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The Lie bracket structure of the string homology on a formal space

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Abstract. In this paper, we consider the Chas–Sullivan loop space homology $\mathbb{H}_*(X^{S^1})$ of a formal elliptic 2-stage Postnikov tower X. We show that the center of the graded Lie algebra $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ on a minimal Sullivan model of X is non-trivial.

Анотація. В роботі розглядаються гомології простору петель Часа-Саллівана $\mathbb{H}_*(X^{S^1})$ формальної еліптичної 2-ступеневої башти Постнікова X. Показано, що центр градуйованої алгебри Лі $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ на мінімальній моделі Саллівана простору X є нетривіальним.

1. Introduction

A *differential graded algebra* is of the form $A = \oplus_{i\geqslant 0}A^i$, where A^i is a vector space over a commutative ring $\Bbbk$, equipped with an associative multiplication

$$A^i \otimes A^j \to A^{i+j}, \qquad x\otimes y \mapsto xy,$$

and a differential $d\colon A^n \to A^{n+1}$ such that

$$d(xy) = (dx)y + (-1)^{|p|}x(dy)$$

for all $x \in A^p$, $y \in A^q$, and $d^2 = 0$. It is said to be *connected* if $H^0(A) \cong \mathbb{Q}$ (for $\Bbbk = \mathbb{Q}$) and commutative if $xy = (-1)^{|p||q|}yx$.

If $V = \oplus_{i\geqslant 1}V^i$, then $\wedge V$ denotes the free commutative graded algebra defined by $\wedge V = T(V)/I$, where $T(V)$ is the tensor algebra on V and I is the ideal generated by

$$v\otimes u - (-1)^{|v||u|}u\otimes v,$$

with $v, u \in V$. A *Sullivan algebra* is a commutative differential graded algebra (*cdga* for short) of the form $(\wedge V, d)$, where $V = \bigcup_{k\geqslant 0} V(k)$ and

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$V(0) \subset V(1)\dots$ such that $dV(0) = 0$ and $dV(k) \subset \wedge V(k-1)$. It is called *minimal* if $dV \subset \wedge^{\geqslant 2} V$.

If (A, d) is a cdga of which the cohomology is connected and finite dimensional in each degree, then there always exists a quasi-isomorphism from a Sullivan algebra $(\wedge V, d)$ to (A, d). To each simply connected space, Sullivan associates a cdga $A_{PL}(X)$ of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X. A minimal Sullivan model of X is a minimal Sullivan model of $A_{PL}(X)$ [2].

We now describe the notion of Hochschild cohomology. The Hochschild cohomology of a differential graded algebra was introduced by Paul G. Hochschild in 1945 [13]. The Hochschild cohomology ring of a differential graded algebra A is a Gerstenhaber algebra, that is, it has both a cup product and a graded Lie bracket, which was first discovered by Gerstenhaber in [12].

An important problem then is to describe the algebraic structure of the Hochschild cohomology $HH^*(A;A)$ of an associative algebra A as a graded vector space, as a ring, and as a graded Lie algebra. As a result, many mathematicians have investigated the Hochschild cohomology ring $HH^*(A;A)$ for various types of algebras (see for instance [14]). In particular, the structure of the Hochschild cohomology ring of a Sullivan algebra $(\wedge V, d)$ has been studied extensively in literature (see for instance, [4, 7, 9–11, 15]). In fact, Hochschild cohomology of a Sullivan algebra corresponds to the free loop space homology $\mathbb{H}_*(X^{S^1})$, where $X^{S^1} = \mathrm{map}(S^1, X)$ is the space of free loops on a closed oriented manifold X of dimension m as Gerstenhaber algebra (see for instance, [1, 4, 6–10]).

A space X and its model $(\wedge V, d)$ are called *elliptic* if and only if V and $H^*(\wedge V, d)$ are both finite dimensional. Topologically, this means that both $\pi_*(X) \otimes \mathbb{Q}$ and $H^*(X;\mathbb{Q})$ are finite-dimensional $\mathbb{Q}$-vector spaces [2, §32]. For instance homogeneous spaces are elliptic.

An elliptic space X is an *n-stage Postnikov tower* if its minimal Sullivan model is given by

$$(\wedge V, d) = (\wedge(V_0 \oplus \cdots \oplus V_{n-1}), d),$$

where $dV_i \subset \wedge(V_0 \oplus \cdots \oplus V_{i-1})$. Every homogeneous space is an elliptic 2-stage Postnikov towers and admits a minimal Sullivan model of the form

$$(\wedge V, d) = (\wedge(V_0 \oplus V_1), d),$$

where V is finite dimensional and $dV_0 = 0$, $dV_1 \subseteq \wedge V_0$. Denote by $\langle v_1, v_2, \dots, v_n\rangle$ the vector space generated by a finite basis $\{v_i\}$ of V and

$$V_0^{\text{even}} = \mathbb{Q}\langle p_1, \dots, p_q\rangle = P, \qquad V_0^{\text{odd}} = \mathbb{Q}\langle w_1, \dots, w_r\rangle = W,$$
$$V_1^{\text{odd}} = \mathbb{Q}\langle y_1, \dots, y_p\rangle = Y,$$

so that

$$(\wedge(V_0 \oplus V_1), d) \xrightarrow{\cong} (\wedge(P \oplus Y), d) \otimes (\wedge W, 0),$$

and $dP = 0$, $dY \subseteq \wedge P$.

A simply connected space B is called *formal* (see [5]) if there is a quasi-isomorphism $(\wedge W, d) \to H^*(\wedge W, d)$, where $(\wedge W, d)$ is the minimal Sullivan model of B. Examples of formal spaces include spheres, projective spaces, homogeneous spaces G/H where G and H have equal rank.

Now, if the elliptic 2-stage Postnikov tower X is a formal, then $p = q$, and we have

$$H^*(\wedge V, d) = \frac{\wedge(p_1, \ldots, p_p)}{(\alpha_1, \ldots, \alpha_p)} \otimes \wedge(w_i, \ldots, w_r),$$

where $(\alpha_1, \ldots, \alpha_p)$ is a regular sequence in $\wedge P$. Hence, X as a formal elliptic 2-stage Postnikov tower admits a minimal Sullivan model of the form

$$(\wedge V, d) = (\wedge(P \oplus Y), d) \otimes (\wedge W, 0),$$

where $dP = 0$, $dy_k = \alpha_k$. If A is a cochain algebra, then $(sA)^n = A^{n+1}$. It is known in literature (see [10, Theorem 13]) that, if X is a simply connected homogeneous space of which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then the graded Lie algebra $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is not nilpotent. However, we could not find to the best of our knowledge any reference in literature that gives a complete description of the Lie bracket structure of the string homology on a space. In this note, we give an explicit description of the partial computation of the Lie bracket structure of the string homology on a formal elliptic 2-stage Postnikov tower. Our main result reads as follows.

Theorem 1.1. *If X is a formal elliptic 2-stage Postnikov tower, then the center of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is non-trivial.*

2. Hochschild cohomology

Let (A, d) be an augmented differential graded cochain algebra over a field $\Bbbk$ of characteristic zero and $\bar{A} = \ker(\varepsilon\colon A \to \Bbbk)$. We define here the Hochschild cohomology through the bar construction of (A, d). If (P, d) is a right differential graded A-module and (N, d) a left graded differential A-module, the definition of the *two sided* (*normalized*) *bar construction* on (A, d) is as follows (see for instance [3, 4]). It is the complex

$$(\mathbb{B}(P; A; N), D) = (\oplus_k \mathbb{B}_k(P; A; N), D)$$

with

$$\mathbb{B}_k(P; A; N) = P \otimes T^k(s\bar{A}) \otimes N, \ k \geqslant 1.$$

A generic element $p[a_1|a_2|\dots|a_k]n$ in $\mathbb{B}_k(P;A;N)$ has (upper) degree $|p|+|n|+\sum_{i=1}^k(|sa_i|)$. If $k=0$, then

$$p[\ ]n = p\otimes 1\otimes n \in P\otimes T^0(s\bar{A})\otimes N.$$

The differential D decomposes into two terms $D = d_0 + d_1$ as follows, $d_0\colon \mathbb{B}_k(P;A;N)\to\mathbb{B}_k(P;A;N)$, with

$$\begin{aligned} d_0(p[a_1|a_2|\dots|a_k]n) &= d(p)[a_1|a_2|\dots|a_k]n \\ &\quad - \sum_{i=1}^k (-1)^{\epsilon_i} p[a_1|a_2|\dots|d(a_i)|\dots|a_k]n \\ &\quad + (-1)^{\epsilon_{k+1}} p[a_1|a_2|\dots|a_k]d(n), \end{aligned}$$

and $d_1\colon \mathbb{B}_k(P;A;N)\to\mathbb{B}_{k-1}(P;A;N)$, is given by

$$\begin{aligned} d_1(p[a_1|a_2|\dots|a_k]n) &= (-1)^{|p|} pa_1[a_2|\dots|a_k]n \\ &\quad + \sum_{i=2}^k (-1)^{\epsilon_i} p[a_1|a_2|\dots|a_{i-1}a_i|\dots|a_k]n \\ &\quad - (-1)^{\epsilon_k} p[a_1|a_2|\dots|a_{k-1}]a_k n, \end{aligned}$$

where $\epsilon_i = |p| + \sum_{j<i}(sa_j)$.

There is a canonical projection $\varphi\colon \mathbb{B}(A;A;A)\to A$ defined by $\varphi([\ ])=1$ and $\varphi([a_1|\dots|a_k])=0$ if $k>0$ which provides a semi-free resolution of A as an A^e-module [3]. Thus, $HH^*(A;A)$ is the homology of the normalized Hochschild cochain complex

$$(C^*(A;A),D) = \mathrm{Hom}_{A^e}(\mathbb{B}(A;A;A),A) \cong (\mathrm{Hom}(T^k(s\bar{A}),A), D_0+D_1).$$

The differential D_0+D_1 is defined as follows [4]:

$$(D_0f)([a_1|a_2|\dots|a_k]) = d(f([a_1|a_2|\dots|a_k])) + \sum_{i=1}^k (-1)^{\bar\epsilon(i)} f([a_1|a_2|\dots|a_k])$$

and

$$\begin{aligned} (D_1f)([a_1|a_2|\dots|a_k]) &= -(-1)^{|sa_1||f|} a_1 f([a_2|\dots|a_k]) \\ &\quad + (-1)^{\bar\epsilon(k)} f([a_1|\dots|a_{k-1}])a_k \\ &\quad + \sum_{i=2}^k (-1)^{\bar\epsilon(i)} f([a_1|\dots|a_{i-1}a_i|\dots|a_k]), \end{aligned}$$

where

$$\bar\epsilon(i) = |f| + |sa_1| + \dots |sa_{i-1}|.$$

As $T^k(s\bar{A})$ is a graded coalgebra, the complex $C^*(A;A)$ is endowed with a cup product, making it a differential graded algebra. Further, there is a

Lie bracket structure on $C^*(A; A)$, giving a Gerstenhaber algebra structure (see [12]). The Lie bracket is defined by the formula

$$\{f \otimes g\} = f \bar{\circ} g - (-1)^{|f||g|} g \bar{\circ} f,$$

where

$$f \bar{\circ} g[a_1|\dots|a_k]) = \sum_{0 \leqslant i \leqslant j \leqslant k} (-1)^{\epsilon(i)} f([a_1|\dots|a_i)|g([a_{i+1}|\dots|a_j])|a_{j+1}|\dots|a_k]),$$

and $\epsilon(i) = |g|(|sa_1|\dots|sa_i|)$.

3. Hochschild cohomology of a Sullivan algebra

To compute the Hochschild cohomology of the minimal Sullivan algebra $(\wedge V, d)$, one considers a relative Sullivan model of the multiplication

$$m \colon (\wedge V \otimes \wedge V, d') \to (\wedge V, d),$$

where $d' = d \otimes 1 + 1 \otimes d$, and the model is given by the following commutative diagram

$$\begin{array}{ccc} (\wedge V \otimes \wedge V, d') & \xrightarrow{\quad m \quad} & (\wedge V, d) \\ & \searrow^{i} & \cong \uparrow \varphi \\ & & (\wedge V \otimes \wedge V \otimes \wedge sV, d), \end{array}$$

where $(sV)^n = V^{n+1}$ and the differential D is defined by

$$d(sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 + \alpha, \quad \alpha \in \wedge V \otimes \wedge V \otimes \wedge^+ sV,$$

and i is the canonical inclusion [2, §15].

Proposition 3.1. [2] *φ is a semi-free resolution of $(\wedge V, d)$ as a $\wedge V \otimes \wedge V$ differential module.*

Therefore, the Hochschild cohomology $HH^*(\wedge V; \wedge V)$ is given by

$$HH^*(\wedge V; \wedge V) \cong H_*(\mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge sV, \wedge V), D').$$

Moreover, $(\mathrm{Hom}_{\wedge V \otimes \wedge V}(\wedge V \otimes \wedge V \otimes \wedge sV, \wedge V), D')$ is isomorphic to

$$(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V), D)$$

where $(\wedge V \otimes \wedge sV, \bar{d})$ is the Sullivan algebra such that

$$\bar{d}v = dv \quad \text{and} \quad \bar{d}(sv) = -S(dv),$$

while S is the unique derivation of $\wedge V \otimes \wedge sV$ defined by

$$S(v) = sv \quad \text{and} \quad S(sv) = 0.$$

Proposition 3.2. *With the above notation, we have an isomorphism*

$$HH^*(\wedge V; \wedge V) \cong H_*(\mathrm{Hom}_{\wedge V}(\wedge V \otimes \wedge sV, \wedge V), D).$$

If $(\wedge V, d)$ is the minimal Sullivan model of X, then $(\wedge V \otimes \wedge sV, \bar{d})$ is a Sullivan model of the free loop space X^{S^1} (see [2, §15]).

Let (A, d) be a differential graded algebra. A *derivation* θ of A of degree k is a linear mapping $\theta\colon A^n \to A^{n-k}$ such that

$$\theta(ab) = \theta(a)b + (-1)^{k|a|} a\theta(b).$$

Denote by $\mathrm{der}_k A$ the vector space of all derivation of degree k and put $\mathrm{der}A = \oplus_k \mathrm{der}_k A$. The Lie bracket induces a graded Lie algebra structure on $\mathrm{der}A$. On the other hand, $(\mathrm{der}A, \delta)$ is a differential graded Lie algebra [16] with the differential δ defined in the usual way by

$$\delta\theta = [d, \theta] = d \circ \theta - (-1)^k \theta \circ d.$$

Furthermore, $\mathrm{der}A$ is a differential graded A-module via the action

$$(a\theta)(x) = a\theta(x).$$

With the grading convention $A_{-n} = A^n$, if $\theta_1 \in \mathrm{der}_k A$ and $a \in A^i$, then $a\theta_1 \in \mathrm{der}_{k-i} A$. That is, $\mathrm{der}A$ is a graded A-module which satisfies the relation

$$[d, a\theta] = (da)\theta + (-1)^{|a|} a[d, \theta].$$

Let $\theta_1, \theta_2 \in \mathrm{der}A$ and $a \in A$. Then

$$[\theta_1, a\theta_2] = \theta_1(a)\theta_2 + (-1)^{|a||\theta_1|} a[\theta_1, \theta_2] \qquad \text{(see [10]).} \tag{3.1}$$

Let $A = (\wedge V, d)$ be a Sullivan algebra, where V is spanned by $\{v_1 \ldots, v_k\}$. Then

$$\mathrm{der}A \cong A \otimes V^*,$$

where V^* is the graded dual of V [10].

Consider $\theta_i = (v_i, 1)$, the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. Then the graded $\wedge V$-module $\mathrm{der} \wedge V$ is spanned by $\{\theta_1, \ldots, \theta_k\}$.

Further, if A is a minimal Sullivan algebra, where V is finite dimensional, then the Hochschild cohomology $HH^*(A; A)$ can be computed in terms of derivations of A. This is another method to compute the $\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ of a simply connected space X for which $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional [10].

4. The main result

The *quaternionic Grassmannian* $G_{k,n}(\mathbb{H})$ of k-dimensional vector subspaces of $\mathbb{H}^n$ is a homogeneous space as

$$G_{k,n}(\mathbb{H}) \cong \mathrm{Sp}(n)/(\mathrm{Sp}(k) \times \mathrm{Sp}(n-k)) \quad \text{for} \quad 1 \leqslant k < n,$$

where $\mathrm{Sp}(n)$ is the symplectic group.

Below we compute the free loop space homology for some homogeneous spaces. Recall that, if X is homogeneous space, then it is an elliptic 2-stage Postnikov tower and its minimal Sullivan model is given by

$$(A, d) = (\wedge(b_1, \ldots, b_n, a_1, \ldots, a_m), d),$$

where $db_i = 0$ and $da_i \in \wedge(b_1, \ldots, b_n)$ (see [5]). Thus, in order to compute the loop space homology of X we consider a complex of the form

$$(A \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n), d)$$

where $z_j = s^{-1}a_j^*$, $u_i = s^{-1}b_i^*$, $dz_j = 0$ and $du_i = \sum \frac{\partial f_j}{\partial b_i} z_j$ with $f_j = db_j$.

Further, if A is a minimal Sullivan model of a simply connected compact oriented m-manifold X such that $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then there is a filtration on $A \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n)$ which yields a spectral sequence of Gerstenhaber algebras for which

$$E^1 = H^*(A) \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n)$$

and which converges to

$$H_*(A \otimes \wedge(z_1, \ldots, z_m, u_1, \ldots, u_n), d) \cong \mathbb{H}_*(X^{S^1}; \mathbb{Q}) \qquad \text{(see [10])}.$$

In particular, if $X = G/K$ is a homogeneous space of which G and K have an equal rank, then

$$\mathbb{H}_*(X^{S^1}; \mathbb{Q}) \cong H_*(H^*(A) \otimes \wedge(Z_0 \oplus Z_1), d),$$

where $dZ_0 = 0$ and $dZ_1 \subset H^+ \otimes Z_0$ (see [10]).

Example 4.1. [5, 10] Let $X = \mathbb{C}P(n)$ of which the minimal model is $A = (\wedge(b_2, a_{2n+1}), d)$, $db_2 = 0$, $da_{2n+1} = b_2^{n+1}$. Thus,

$$\mathbb{H}_*(\mathbb{C}P(n), \mathbb{Q}) \cong H_*((\wedge b_2)/(b_2^{n+1}) \otimes \wedge(z_1, z_{2n}), d),$$
$$dz_{2n} = 0, \quad dz_1 = (n+1)b_2^n z_{2n},$$

while homology classes are

$$\{b^j z_{2n}^k, b^i z_1, b^i z_1 z_{2n}^k,\ k \geqslant 0,\ 0 \leqslant j \leqslant n-1,\ 1 \leqslant i \leqslant n\}.$$

Example 4.2. We consider the Sullivan minimal model of

$$X = \mathrm{Sp}(5)/(\mathrm{Sp}(2) \times \mathrm{Sp}(3))$$

which is given by

$$A = (\wedge(b_4, b_8, a_{15}, a_{19}), d),$$

where $db_i = 0$, and

$$da_{15} = 2b_8 b_4^2 + b_4^4 - b_8^2, \qquad da_{19} = 2b_8 b_4^3 + b_8^2 b_4.$$

Consider the ideal

$$I = \left(2b_8 b_4^2 + b_4^4 - b_8^2, 2b_8 b_4^3 + b_8^2 b_4\right).$$

It follows that, $H^*(\wedge V, d) = \wedge(b_4, b_8)/I$. Hence, there is a quasi-isomorphism

$$f\colon (\wedge(b_4, b_8, a_{15}, a_{19}), d) \xrightarrow{\cong} H^*(\wedge V, d).$$

Thus, Sp(5)/(Sp(2) × Sp(3)) is formal. The rational cohomology is given by classes of

$$\left\{\begin{matrix} 1, b_4, b_4^2, b_8, b_4^3, b_4b_8, b_8b_4^2, b_4^4, [2b_8b_4^2 + b_4^4] = [b_8^2], \\ [b_4^5] = [-2b_4^3b_8] = [b_8^2b_4], [b_4^6] = [b_4^2b_8^2] = [b_8^3] \end{matrix}\right\}$$

Thus, to compute the loop space homology of $X = \mathrm{Sp}(5)/(\mathrm{Sp}(2) \times \mathrm{Sp}(3))$ we consider a complex of the form

$$(A \otimes \wedge(z_{14}, z_{18}, u_3, u_7), d),$$

where $dz_i = 0$, and

$$du_3 = (4b_4b_8 + 4b_4^3)z_{14} + (6b_8b_4^2 + b_8^2)z_{18},$$
$$du_7 = (2b_4^2 - 2b_8)z_{14} + (2b_4^3 + 2b_4b_8)z_{18}.$$

In some lower degrees, the loop space homology is given by classes of

$$\left\{\begin{matrix} 1, z_{14}, z_{18}, b_4, b_4^2, b_8, b_4^3, b_4b_8, b_8b_4^2, b_4^4, [2b_8b_4^2 + b_4^4] = [b_8^2], \\ b_4z_{14}, b_4^2z_{14}, b_8z_{14}, b_4^3z_{14}, b_4b_8z_{14}, b_8b_4^2z_{14}, b_4^4z_{14}, \\ [b_4^5] = [-2b_4^3b_8] = [b_8^2b_4], [b_4^6] = [b_4^2b_8^2] = [b_8^3], b_8^2z_{14}, b_4^5z_{14}, b_4^6z_{14}, \\ b_4z_{18}, b_4^2z_{18}, b_8z_{18}, b_4^3z_{18}, b_4b_8z_{18}, b_8b_4^2z_{18}, b_4^4z_{18}, b_8^2z_{18}, b_4^5z_{18}, b_4^6z_{18}, \\ (4b_4b_8 + 4b_4^3)z_{14}, (6b_8b_4^2 + b_8^2)z_{18}, (2b_4^2 - 2b_8)z_{14}, (2b_4^3 + 2b_4b_8)z_{18} \end{matrix}\right\}$$

Definition 4.3. Let L be a Lie algebra. The *center* $Z(L)$ is defined by

$$Z(L) = \{x \in L : [x, y] = 0, \forall y \in L\}.$$

In [10, Theorem 13], it is shown that, if X is a simply connected homogeneous space such that $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, then the graded Lie algebra $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is not nilpotent. In addition, we establish our main result.

Theorem 4.4. *If X is a formal elliptic 2-stage Postnikov tower, then the center of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$ is non-trivial.*

Proof. Recall that a formal elliptic 2-stage Postnikov tower admits a minimal Sullivan model of the form

$$A = (\wedge V, d) = (\wedge(P \oplus Y), d) \otimes (\wedge W, 0),$$

where $dP = 0$, $dy_k = \alpha_k$.

Further, there is filtration (see [10]) on $(A \otimes \wedge s^{-1}V^*, d)$ which yields a spectral sequence of Gerstenhaber algebras for which $E^1 = H^*(A) \otimes \wedge Z$, where $Z = Z_0 \oplus Z_1$, and $Z_0 = s^{-1}V_1^*$, $Z_1 = s^{-1}V_0^*$. Let $L = s^{-1}\text{der}A$. Given $a \in \wedge_A^0 L = A$, $\theta_1, \theta_2 \in \text{der}A$, and for $x, \beta \in L$, and using

$$[\theta_1, a\theta_2] = \theta_1(a)\theta_2 + (-1)^{|a||\theta_1|}a[\theta_1, \theta_2] \tag{4.1}$$

we have

$$\{x, a\} = -(-1)^{|a|}(sx)(a). \tag{4.2}$$

Hence,

$$\{x, a\beta\} = \{x, a\}\beta + (-1)^{|a|(|x|+1)}a\{x, \beta\}, \quad \text{(see [10])}. \tag{4.3}$$

Moreover $\wedge_A L$ and $(A \otimes \wedge(Z_0 \oplus Z_1), d)$ are isomorphic as differential Gerstenhaber algebras. It is sufficient to check that $\wedge Z_0$ is abelian in $(H^*(A) \otimes \wedge Z, d)$. It follows from equation (4.2), that for all $z_i, z_j \in Z_0$ we have

$$\{z_i, z_j\} = -(-1)^{|z_i|}s(z_i)(z_j) = 0.$$

Furthermore, using (4.3), for all $a_i \neq 0 \in H^*(A)$,

$$v_i \neq 0 \in H^*(A \otimes \wedge(Z_0 \oplus Z_1)) \quad \text{and} \quad z_i \in Z_0$$

one gets

$$\begin{aligned}\{a_i z_i, v_i\} &= a_i\{z_i, v_i\} + (-1)^{|a_i|(|v_i|+1)}z_i\{a_i, v_i\} \\ &= 0, \quad \text{as } \{v_i, z_i\} = \{a_i, v_i\} = 0, \quad \text{using (4.2)}.\end{aligned}$$

Also, using (4.2), if $x_i \neq 0$ is a cocycle in $H^*(A) \otimes \wedge^+(Z_0 \oplus Z_1)$, then $\{Z_0, x_i\} = 0$. Hence $\wedge Z_0$ is abelian. Thus, the algebra $\wedge Z_0$ is in the center of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$. □

Remark 4.5. The formality condition on X is necessary for $\wedge Z_0$ to be in the center of $s\mathbb{H}_*(X^{S^1}; \mathbb{Q})$. We consider the minimal Sullivan model of the non-formal homogeneous space $X = \text{Sp}(6)/\text{SU}(6)$ which is given by

$$A = (\wedge(x_6, x_{10}, b_{15}, b_{19}, b_{23}), d),$$

$$dx_i = 0, \quad db_{15} = x_6x_{10}, \quad db_{19} = x_{10}^2, \quad db_{23} = x_6^4.$$

The rational cohomology is given by classes of

$$\left\{\begin{matrix} 1, x_6, x_{10}, x_6^2, x_6^3, x_6b_{19} - x_{10}b_{15}, x_6^2b_{19} - x_6x_{10}b_{15}, \\ x_6^3b_{15} - x_{10}b_{23}, x_6^3b_{19} - x_6^2x_{10}b_{15}, x_6^4b_{19} - x_6^3x_{10}b_{15} \end{matrix}\right\}$$

The loop space homology of $X = \text{Sp}(6)/\text{SU}(6)$ is computed from the complex

$$(A \otimes \wedge(z_{14}, z_{18}, z_{22}, b_5, b_9), d),$$

$$dz_i = 0, db_5 = x_{10}z_{14} + 4x_6^3z_{22}, db_9 = x_6z_{14} + 2x_{10}z_{18},$$

which is isomorphic to $(A \otimes (Z_0 \oplus Z_1), d)$, where $dZ_0 = 0$, $dZ_1 \subseteq A \otimes Z_0$. It contains $H^*(X) \otimes \wedge(z_{14}, z_{18}, z_{22})/I$ where I is the ideal generated by $\{db_5, db_9\}$. In some lower degrees, the loop space homology is given by classes of

$$\left\{\begin{array}{c} 1, z_{14}, z_{18}, z_{22}, x_6, x_{10}, x_6^2, x_6^3, x_6 b_{19} - x_{10} b_{15}, \\ x_6^2 b_{19} - x_6 x_{10} b_{15}, x_6^3 b_{15} - x_{10} b_{23}, x_6^3 b_{19} - x_6^2 x_{10} b_{15}, \\ x_6^4 b_{19} - x_6^3 x_{10} b_{15}, x_6 z_{14}, x_6 z_{18}, x_6 z_{22}, x_{10} z_{14}, x_{10} z_{18}, x_{10} z_{22}, \\ x_6^2 z_{14}, x_6^2 z_{18}, x_6^2 z_{22}, x_6^3 z_{14}, x_6^3 z_{18}, x_6^3 z_{22}, x_{10} z_{14}, 4x_6^3 z_{22}, \\ x_6 z_{14}, 2x_{10} z_{18}, x_6 b_5 - (z_{14} b_{15} + 4 z_{22} b_{23}), \\ x_{10} b_5 - (z_{14} b_{19} + 4x_6^2 z_{22} b_{15}), x_{10} b_9 - (z_{14} b_{15} + 2 z_{18} b_{19}) \end{array}\right\}$$

Then for $z_i \in Z_0$, $x_i \neq 0 \in H^*(A \otimes \wedge(Z_0 \oplus Z_1), d)$. The non-zero brackets for $k \geqslant 1$ include

$$\{z_{14}, (x_6 b_{19} - x_{10} b_{15}) z_i^k\} = x_{10} z_i^k, \qquad \{z_{14}, (x_6^3 b_{15} - x_{10} b_{23}) z_i^k\} = -x_6^3 z_i^k,$$
$$\{z_{18}, (x_6 b_{19} - x_{10} b_{15}) z_i^k\} = -x_6 z_i^k, \quad \{z_{18}, (x_6^2 b_{19} - x_6 x_{10} b_{15}) z_i^k\} = -x_6^2 z_i^k,$$
$$\{z_{18}, (x_6^3 b_{19} - x_6^2 x_{10} b_{15}) z_i^k\} = -x_6^3 z_i^k, \qquad \{z_{22}, (x_6^3 b_{15} - x_{10} b_{23}) z_i^k\} = x_{10} z_i^k.$$

Hence, $\wedge Z_0$ is not in the center of $(H^*(A \otimes \wedge(Z_0 \oplus Z_1)), d)$.

Definition 4.6. Let L be a Lie algebra. Set $L^{(0)} := L$ and, for $k \geqslant 1$, define the k-th derived algebra of L as

$$L^{(k)} := [L^{(k-1)}, L^{(k-1)}].$$

Then L is called solvable if $L^{(k)} = 0$ for some k.

Proposition 4.7. *Let*

$$L = \big(\wedge(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r), d = 0\big) \subseteq (H^*(A) \otimes \wedge Z, d),$$

where $|w_i|$ is odd, and $z_i = s^{-1} w_i^*$. Then L is solvable.

Proof. Let $a, b \in L$. Then we have

$$\{a, b\} = \sum_i (-1)^{|a|} \frac{\partial^2}{\partial w_i \partial z_i}(ab).$$

Hence, if

$$a \in \wedge^k(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r) \quad \text{and} \quad b \in \wedge^l(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r),$$

with $k + l \leqslant r$. Then, $\{a, b\} \subseteq \wedge^{k+l-1}(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r)$ and so

$$L^{(1)} = \{L, L\} \subseteq \wedge^{\leqslant r-1}(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r),$$
$$L^{(2)} = \{L^{(1)}, L^{(1)}\} \subseteq \wedge^{\leqslant r-2}(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r),$$

$$L^{(3)} = \{L^{(2)}, L^{(2)}\} \subseteq \wedge^{\leqslant r-3}(w_1, \dots, w_r) \otimes \wedge(z_1, \dots, z_r).$$

Continuing this process yields $L^{(r)} = \{L^{(r-1)}, L^{(r-1)}\} \subseteq \wedge(z_1, \dots, z_r)$, which implies that $L^{(r+1)} = \{L^{(r)}, L^{(r)}\} = 0$, and L is solvable. □

In conclusion, we have the following result.

Corollary 4.8. *If X is a formal elliptic 2-stage Postnikov tower, then*
(1) $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ *is not nilpotent,*
(2) *the center of* $s\mathbb{H}_*(X^{S^1};\mathbb{Q})$ *is non-trivial.*

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