} - a_{\alpha\beta} R^{\alpha\ \ \beta}_{.\,ij\,.}, \tag{4.5}$$

where $\mu = \lambda_{\alpha,\beta} g^{\alpha\beta}$ and $R^{h\ \ k}_{\ ij} = R^{h}_{ij\alpha} g^{\alpha\ k}_{\ .}$. Hence

$$(n-1)\mu_{,i} = 2(n+1)\lambda_\alpha R^{\alpha}_{i} + a_{\alpha\beta}(2R^{\alpha\ \ \beta}_{.\,i,\,.} - R^{\alpha\beta}_{\ \ \ ,i}). \tag{4.6}$$

The solutions (4.2) and (4.3) are related as follows:

$$a_{ij} = e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \qquad \lambda_i = -e^{2\varphi} \bar{g}^{\alpha\beta} g_{\alpha i} \varphi_\beta.$$

The system of equations (4.3), (4.5) and (4.6) gives a possibility to answer the fundamental question: whether a given pseudo-Riemannian space V_n admits a geodesic mapping to the pseudo-Riemannian space $\bar{V}_n$. The question boils down to studying the conditions for the integration of these equations and their differential extensions.

Consider a geodesic mappings of spaces of split curvature. It is well known, that they belong to the class of semisymmetric pseudo-Riemannian spaces. Semi-symmetric pseudo-Riemannian spaces admit geodesic mappings only if they are equidistant, and the condition (3.1) is satisfied by the vector λ_i from the linear form of the basic equations, namely:

$$\lambda_{i,j} = \mu g_{ij}, \qquad \mu_{,i} = 0.$$

Now, integrability conditions will be written as follows:

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha j} R^{\alpha}_{ikl} = 0, \tag{4.7}$$

$$\lambda_\alpha R^{\alpha}_{ijk} = 0.$$

Multiplying the equation (4.7) by the vector a^l, wrapping it by the index l, and taking into account the condition (2.9), we obtain

$$a_{\alpha i} a^{\alpha} R_{jk} - a_{\alpha i} R^{\alpha}_{k} a_j + a_{\alpha j} a^{\alpha} R_{ik} - a_{\alpha j} R^{\alpha}_{k} a_i = 0. \tag{4.8}$$

Symmetrizing further the latter identity by the indices i, k and taking to account that $a_{\alpha i}R^{\alpha}_{j} = a_{\alpha j}R^{\alpha}_{i}$, we get that

$$a_{\alpha i}a^{\alpha}R_{jk} - a_{\alpha k}a^{\alpha}R_{ji} - a_{\alpha j}R^{\alpha}_{k}a_{i} + a_{\alpha j}R^{\alpha}_{i}a_{k} = 0.$$

Let us rename the indices j and k

$$a_{\alpha i}a^{\alpha}R_{jk} - a_{\alpha j}a^{\alpha}R_{ki} - a_{\alpha k}R^{\alpha}_{j}a_{i} + a_{\alpha k}R^{\alpha}_{i}a_{j} = 0. \tag{4.9}$$

Adding equations (4.9) and (4.8) we obtain that

$$a_{\alpha i}a^{\alpha}R_{jk} - a_{\alpha k}R^{\alpha}_{j}a_{i} = 0.$$

Wrapping this identity by indices j and k we get that

$$a_{\alpha i}a^{\alpha} = \overset{3}{\tau} a_{i}, \tag{4.10}$$

where $\overset{3}{\tau} = a^{\alpha\beta}R_{\alpha\beta} - R$. This finally gives the following equation holds for the vector b_i:

$$a_{\alpha i}b^{\alpha} = \overset{3}{\tau} b_{i}.$$

Thus we proved the following theorem:

Theorem 4.2. *The vectors a_i and b_i are the eigenvectors of the tensor matrix with the same invariant $\overset{3}{\tau}$.*

By differentiating (4.10) we obtain

$$\lambda_{i}a_{j} + a_{\alpha i}a^{\alpha}_{,j} = \overset{3}{\tau}_{,j} a_{i} + \overset{3}{\tau} a_{i,j}. \tag{4.11}$$

Multiplying (4.11) by vector a^i and wrapping it up by index i yields

$$\overset{3}{\tau}_{,j} a_{\alpha}a^{\alpha} = 0.$$

Similar identity holds for the vector b^i:

$$\overset{3}{\tau}_{,j} b_{\alpha}b^{\alpha} = 0.$$

It followss that if at least one of the vectors a_i or b_i is non-isotropic, then $\overset{3}{\tau}_{,i} = 0$.

Corollary 4.3. *For nontrivial geodesic mappings of spaces of split curvature of type* (A) *and* (B)*, the vectors a_i and b_i are eigenvectors of the tensor matrix a_{ij} with the same constant $\overset{3}{\tau}$.*

Theorem 4.2 allows to calculate the value of $\overset{3}{\tau}$ after substituting the conditions (2.3) and (2.4)

$$\overset{3}{\tau} = a^{\alpha\beta}R_{\alpha\beta} - R = (\overset{3}{\tau} - 1)R.$$

If, in addition, $R \neq 1$, then

$$\overset{3}{\tau} = \frac{R}{R-1}.$$

Now, taking into account Corollary 4.3, we see that $R_{,i} = 0$, so we arrived at the following theorem

Theorem 4.4. *Every space of split curvature admitting nontrivial geodesic mappings is a spaces of constant scalar curvature.*

Thus, for a space of type (B) and (C) the scalar curvature is always zero, and for a space of type (A) the scalar curvature is constant (by necessary condition) if that space admits nontrivial geodesic mappings.

5. Conclusion

The spaces of split curvature are divided into three types (A), (B) and (C). If they admit nontrivial geodesic mappings, they are equidistant spaces.

Pseudo-Riemannian spaces of type (A) are internally characterized by the objects defined by a metric tensor. If they admit nontrivial geodesic mappings, then their scalar curvature is constant.

Spaces of type (B) are almost Einstein spaces. Geodesic mappings of such spaces were studied in [5, 11, 13–15, 17] and those results agree with the results of the present paper.

Spaces of type (C) are Einstein spaces. Geodesic mappings of Einstein spaces were studied in [3, 8, 9, 21, 24]. In particular, it is known that four-dimensional Einstein spaces distict from spaces of constant curvature do not allow nontrivial geodesic mappings. Therefore, there are no four-dimensional spaces of type (C) that admit nontrivial geodesic mappings. On the other hand, there are pseudo-Riemannian spaces admitting nontrivial geodesic mappings to spaces of type (C) for $n > 4$, which admit $\varphi(Ric)$ vector fields.

The geometric properties of such pseudo-Riemannian spaces were studied in [4, 7, 12, 16, 18]. The spaces of split curvature [1, 10, 19] include pseudo-Riemannian spaces admitting $(n-2)$ equivariant vector fields and having maximum mobility with respect to nontrivial geodesic mappings.

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