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Regularities of the theory of quasi-geodesic mappings of special parabolic spaces

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Abstract. We study quasi-geodesic mappings (QGM) of generalized recurrent-parabolic spaces $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h)$. QGM can be of two types: general and canonical. This article examines the QGM of the general type. Earlier, we considered the fundamental questions of the theory of QGM of generalized-recurrent-parabolic spaces. We proved theorems that allow for any generalized-recurrent-parabolic space (V_n, g_{ij}, F_i^h) to either find all spaces $(\overline{V}_n, \bar{g}_{ij}, F_i^h)$ on which V_n admits QGM of the general form, or prove that there are no such spaces.

In this article, we constructed a Γ-transformation that makes it possible to obtain from a pair of generalized-recurrent-parabolic spaces that are in a quasi-geodesic mapping, an infinite sequence of pairs of other generalized-recurrent-parabolic spaces, which are also in a quasi-geodesic mapping.

Анотація. В роботі досліджеються квазі-геодезичні відображення (КГВ) узагальнено-рекурентно-параболічних просторів

$$f : (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h).$$

КГВ можуть бути двох типів: загального виду і канонічні. У цій статті досліджуються КГВ основного типу. Раніше ми розглянули фундаментальні питання теорії КГВ узагальнено-рекурентно-параболічних просторів. Ми довели теореми, які дозволяють для будь-якого узагальнено-рекурентно-параболічного простору (V_n, g_{ij}, F_i^h) або знайти всі такі простори $(\overline{V}_n, \bar{g}_{ij}, F_i^h)$, на які V_n допускає КГВ загального виду, або довести, що таких просторів немає.

В даній статті ми побудували Γ-перетворення, яке дає змогу із пари узагальнено-рекурентно-параболічних просторів, що знаходяться в квазі-геодезичному відображенні, отримати нескінченну послідовність пар інших узагальнено-рекурентно-параболічних просторів, які також знаходяться в квазі-геодезичному відображенні.

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1. Introduction

1.1. This article is devoted to the study of diffeomorphisms of pseudo-Riemannian spaces that belong to the intersection of classes of quasi-geodesic mappings (QGM) [5,12–14,16,17,21,25] with the reciprocity condition and almost-geodesic mappings of the second type [2,3,11,20,28–30]. We mean that QGM

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h)$$

satisfies the *reciprocity* condition if the inverse mapping f^{-1} is also QGM.

The fundamental equations of such a mapping f in a common coordinate system (x^i) with respect to the mapping f has the form [16]

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}(x)\delta_{j)}^h + \phi_{(i}(x)F_{j)}^h(x), \tag{1.1}$$

$$F_i^h(x) = \overline{F}_i^h(x),$$

$$g_{i\alpha}F_j^\alpha = -g_{j\alpha}F_i^\alpha, \qquad \bar{g}_{i\alpha}F_j^\alpha = -\bar{g}_{j\alpha}F_i^\alpha, \tag{1.2}$$

$$F_{(i,j)}^h = q_{(i}F_{j)}^h, \tag{1.3}$$

$$F_\alpha^h F_i^\alpha = e\delta_i^h, \qquad e = 0, \pm 1, \tag{1.4}$$

where $i, h, j, \ldots = 1, 2, \ldots, n$, Γ_{ij}^h, $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of V_n, $\overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$, $q_i(x)$ are certain covectors; $F_i^h(x)$ is affinor; brackets (i, j) denote the symmetrization with respect to the corresponding indices; comma «,» is a sign of the covariant derivative in respect to the connection of V_n.

If in (1.1) $\phi_i = 0$ and $\psi_i \neq 0$, then the quasi-geodesic mapping is called a *geodesic mapping*, [8–10, 28]. In the case $\phi_i \neq 0$ and $\psi_i = 0$, the quasi-geodesic mapping is called *canonical*, [13–15, 24]. Also, if in (1.1) $\phi_i = 0$ and $\psi_i = 0$, the QGM is called trivial.

Equation (1.1) characterizes F-planar mappings which started to study J. Mikeš and N. S. Sinyukov [19]. These results were specified in [7].

An affinor structure satisfying condition (1.4) is called [20]:

- *elliptic* if $e = -1$,
- *hyperbolic* if $e = +1$,
- *m-parabolic* when $e = 0$, $\operatorname{rank} F = m$ $(2m < n)$,
- *parabolic* when $e = 0$, $\operatorname{rank} F = m$ $(2m = n)$.

1.2. We call an affinor structure F_i^h that satisfies conditions (1.3) a *generalized-recurrent structure* (of elliptic, hyperbolic, or parabolic type) [16].

If in (1.3) $q_i = 0$, the affinor F_i^h defines a *K-structure*, [1, 12, 29].

In [17], a recurrent-parabolic structure was introduced, which is determined by the conditions:

$$F^h_\alpha F^\alpha_i = 0, \qquad g_{i\alpha} F^\alpha_j = -g_{j\alpha} F^\alpha_i, \qquad F^h_{i,j} = q_j F^h_i.$$

The articles [15, 17, 24] are devoted to some issues that concern quasi-geodesic mappings of recurrent-parabolic spaces.

The K-structure and the recurrent-parabolic structure are the special cases of a generalized-recurrent structure.

In the context of types of recurrences and methods for extracting special spaces with structure, the papers [6, 26] are of interest.

In [16, 25] the properties of a generalized-recurrent structure of parabolic type were studied.

We call the vector q_i in (1.3) the *generalized recurrence vector* of the structure F^h_i, and in the case $F^h_{i,j} = q_j F^h_i$, it will be called the *recurrence vector*. Note that under the condition that q_i is gradient, the affinor

$$\tilde{F}^h_i = e^{-q} F^h_i, \quad \text{where} \quad q_i = \frac{\partial q(x)}{\partial x^i},$$

defines a K-structure in the generalized-recurrent space (V_n, g_{ij}, F^h_i), and a Kählerian structure in the recurrent-parabolic space. Shortly, in this case $(V_n, g_{ij}, \tilde{F}^h_i)$ is a parabolic Kähler space, see [18, 20]. The studied mappings are holomorphically projective mappings between parabolic Kähler spaces. These problems were considered in [4, 18, 22, 23] and also in the dissertation by Shiha [27].

1.3. Let us define an operation of contraction with an affinor, which is called *conjugation* with respect to the corresponding indices and is denoted as follows

$$T^{\cdots}_{j_1\cdots j_{k-1}\alpha j_{k+1}\cdots j_r} F^\alpha_i = T^{\cdots}_{j_1\cdots j_{k-1}\bar{i} j_{k+1}\cdots j_r},$$
$$T^{j_1\cdots j_{k-1}\alpha j_{k+1}\cdots j_r}_{\cdots} F^h_\alpha = T^{j_1\cdots j_{k-1}\bar{h} j_{k+1}\cdots j_r}_{\cdots}.$$

1.4. In [16] it was proved that the image of a generalized-recurrent space under QGM is also a generalized-recurrent space, that is

$$F^h_{(i|j)} = \tilde{q}_{(j} F^h_{i)} \quad \text{and} \quad \tilde{q}_i = q_i - \psi_i + \phi_{\bar{i}},$$

where «|» is a sign of a covariant derivative in respect to the connection of $\overline{V}_n$. In other words, the affinor F^h_i in the space $\overline{V}_n$ also defines a generalized-recurrent structure.

Under the condition $\tilde{q}_i = q_i$ we say that *QGM preserves the generalized recurrence vector*. In this case, the vectors ψ_i and ϕ_i in the basic QGM equations (1.1) are related as follows:

$$\psi_i = \phi_{\bar{i}} \tag{1.5}$$

and ψ_i is locally a gradient:

$$\psi_i = \frac{\partial\psi(x)}{\partial x^i}, \qquad \psi(x) = \frac{1}{2(n+2)} \log \frac{|\bar{g}|}{|g|}.$$

In this paper, we consider quasi-geodesic mappings which preserve the corresponding generalized recurrence vectors.

The investigation is carried out in tensor form, locally, in the class of real sufficiently smooth functions.

2. Invariant transformation of generalized-recurrent-parabolic spaces that are in a quasi-geodesic mapping

2.1. Assume that a generalized-recurrent-parabolic space (V_n, g_{ij}, F_i^h) admits a QGM onto $(\overline{V}_n, \bar{g}_{ij}, F_i^h)$.

Since $\bar{g}_{ij|k} = 0$ in $\overline{V}_n$,the equation (1.1) can be written in an equivalent form

$$\begin{gathered} \bar{g}_{ij,k} = 2\phi_{\bar{k}}\bar{g}_{ij} + \phi_{\bar{i}}\bar{g}_{jk} + \phi_{\bar{j}}\bar{g}_{ik} + \phi_i\overline{F}_{jk} + \phi_j\overline{F}_{ik}, \\ \bar{F}_{ik} = \bar{g}_{i\alpha}F_k^\alpha. \end{gathered} \tag{2.1}$$

Here «|» is a sign of the covariant derivative in respect to the connection of $\overline{V}_n$.

Let us introduce the nondegenerate tensor

$$\overset{1}{g}_{ij} = e^{2\psi}\bar{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}. \tag{2.2}$$

Since $\bar{g}_{i\alpha}\bar{g}^{\alpha h} = \delta_i^h$, we have that

$$\bar{g}_{i\alpha,k}\bar{g}^{\alpha h} = -\bar{g}_{i\alpha}\bar{g}^{\alpha h}_{,k}.$$

Therefore, it follows from (2.1) and (2.2) that

$$\overset{1}{g}_{ij,k} = -\overset{1}{\phi}_{\bar{i}}g_{jk} - \overset{1}{\phi}_{\bar{j}}g_{ik} - \overset{1}{\phi}_iF_{jk} - \overset{1}{\phi}_jF_{ik}, \tag{2.3}$$

where

$$\begin{gathered} \overset{1}{\phi}_i = e^{2\psi}\phi_\gamma\bar{g}^{\gamma\alpha}g_{\alpha i}, \\ F_{ik} = g_{i\alpha}F_k^\alpha \end{gathered} \tag{2.4}$$

It is easy to check that in view of (1.2) and (1.5), $\overset{1}{\phi}_{\bar{i}}$ is gradient and

$$\overset{1}{g}_{i\bar{j}} = -\overset{1}{g}_{j\bar{i}}, \qquad \det\left\|\overset{1}{g}_{ij}\right\| \neq 0.$$

2.2. Consider a pseudo-Riemannian space $\overset{1}{V}_n$ in which the tensor $\overset{1}{g}_{ij}$ is its metric tensor.

Let us rewrite the left side of (2.3) via the definition of the covariant derivative in V_n:

$$\frac{\partial \overset{1}{g}_{ij}}{\partial x^k} = \overset{1}{g}_{\alpha j}\Gamma^{\alpha}_{ki} + \overset{1}{g}_{\alpha i}\Gamma^{\alpha}_{kj} - \overset{1}{\phi}_{\overline{i}} g_{jk} - \overset{1}{\phi}_{\overline{j}} g_{ik} - \overset{1}{\phi}_i F_{jk} - \overset{1}{\phi}_j F_{ik}.$$

Calculate now the first kind Christoffel symbols of $\overset{1}{V}_n$ using the obtained equality:

$$\overset{1}{\Gamma}_{ij,k} = \overset{1}{g}_{\alpha k}\Gamma^{\alpha}_{ij} - \overset{1}{\phi}_{\overline{k}} g_{ij} - \overset{1}{\phi}_i F_{kj} - \overset{1}{\phi}_j F_{ki}.$$

Contraction this equation with $\overset{1}{g}{}^{hk} = e^{-2\psi}\bar{g}_{\alpha\beta} g^{\alpha k} g^{\beta h}$ over indice k gives us:

$$\overset{1}{\Gamma}{}^h_{ij} = \Gamma^h_{ij} - \overset{1}{\phi}_{\overline{\beta}} \overset{1}{g}{}^{h\beta} g_{ij} - \overset{1}{\phi}_i F_{\beta j} \overset{1}{g}{}^{h\beta} - \overset{1}{\phi}_j F_{\beta i} \overset{1}{g}{}^{h\beta}, \tag{2.5}$$

where $\overset{1}{\Gamma}{}^h_{ij}$ are the second kind Christoffel symbols of the space $\overset{1}{V}_n$.

2.3. Consider the following tensor:

$$B^h_i = e^{2\psi}\bar{g}^{h\alpha} g_{\alpha i}, \tag{2.6}$$

and its inverse tensor

$$\tilde{B}^h_i = e^{-2\psi} g^{h\alpha}\bar{g}_{\alpha i}. \tag{2.7}$$

In accordance with equalities (1.2), (1.5), (2.2) and (2.4) the following relations hold true:

$$\overset{1}{F}{}^h_i := F^h_\alpha \tilde{B}^\alpha_i = F^\alpha_i \tilde{B}^h_\alpha, \tag{2.8}$$

$$\overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0, \tag{2.9}$$

$$\overset{1}{\phi}_{\overline{\beta}} \overset{1}{g}{}^{h\beta} = \psi_\alpha g^{\alpha h} = \psi^h, \qquad F_{\beta j} \overset{1}{g}{}^{h\beta} = F^\alpha_j \tilde{B}^h_\alpha = \overset{1}{F}{}^h_j.$$

Now (2.5) can be rewritten as follows:

$$\overset{1}{\Gamma}{}^h_{ij} = \Gamma^h_{ij} - \psi^h g_{ij} - \overset{1}{\phi}_i \overset{1}{F}{}^h_j - \overset{1}{\phi}_j \overset{1}{F}{}^h_i. \tag{2.10}$$

2.4. Consider further a pseudo-*Riemannian space* $\overset{1}{\overline{V}}_n$ with a metric tensor

$$\overset{1}{\overline{g}}_{ij} = e^{2\psi} g_{ij}. \tag{2.11}$$

It is known that this dependence determines the conformal mapping of V_n onto $\overline{V}^{1}_n$, so the following equality will hold for Christoffel symbols [28]:

$$\overline{\Gamma}^{1}{}^h_{ij} = \Gamma^h_{ij} - \psi^h g_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i.$$

Substituting (2.10) into the latter equality, we get that

$$\overline{\Gamma}^{1}{}^h_{ij} = \overset{1}{\Gamma}{}^h_{ij} + \overset{1}{\psi}_i \delta^h_j + \overset{1}{\psi}_j \delta^h_i + \overset{1}{\phi}_i \overset{1}{F}{}^h_j + \overset{1}{\phi}_j \overset{1}{F}{}^h_i, \tag{2.12}$$

where $\overset{1}{\psi}_i = \psi_i$.

Relations (2.12) mean that between $\overset{1}{V}_n$ and $\overline{V}^{1}_n$ there is a QGM corresponding to the affinor $\overset{1}{F}{}^h_i$.

From the definition of tensors B^h_i, $\tilde{B}^h_i$ and equalities (1.2), (1.5), (2.2), (2.4) and (2.11), we get that

$$\begin{aligned} &\overset{1}{\phi}_i = \phi_\alpha B^\alpha_i, && \overset{1}{\psi}_i = \psi_i = \phi_\alpha F^\alpha_i = \overset{1}{\phi}_\alpha \overset{1}{F}{}^\alpha_i, \\ &\overset{1}{g}_{i\alpha} \overset{1}{F}{}^\alpha_j = -\overset{1}{g}_{j\alpha} \overset{1}{F}{}^\alpha_i, && \overline{g}^{1}_{i\alpha} \overset{1}{F}{}^\alpha_j = -\overline{g}^{1}_{j\alpha} \overset{1}{F}{}^\alpha_i. \end{aligned} \tag{2.13}$$

2.5. Consider the following tensor

$$\overset{1}{F}_{ij} = \overset{1}{g}_{i\alpha} \overset{1}{F}{}^\alpha_j.$$

Taking into account the definition of $\overset{1}{g}_{ij}$ and $\overset{1}{F}{}^h_j$ we get that $\overset{1}{F}_{ij} = F_{ij}$.

Let us find the relationship between the covariant derivatives of the tensor $\overset{1}{F}_{ij}$ in the spaces $\overset{1}{V}_n$ and V_n taking into account (2.10) and (2.13):

$$\begin{aligned} \overset{1}{F}_{ij1k} &= \frac{\partial \overset{1}{F}_{ij}}{\partial x^k} - \overset{1}{F}_{\alpha j} \overset{1}{\Gamma}{}^\alpha_{ki} - \overset{1}{F}_{i\alpha} \overset{1}{\Gamma}{}^\alpha_{kj} \\ &= \frac{\partial F_{ij}}{\partial x^k} - F_{\alpha j} \overset{1}{\Gamma}{}^\alpha_{ki} - F_{i\alpha} \overset{1}{\Gamma}{}^\alpha_{kj} \\ &= F_{ij,k} + F_{\alpha j}\left(\psi^\alpha g_{ki} + \overset{1}{\phi}_i \overset{1}{F}{}^\alpha_k + \overset{1}{\phi}_k \overset{1}{F}{}^\alpha_i\right) + \\ &\qquad + F_{i\alpha}\left(\psi^\alpha g_{kj} + \overset{1}{\phi}_j \overset{1}{F}{}^\alpha_k + \overset{1}{\phi}_k \overset{1}{F}{}^\alpha_j\right) = F_{ij,k}. \end{aligned} \tag{2.14}$$

Here «1» is a sign of the covariant derivative in respect to the connection of $\overset{1}{V}_n$.

Note that (1.3) are equivalent to the following relations:

$$F_{i(j,k)} = F_{i(j} q_{k)}.$$

Hence, it follows from (2.14) that

$$\overset{1}{F}_{i(j1k)} = \overset{1}{F}_{i(j}q_{k)}.$$

Contraction this equation with $\overset{1}{g}{}^{ih}$ over the index i gives us the following identity:

$$\overset{1}{F}{}^{h}_{(j1k)} = q_{(j}\overset{1}{F}{}^{h}_{k)}. \tag{2.15}$$

Obtained relations indicate that if the space V_n is generalized-recurrent with respect to the affinor F_i^h, then $\overset{1}{V}_n$ will be the same with respect to $\overset{1}{F}{}^h_i$.

So, we have come to the conclusion that the mapping

$$f_1\colon (\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i) \to (\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i)$$

is characterized by the following conditions:

$$\overset{1}{\overline{\Gamma}}{}^h_{ij} = \overset{1}{\Gamma}{}^h_{ij} + \overset{1}{\psi}_i\delta^h_j + \overset{1}{\psi}_j\delta^h_i + \overset{1}{\phi}_i\overset{1}{F}{}^h_j + \overset{1}{\phi}_j\overset{1}{F}{}^h_i,$$

$$\overset{1}{g}_{i\alpha}\overset{1}{F}{}^\alpha_j = -\overset{1}{g}_{j\alpha}\overset{1}{F}{}^\alpha_i, \qquad \overset{1}{\bar{g}}_{i\alpha}\overset{1}{F}{}^\alpha_j = -\overset{1}{\bar{g}}_{j\alpha}\overset{1}{F}{}^\alpha_i,$$

$$\overset{1}{F}{}^h_\alpha\overset{1}{F}{}^\alpha_i = 0,$$

$$\overset{1}{\psi}_i = \overset{1}{\phi}_\alpha\overset{1}{F}{}^\alpha_i, \qquad \overset{1}{F}{}^h_{(j1k)} = q_{(j}\overset{1}{F}{}^h_{k)}.$$

This indicates that f_1 is a quasi-geodesic mapping of generalized-recurrent-parabolic spaces $(\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i)$, $(\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i)$ which preserves the generalized recurrence vector q_i. Thus, we have proved the following:

Theorem 2.6. *If there is a non-trivial QGM of generalized-recurrent-parabolic spaces* $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h)$*, which corresponds to the affinor* F_i^h *and the vector* ϕ_i*, then it generates another non-trivial QGM of other generalized-recurrent-parabolic spaces*

$$f_1\colon (\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i) \to (\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i),$$

which corresponds to the affinor $\overset{1}{F}{}^h_i$ *and the vector* $\overset{1}{\phi}_i$ *and preserves the generalized recurrence vector* q_i*. The tensors* $\overset{1}{g}_{ij}$, $\overset{1}{\bar{g}}_{ij}$, $\overset{1}{\phi}_i$, $\overset{1}{F}{}^h_i$ *are given by the formulas* (2.2)*,* (2.11)*,* (2.4) *and* (2.8)*.*

The found law, represented by formulas (2.2), (2.11), (2.4) and (2.8), transfers a pair of generalized-recurrent-parabolic spaces

$$(V_n, g_{ij}, F_i^h) \qquad \text{and} \qquad (\overline{V}_n, \bar{g}_{ij}, F_i^h),$$

that are in the QGM, into a new pair of generalized-recurrent-parabolic spaces

$$(\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i) \qquad \text{and} \qquad (\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i),$$

connected by the same mapping. Similarly to how this is done in the theory of geodesic mappings of Riemannian spaces [28] we call this law an *invariant transformation* (Γ*-transformation*) *of generalized-recurrent-parabolic spaces that are in the* QGM and denote it by

$$\Gamma(g, \bar{g}, \phi, F) = (\overset{1}{g}, \overset{1}{\bar{g}}, \overset{1}{\phi}, \overset{1}{F}).$$

Here by $(g, \bar{g}, \phi, F)$ we denote the set of tensors that provide the QGM f, and by $(\overset{1}{g}, \overset{1}{\bar{g}}, \overset{1}{\phi}, \overset{1}{F})$ the set of tensors that provide the QGM f_1.

In view of (2.14), as a consequence of Theorem 2.6 we obtain the following result:

Corollary 2.7. *If there exists a non-trivial* QGM *of recurrent-parabolic* (*in particular parabolic Kähler*) *spaces*

$$f\colon (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \bar{g}_{ij}, F^h_i),$$

which corresponds to the affinor F^h_i *and the vector* ϕ_i*, then it generates another non-trivial* QGM *of other recurrent-parabolic* (*in particular parabolic Kähler*) *spaces*

$$f_1\colon (\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i) \to (\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i),$$

which corresponds to the affinor $\overset{1}{F}{}^h_i$ *and the vector* $\overset{1}{\phi}_i$ *and preserves the generalized recurrence vector* q_i*. The tensors* $\overset{1}{g}_{ij}$, $\overset{1}{\bar{g}}_{ij}$, $\overset{1}{\phi}_i$, $\overset{1}{F}{}^h_i$ *are given by the formulas* (2.2), (2.11), (2.4) *and* (2.8).

3. Properties of the Γ-transformation of generalized-recurrent-parabolic spaces that are in a quasi-geodesic mapping

3.1. In view of (2.6) and (2.7) the metric tensors $\overset{1}{g}_{ij}$ and $\overset{1}{\bar{g}}_{ij}$ of the spaces $\overset{1}{V}_n$ and $\overset{1}{\overline{V}}_n$, vector $\overset{1}{\phi}_i$, and afinor $\overset{1}{F}{}^h_i$ which were obtained from the metric tensors g_{ij} and $\bar{g}_{ij}$ of the spaces V_n and $\overline{V}_n$, vector ϕ_i and F^h_i as a result of the Γ-transformation, can be written in the form

$$\overset{1}{g}_{ij} = B^\alpha_i g_{\alpha j}, \qquad \overset{1}{\bar{g}}_{ij} = e^{2\psi} g_{ij}, \tag{3.1}$$
$$\overset{1}{\phi}_i = \phi_\alpha B^\alpha_i, \qquad \overset{1}{F}{}^h_i = F^h_\alpha \tilde{B}^\alpha_i.$$

Since, according to Theorem 2.6, the space $\overset{1}{V}_n$ admits a QGM onto $\overset{1}{\overline{V}}_n$ corresponding to the affinor $\overset{1}{F}{}^h_i$ and the vector $\overset{1}{\phi}_i$, then there exists a Γ-transformation

$$\Gamma(\overset{1}{g}, \overset{1}{\bar{g}}, \overset{1}{\phi}, \overset{1}{F}) = (\overset{2}{g}, \overset{2}{\bar{g}}, \overset{2}{\phi}, \overset{2}{F}).$$

As a result of this transformation we will obtain the space $(\overset{2}{V}_n, \overset{2}{g}_{ij}, \overset{2}{F}{}^h_i)$ that admits a QGM onto $(\overset{2}{\overline{V}}_n, \overset{2}{\bar{g}}_{ij}, \overset{2}{F}{}^h_i)$, corresponding to the affinor $\overset{2}{F}{}^h_i$ and the vector $\overset{2}{\phi}_i$, where

$$\overset{2}{g}_{ij} = \overset{1}{B}{}^\alpha_i \overset{1}{g}_{\alpha j}, \qquad \overset{2}{\bar{g}}_{ij} = e^{2\psi} \overset{1}{g}_{ij}, \tag{3.2}$$

$$\overset{2}{\phi}_i = \overset{1}{\phi}_\alpha \overset{1}{B}{}^\alpha_i, \qquad \overset{2}{F}{}^h_i = \overset{1}{F}{}^h_\alpha \overset{1}{\tilde{B}}{}^\alpha_i. \tag{3.3}$$

By analogy with (2.6) and (2.7), we denote here

$$\overset{1}{B}{}^h_i = e^{2\psi} \overset{1}{\bar{g}}{}^{h\alpha} \overset{1}{g}_{\alpha i}, \qquad \overset{1}{\tilde{B}}{}^h_i = e^{-2\psi} \overset{1}{g}{}^{h\alpha} \overset{1}{\bar{g}}_{\alpha i}.$$

Comparing these relations with (2.6), (2.7) and (3.1), we come to the conclusion that

$$\overset{1}{B}{}^h_i = e^{2\psi} \overset{1}{\bar{g}}{}^{h\alpha} \overset{1}{g}_{\alpha i} = e^{2\psi}(e^{-2\psi} g^{h\alpha})(B^\beta_i g_{\beta\alpha}) = B^h_i.$$

In exactly the same way we also get that $\overset{1}{\tilde{B}}{}^h_i = \tilde{B}^h_i$. Thus, we get the following

Theorem 3.2. *The tensors B^h_i and its inverse $\tilde{B}^h_i$, which are defined by formulas* (2.6) *and* (2.7)*, are invariant with respect to the* Γ*-transformation.*

3.3. Taking into account Theorem 3.2, equations (3.2), (3.3) can be rewritten as follows:

$$\overset{2}{g}_{ij} = B^\beta_i B^\alpha_\beta g_{\alpha j}, \qquad \overset{2}{\bar{g}}_{ij} = e^{2\psi} B^\alpha_i g_{\alpha j},$$

$$\overset{2}{\phi}_i = \phi_\alpha B^\alpha_\beta B^\beta_i, \qquad \overset{2}{F}{}^h_i = F^h_\alpha \tilde{B}^\alpha_\beta \tilde{B}^\beta_i.$$

It is easy to see that the $\Gamma(\overset{2}{g}, \overset{2}{\bar{g}}, \overset{2}{\phi}, \overset{2}{F})$-transformation leads us to the set $(\overset{3}{g}, \overset{3}{\bar{g}}, \overset{3}{\phi}, \overset{3}{F})$:

$$\overset{3}{g}_{ij} = B^\gamma_i B^\beta_\gamma B^\alpha_\beta g_{\alpha j}, \qquad \overset{3}{\bar{g}}_{ij} = e^{2\psi} B^\beta_i B^\alpha_\beta g_{\alpha j},$$

$$\overset{3}{\phi}_i = \phi_\alpha B^\alpha_\beta B^\beta_\gamma B^\gamma_i, \qquad \overset{3}{F}{}^h_i = F^h_\alpha \tilde{B}^\alpha_\beta \tilde{B}^\beta_\gamma \tilde{B}^\gamma_i.$$

Denoting the s-th degree of the Γ-transformation by Γ^s, we get that

$$\Gamma^s\,(g,\bar{g},\phi,F) = (\overset{s}{g},\overset{s}{\bar{g}},\overset{s}{\phi},\overset{s}{F}),$$

$$\overset{s}{g}_{ij} = \overset{(s)}{B}{}_i^\alpha g_{\alpha j}, \qquad \overset{s}{\bar{g}}_{ij} = e^{2\psi}\overset{(s-1)}{B}{}_i^\alpha\, g_{\alpha j}. \tag{3.4}$$

$$\overset{s}{\phi}_i = \phi_\alpha \overset{(s)}{B}{}_i^\alpha, \qquad \overset{s}{F}{}_i^h = F_\alpha^h \overset{(s)}{\tilde{B}}{}_i^\alpha, \tag{3.5}$$

where $\overset{(s)}{B}{}_i^h$ is the s-th degree of the afinor B_i^h and $\overset{(0)}{B}{}_i^h = \delta_i^h$.

3.4. Naturally, the question arises whether the sequence of spaces (3.4) will close, that is, is there a number s for which

$$\Gamma^s\,(g,\bar{g},\phi,F) = (g,\bar{g},\phi,F)\,.$$

In this case:

$$\overset{s}{g}_{ij} = g_{ij}$$

Hence, taking into account (3.4), we obtain that:

$$\overset{(s)}{B}{}_i^h = \delta_i^h.$$

Therefore,

$$|\overset{(s)}{B}{}_i^h| = 1, \qquad |\overset{(s)}{\tilde{B}}{}_i^h| = 1, \qquad |\tilde{B}_i^h| = 1.$$

In accordance with (2.7), taking into account the properties of the determinant, we get that:

$$|\tilde{B}_i^h| = e^{-2\psi n}\frac{|\bar{g}|}{|g|} = 1.$$

Taking logarithm of the last equality and differentiating the result with respect to x^i, we find that

$$2(n-1)(n+2)\psi_i = 0.$$

Hence, $\psi_i = 0$ and QGM is a canonical. Since we considered QGM of the main type ($\psi_i \neq 0$), we come to the conclusion that the sequence of spaces (3.4) is not closed.

Thus, the following theorem holds:

Theorem 3.5. *Suppose there exists a non-trivial QGM of generalized-recurrent-parabolic spaces $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h)$, which corresponds to the affinor F_i^h and the vector ϕ_i. Then it generates an infinite* *sequence of non-trivial QGM of other generalized-recurrent-parabolic spaces*

$$(\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i) \overset{f_1}{\to} (\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i),$$
$$\downarrow$$
$$(\overset{2}{V}_n, \overset{2}{g}_{ij}, \overset{2}{F}{}^h_i) \overset{f_2}{\to} (\overset{2}{\overline{V}}_n, \overset{2}{\bar{g}}_{ij}, \overset{2}{F}{}^h_i),$$
$$\downarrow$$
$$\dots$$
$$\downarrow$$
$$(\overset{s}{V}_n, \overset{s}{g}_{ij}, \overset{s}{F}{}^h_i \overset{f_s}{\to} (\overset{s}{\overline{V}}_n, \overset{s}{\bar{g}}_{ij}, \overset{s}{F}{}^h_i),$$
$$\downarrow$$
$$\dots$$

which correspond to the affinor $\overset{s}{F}{}^h_i$ *and the vector* $\overset{s}{\phi}_i$ *and also preserve the generalized recurrence vector* q_i*. The tensors* $\overset{s}{g}_{ij}$, $\overset{s}{\bar{g}}_{ij}$, $\overset{s}{\phi}_i$, $\overset{s}{F}{}^h_i$, *are given by the formulas* (3.4) *and* (3.5)*.*

Such invariant transformation was constructed in 1961 by N. S. Sinyukov for the geodesic mapping of Riemannian spaces [28]. In the present article, for the first time, it was possible to generalize this result of N. S. Sinyukov for another classes of mappings. In particular, holomorphic-projective mappings of parabolic-Kählerian spaces are a special case of the QGM of generalized-recurrent-parabolic spaces. Therefore, the results obtained in this article can be successfully applied to holomorphic-projective mappings of parabolic-Kählerian spaces, which were studied in detail in the papers [4, 18, 22, 23] and also in the dissertation by M. Shiha [27].

4. Affinor structures generated by the Γ-transformation

Assume that a generalized-recurrent-parabolic space (V_n, g_{ij}, F^h_i) admits a QGM onto $(\overline{V}_n, \bar{g}_{ij}, F^h_i)$.

Consider the following transformation

$$\Gamma\,(g, \bar{g}, \phi, F) = (\overset{1}{g}, \overset{1}{\bar{g}}, \overset{1}{\phi}, \overset{1}{F}).$$

Let us investigate certain properties of the structure $\overset{1}{F}{}^h_i$ in V_n and $\overline{V}_n$.

Notice that due to (1.2), (2.7), (2.8) and (2.9) the following obvious relations hold:

$$\overset{1}{F}{}^h_\alpha \overset{1}{F}{}^\alpha_i = 0, \tag{4.1}$$

$$g_{i\alpha}\overset{1}{F}{}^{\alpha}_{j} = -g_{j\alpha}\overset{1}{F}{}^{\alpha}_{i}, \qquad \bar{g}_{i\alpha}\overset{1}{F}{}^{\alpha}_{j} = -\bar{g}_{j\alpha}\overset{1}{F}{}^{\alpha}_{i}. \tag{4.2}$$

Taking into account (2.9), (2.10) and (2.13) the relationship between the covariant derivatives of the tensor $\overset{1}{F}{}^{h}_{i}$ in the spaces $\overset{1}{V}_n$ and V_n can be represented in the following form:

$$\begin{aligned}\overset{1}{F}{}^{h}_{i1k} &= \frac{\partial \overset{1}{F}{}^{h}_{i}}{\partial x^k} + \overset{1}{F}{}^{\alpha}_{i}\overset{1}{\Gamma}{}^{h}_{k\alpha} - \overset{1}{F}{}^{h}_{\alpha}\overset{1}{\Gamma}{}^{\alpha}_{ki} \\ &= \overset{1}{F}{}^{h}_{i,k} - \overset{1}{F}{}^{\alpha}_{i}(\psi^h g_{\alpha k} + \overset{1}{\phi}_k \overset{1}{F}{}^{h}_{\alpha} + \overset{1}{\phi}_\alpha \overset{1}{F}{}^{h}_{k}) \\ &\qquad + \overset{1}{F}{}^{h}_{\alpha}(\psi^\alpha g_{ki} + \overset{1}{\phi}_i \overset{1}{F}{}^{\alpha}_{k} + \overset{1}{\phi}_k \overset{1}{F}{}^{\alpha}_{i}) \\ &= \overset{1}{F}{}^{h}_{i,k} - e^{-2\psi}\psi^h \overline{F}_{ki} - \psi_i \overset{1}{F}{}^{h}_{k}.\end{aligned} \tag{4.3}$$

From this equality it follows that

$$\overset{1}{F}{}^{h}_{(i,k)} = \overset{1}{F}{}^{h}_{(i1k)} + \psi_{(i}\overset{1}{F}{}^{h}_{k)}. \tag{4.4}$$

In accordance with (2.15), if the space (V_n, g_{ij}, F^h_i), admitting a QGM onto $(\overline{V}_n, \bar{g}_{ij}, F^h_i)$, is a generalized-recurrent-parabolic (in particular, Kählerian, K-space or recurrent-parabolic), then as a result of the Γ-transformation we obtain a pair of spaces connected by QGM and beign also generalized-recurrent-parabolic (Kählerian, K-space or recurrent-parabolic).

According to (4.3) and (4.4) we conclude that if F^h_i defines a generalized-recurrent-parabolic structure in the space (V_n, g_{ij}, F^h_i), that is, the following condition

$$F^h_{(i,j)} = q_{(i}F^h_{j)}$$

is satisfied, then $\overset{1}{F}{}^{h}_{i}$ in V_n also defines a generalized-recurrent-parabolic structure with another generalized recurrence vector

$$\overset{1}{F}{}^{h}_{(i,k)} = \tilde{q}_{(i}\overset{1}{F}{}^{h}_{k)}, \qquad \tilde{q}_i = q_i + \psi_i. \tag{4.5}$$

Further, if F^h_i defines a recurrent-parabolic structure in (V_n, g_{ij}, F^h_i), that is, the condition

$$F^h_{i,j} = q_i F^h_j$$

holds, then $\overset{1}{F}{}^{h}_{i}$ in V_n defines a generalized-recurrent-parabolic structure being not recurrent-parabolic with a generalized recurrence vector $\tilde{q}_i = q_i + \psi_i$.

Finally, if F_i^h defines a parabolic K-structure (in particular, parabolic Kählerian) in V_n, that is, the condition $F^h_{(i,j)} = 0$ ($F^h_{i,j} = 0$) is satisfied, then $\overset{1}{\tilde{F}}{}^h_i = e^{-\psi}\overset{1}{F}{}^h_i$ in V_n defines a parabolic K-structure, which is not Kählerian:

$$\overset{1}{\tilde{F}}{}^h_{(i,k)} = 0. \tag{4.6}$$

Taking into account (4.1), (4.2), (4.5) and (4.6) we get the following result:

Theorem 4.1. *Suppose there exists a non-trivial QGM of generalized-recurrent-parabolic* (*in particular, recurrent-parabolic*) *spaces*

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h),$$

which corresponds to the affinor F_i^h *and the vector* ϕ_i*. Then the tensor* $\overset{1}{F}{}^h_i$ *in* V_n*, given by formulas* (2.7) *and* (2.8), *defines another generalized-recurrent-parabolic structure that is not recurrent-parabolic.*

Theorem 4.2. *Suppose there exists a non-trivial QGM of parabolic K-spaces* (*in particular, parabolic Kählerian*) $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h)$, *which corresponds to the affinor* F_i^h *and the vector* ϕ_i*. Then the tensor*

$$\overset{1}{\tilde{F}}{}^h_i = e^{-\psi}\overset{1}{F}{}^h_i$$

in V_n *defines another parabolic K-structure that is not Kählerian.*

Note that in $(\overline{V}_n, \bar{g}_{ij}, F_i^h)$ the equalities (4.1) and (4.2) are satisfied and the equalities similar to (4.5) and (4.6) are not satisfied. This means that the parabolic affinor structure $\overset{1}{F}{}^h_i$ in V_n does not satisfy the required differential conditions.

5. Conclusion

In this article, we constructed a Γ-transformation allowed to obtain from a pair of generalized-recurrent-parabolic spaces that are in a quasi-geodesic mapping, an infinite sequence of pairs of other generalized-recurrent-parabolic spaces, which are also in quasi-geodesic mappings. The found law, represented by formulas (2.2), (2.11), (2.4) and (2.8), transfers a pair of generalized-recurrent-parabolic spaces (V_n, g_{ij}, F_i^h) and $(\overline{V}_n, \bar{g}_{ij}, F_i^h)$, that are in the QGM, into a new pair of generalized-recurrent-parabolic spaces

$$(\overset{1}{V}_n, \overset{1}{g}_{ij}, \overset{1}{F}{}^h_i) \quad \text{and} \quad (\overset{1}{\overline{V}}_n, \overset{1}{\bar{g}}_{ij}, \overset{1}{F}{}^h_i),$$

connected by the same mapping. Similarly to the theory of geodesic mappings of Riemannian spaces [28], we call this law an *invariant transformation* (Γ-transformation) of generalized-recurrent-parabolic spaces that are in the QGM and denote it by

$$\Gamma\,(g,\bar{g},\phi,F) = (\overset{1}{g},\overset{1}{\bar{g}},\overset{1}{\phi},\overset{1}{F}).$$

Moreover, if there is a non-trivial QGM of generalized-recurrent-parabolic (recurrent-parabolic or parabolic Kählerian) spaces

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \bar{g}_{ij}, F_i^h),$$

which corresponds to the affinor F_i^h and the vector ϕ_i, then it generates an infinite sequence of non-trivial QGM of other generalized-recurrent-parabolic (recurrent-parabolic or parabolic Kählerian) spaces

$$(\overset{s}{V}_n, \overset{s}{g}_{ij}, \overset{s}{F}{}_i^h) \xrightarrow{f_s} (\overset{s}{\overline{V}}_n, \overset{s}{\bar{g}}_{ij}, \overset{s}{F}{}_i^h),$$

which correspond to the affinor $\overset{s}{F}{}_i^h$ and the vector $\overset{s}{\phi}_i$ and also preserve the generalized recurrence vector q_i. That is, the equality

$$\Gamma^s\,(g,\bar{g},\phi,F) = (g,\bar{g},\phi,F)$$

is not satisfied for any value of s.

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