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# Uncountable group of continuous transformations of unit segment preserving tails of $Q_2$-representation of numbers

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**Abstract.** We consider two-base $Q_2$-representation of numbers of segment $[0;1]$:

$$\Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \equiv x = \alpha_1 q_{1-\alpha_1} + \sum_{k=2}^{\infty} \alpha_k q_{1-\alpha_k} \prod_{i=1}^{k-1} q_{\alpha_i},$$

which is defined by two bases $q_0 \in (0;1)$, $q_1 = 1 - q_0$ and an alphabet $A = \{0,1\}$, $(\alpha_n) \in A \times A \times \ldots$. It is a generalization of classic binary representation ($q_0 = \frac{1}{2}$).

In the article we prove that the set of all continuous bijections of segment $[0;1]$ preserving "tails" of $Q_2$-representation of numbers forms an uncountable non-abelian group with respect to composition such that it is a subgroup of the group of continuous transformations preserving frequencies of digits of $Q_2$-representation of numbers.

Construction of such transformations (bijections) is based on the left and right shift operators for digits of $Q_2$-representation of numbers.

**Анотація.** Двоосновне $Q_2$-зображення чисел відрізка $[0;1]$:

$$\Delta_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \equiv x = \alpha_1 q_{1-\alpha_1} + \sum_{k=2}^{\infty} \alpha_k q_{1-\alpha_k} \prod_{i=1}^{k-1} q_{\alpha_i},$$

визначається двома основами $q_0 \in (0;1)$ та $q_1 = 1 - q_0$ і алфавітом $A = \{0,1\}$, $(\alpha_n) \in A \times A \times \ldots$. Воно є узагальненням класичного двійкового зображення ($q_0 = \frac{1}{2}$).

У даній роботі доводиться, що множина всіх неперервних бієкцій відрізка $[0;1]$, які зберігають «хвости» $Q_2$-зображення чисел, відносно операції «композиція» (суперпозиція) утворює незліченну некомутативну групу, яка є підгрупою групи неперервних перетворень, що зберігають

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частоти цифр $Q_2$-зображення чисел. Основою конструкції вказаних перетворень (біекцій) є оператори лівостороннього та правостороннього зсувів цифр $Q_2$-зображення чисел.

## 1. Introduction

The set of all continuous transformations of segment $[0;1]$ (bijections of $[0;1]$ to itself) consists of continuous strictly monotonic functions $f$ such that $f(0) = 0$ and $f(1) = 1$ or $f(0) = 1$ and $f(1) = 0$ including absolutely continuous and singular functions as well as nontrivial mixtures of absolutely continuous and singular functions, functions with structurally and metrically fractal graphs, etc. This set forms an infinite non-abelian group with respect to composition of transformations, which has a number of interesting uncountable subgroups.

This paper is devoted to one of these subgroups (it depends on parameter $q_0 \in (0;1)$), and therefore we discuss a continuum set of subgroups.

A two-symbol system of encoding (representation) of real numbers is a powerful tool for development of many mathematical theories including number metric theory, theory of continuous locally complicated functions, theory of singular probability measures, dynamical systems theory, fractal analysis and fractal geometry [2, 8, 9].

The classic binary numeral system is the most widely used and not yet surpassed in applications. There are various generalizations and analogs of it [8] (e.g. $Q_2$-representation, Markov representation, negabinary representation, Fibonacci representation, $G_2$-representation, mediant representation, $A_2$-continued fraction representation [5, 6], representation of numbers by Denjoy's fraction, etc.).

The $Q_2$-representation of numbers is one of the simplest self-similar generalizations of binary system. It is defined by positive bases $q_0 \in (0;1)$, $q_1 = 1 - q_0$ and the traditional binary alphabet $A = \{0, 1\}$. In this system, a number is encoded by a sequence $(a_n)$ of zeros and ones, i.e., by an element of the space $L = A \times A \times \dots \times A \times \dots$ of sequences of elements of the alphabet.

It is known that for any number $x \in [0;1]$ there exists a sequence $(\alpha_n) \in L$ such that

$$x = \beta_{\alpha_1} + \sum_{k=2}^{\infty} \beta_{\alpha_k} \prod_{i=1}^{k-1} q_{\alpha_i} \equiv \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}, \qquad \beta_{\alpha_i} = \alpha_i q_{1-\alpha_i}. \tag{1.1}$$

In this case, the symbolic notation $\Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ is called the *$Q_2$-representation* of the number $x$, and a number $\alpha_n = \alpha_n(x)$ is called its $n$-th digit.

Most numbers have a unique $Q_2$-representation, and only a countable set of numbers have two representations:

$$\Delta^{Q_2}_{c_1c_2\ldots c_{m-1}1(0)} = \Delta^{Q_2}_{c_1c_2\ldots c_{m-1}0(1)} \tag{1.2}$$

for some $m$. Numbers with representations (1.2) are called $Q_2$*-binary*, and those that have a unique representation are called $Q_2$*-unary*.

Note that $0 = \Delta^{Q_2}_{(0)}$ and $1 = \Delta^{Q_2}_{(1)}$ have a unique $Q_2$-representation and are not $Q_2$-binaryİf $q_0$ is a rational number, then a $Q_2$-binary number is a rational number, but not every $Q_2$-unary number is irrational, and all numbers with periodic $Q_2$-representations are rational.

The *rank* of a $Q_2$-binary number (1.2) is the positive integer $m$.

Following [8], representations of numbers

$$x_1 = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \qquad \text{and} \qquad x_2 = \Delta^{Q_2}_{c_1c_2\ldots c_n\ldots}$$

are said to *have equal tails* whenever there are numbers $k$ and $m$ such that $\alpha_{k+j} = c_{m+j}$ for any $j \in N$.

We denote this by

$$x_1 = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots} \sim \Delta^{Q_2}_{c_1c_2\ldots c_n\ldots} = x_2.$$

It is obvious that the relation "to have equal tails" is an equivalence relation. The class of all numbers that have equal tails of $Q_2$-representations (equivalence class with respect to the $\sim$ relation) is called a *tail set*. Tail sets with representatives $0 = \Delta^{Q_2}_{(0)}$ and $1 = \Delta^{Q_2}_{(1)}$ are considered equal, since $Q_2$-binary points *have* two representations with different tails. Each tail set is countable and everywhere dense in the segment $[0;1]$, and the set of tail sets is a continuum set. We do not know any meaningful metrizations of the quotient set $[0;1]/_\sim$.

If the function (transformation of a unit segment) $f$ has the property $f(x) \sim x$ for any $x \in [0;1]$, then we say that the function (transformation) $f$ *preserves tails of the* $Q_2$*-representation*. If in addition $f$ is a continuous transformation of the unit segment, then it is obvious that its inverse transformation must also preserve tails of the $Q_2$-representation of numbers.

## 2. Shift operators for digits of $Q_2$-representation of numbers

**Remark 2.1.** To ensure that the following functions be well defined by the corresponding equalities, we will use only one of the two representations of $Q_2$-binary numbers, namely, the one with period (1).

We define the left shift operator $\omega$ of the digits of the $Q_2$-representation of numbers by equality:

$$\omega\left(x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}\right) = \Delta^{Q_2}_{\alpha_2\ldots\alpha_n\ldots}. \tag{2.1}$$

Since
$$\omega(\Delta^{Q_2}_{0(1)}) = \Delta^{Q_2}_{(1)} = 1 \neq \omega(\Delta^{Q_2}_{1(0)}) = \Delta^{Q_2}_{(0)} = 0,$$
we see that Remark 2.1 is appropriate.

Also, since
$$x = \beta_{\alpha_1(x)} + q_{\alpha_1(x)}(\beta_{\alpha_2(x)} + \sum_{k=3}^{\infty} \beta_{\alpha_k(x)} \prod_{i=2}^{k-1} q_{\alpha_i(x)}) = \beta_{\alpha_1(x)} + q_{\alpha_1(x)}\omega(x),$$
we have that
$$\omega(x) = \frac{1}{q_{\alpha_1(x)}} x - \frac{\beta_{\alpha_1(x)}}{q_{\alpha_1(x)}} = \begin{cases} \frac{1}{q_0}x, & \text{if } 0 \leqslant x \leqslant q_0, \\ \frac{1}{q_1}x - \frac{q_0}{q_1}, & \text{if } q_0 < x \leqslant 1. \end{cases}$$
Therefore, the left shift operator is a piecewise-linear function with a single discontinuity point $x_0 = q_0$, where the function has a jump of magnitude 1.

The $n$-times left shift operator $y = \omega^n(x)$ of the digits of the $Q_2$-representation of numbers is defined by the equality:
$$\omega^n(x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\alpha_{n+1}\ldots}) = \omega(\omega^{n-1}(x)) = \Delta^{Q_2}_{\alpha_{n+1}\alpha_{n+2}\ldots}.$$
It is a piecewise-linear function with $2^{n-1}$ discontinuity points having the following analytic expression:
$$\begin{aligned} \omega^n(x) &= \frac{1}{q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_n}} x - \frac{\beta_{\alpha_1}}{q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_n}} - \frac{\beta_{\alpha_2}}{q_{\alpha_2}\cdots q_{\alpha_n}} - \cdots - \frac{\beta_{\alpha_n}}{q_{\alpha_n}}, \\ &= \frac{1}{q_{\alpha_1(x)}q_{\alpha_2(x)}\cdots q_{\alpha_n(x)}} x - \sum_{i=1}^{n} \frac{\beta_{\alpha_i(x)}}{q_{\alpha_i(x)}\cdots q_{\alpha_n(x)}}, \end{aligned}$$
because
$$x = \Delta^{Q_2}_{\alpha_1\ldots\alpha_n\ldots} = \beta_{\alpha_1} + \beta_{\alpha_2}q_{\alpha_1} + \cdots + \beta_{\alpha_n}q_{\alpha_1}\cdots q_{\alpha_{n-1}} + q_{\alpha_1}q_{\alpha_2}\cdots q_{\alpha_n}\omega^n(x).$$

**Lemma 2.2.** *The equation $\omega^n(x) = x$ has $2^n$ solutions: $x = \Delta^{Q_2}_{(\alpha_1\alpha_2\ldots\alpha_n)}$, where $\alpha_k \in A$ are any independent digits of the alphabet, and there are no $Q_2$-binary numbers among the solutions.*

*Proof.* Indeed, if $x = \Delta^{Q_2}_{\alpha_1\ldots\alpha_n\ldots}$ is the solution to the equation, then according to the definition of the operator $y = \omega^n(x)$ we have
$$\alpha_1 = \alpha_{n+1}, \quad \alpha_2 = \alpha_{n+2}, \quad \ldots, \quad \alpha_n = \alpha_{2n},$$
$$\alpha_{n+1} = \alpha_{2n+1} = \alpha_1, \quad \ldots, \quad \alpha_{2n} = \alpha_{3n} = \alpha_n.$$
So, $x = \Delta^{Q_2}_{(\alpha_1\alpha_2\ldots\alpha_n)}$. And the fact that $\Delta^{Q_2}_{(\alpha_1\ldots\alpha_n)}$, where $\alpha_k \in A$ are free variables, is the root of the equation is obvious.

According to Remark 2.1 the operator $\omega$ is well defined, namely, its value is calculated by formula (2.1) from a representation with period 1. Then, for $n \geqslant m$, we have $\omega^n(\Delta^{Q_2}_{c_1\ldots c_{m-1}0(1)}) = 1 = \omega^n(1)$, but 1 is not a $Q_2$-binary number.

Since the image of a $Q_2$-binary number under the mapping $\omega^n$ is a $Q_2$-binary number of lower rank, and $Q_2$-binary numbers of different ranks are never equal, we see that any root of the equation $\omega^n(x) = x$ cannot be a $Q_2$-binary number. □

The right shift operator $\delta_i$ with parameter $i$ of digits of the $Q_2$-representation of numbers is defined by equality:

$$\delta_i(x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) := \Delta^{Q_2}_{i\alpha_1\alpha_2\ldots\alpha_n\ldots}, \quad i \in A.$$

More generally, the right shift operator with a set of parameters $(i_1, \ldots, i_n)$, $i_k \in A$, of digits of the $Q_2$-representation of numbers is defined by equality:

$$\begin{aligned}\delta_{i_1\ldots i_n}(x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}) &:= \Delta^{Q_2}_{i_1\ldots i_n\alpha_1\alpha_2\ldots} \\ &= \Big(\prod_{j=1}^{n} q_{i_j}\Big)x + \beta_{i_1} + \sum_{k=2}^{n} \beta_{i_k} \prod_{j=1}^{k-1} q_{i_j}.\end{aligned}$$

**Lemma 2.3.** *The equation $\delta_{i_1\ldots i_n}(x) = x$ has a unique root: $x = \Delta^{Q_2}_{(i_1\ldots i_n)}$.*

*Proof.* It is obvious that the number $x = \Delta^{Q_2}_{(i_1\ldots i_n)}$ is the root of the equation. It remains to prove the uniqueness. Let $x = \Delta^{Q_2}_{\alpha_1\alpha_2\ldots\alpha_n\ldots}$ be the solution of the equation. Then according to the definition of the right shift operator $y = \delta_i(x)$ we have

$$\begin{aligned}&\alpha_1 = i_1, \qquad \alpha_2 = i_2, \qquad \ldots, \qquad \alpha_n = i_n, \\ &\alpha_{n+1} = \alpha_1 = i_1, \qquad \ldots, \qquad \alpha_{2n} = \alpha_n = i_n.\end{aligned}$$

So, $x = \Delta^{Q_2}_{(i_1\ldots i_n)}$. □

**Theorem 2.4.** *A system of equations with parameters $(i_1, \ldots, i_n)$*

$$\omega^n(x) = x = \delta_{i_1\ldots i_n}(x) \tag{2.2}$$

*has a unique solution $x = \Delta^{Q_2}_{(i_1\ldots i_n)}$.*

This statement follows from the two previous lemmas.

**Lemma 2.5.** *The equation*

$$\omega^k(x) = \delta_{i_1\ldots i_n}(x) \tag{2.3}$$

*has exactly $2^k$ solutions:*

$$x = \Delta^{Q_2}_{(\alpha_1\ldots\alpha_k i_1\ldots i_n)}, \tag{2.4}$$

*where* $\alpha_1, \ldots, \alpha_k$ *are independent variables taking the values* 0 *and* 1*.*

*Proof.* The following cases are possible: 1) $k > n$; 2) $k = n$; 3) $k < n$. But we will not distinguish between them. Let $x = \Delta^{Q_2}_{\alpha_1\alpha_1\ldots\alpha_n\ldots}$ be a solution of equation (2.3). Then

$$\begin{gathered} \alpha_{k+1} = i_1, \qquad \alpha_{k+2} = i_2, \qquad \ldots, \qquad \alpha_{k+n} = i_n, \\ \alpha_{k+n+1} = \alpha_1, \qquad \alpha_{k+n+2} = \alpha_2, \qquad \ldots, \qquad \alpha_{k+n+k} = \alpha_k, \\ \alpha_{2k+n+1} = \alpha_{k+1} = i_1, \qquad \ldots, \qquad \alpha_{2k+n+n} = \alpha_{k+n} = i_n, \end{gathered}$$

and so on. Hence, $x = \Delta^{Q_2}_{(\alpha_1\ldots\alpha_k i_1\ldots i_n)}$. It is obvious that the number $x$ of the form (2.4) is the root of the equation. □

Notice that the functions $\omega^k$ and $\delta_{i_1\ldots i_n}$ preserve tails of the $Q_2$-representation of numbers, but are not transformations of the segment $[0;1]$. However they can be used to construct continuous transformations preserving tails.

## 3. Group of continuous bijections

**Theorem 3.1.** *The set $W$ of all continuous transformations (bijections) of the segment $[0;1]$ preserving tails of $Q_2$-representation of numbers with respect to the operation "composition of transformations" is an uncountable non-commutative group, which is a subgroup of the group of transformations that preserve the frequencies of digits in the $Q_2$-representation of numbers.*

*Proof.* It is well known that the set of all transformations of a set, in particular a segment, with respect to the operation "composition" forms a group. Let us use the subgroup criterion.

It is obvious that the set $W$ is closed with respect to the operation "$\circ$", and the inverse of $g \in W$ also belongs to $W$. So $(W, \circ)$ is a group. Let us prove its non-commutativity.

Consider two continuous piecewise-linear transformations

$$f_1(x) = \begin{cases} \omega(x), & \text{if } 0 \leqslant x \leqslant x_0 = \Delta^{Q_2}_{(01)}, \\ \delta_1(x), & \text{if } x_0 \leqslant x \leqslant 1, \end{cases}$$

$$f_2(x) = \begin{cases} \omega^2(x), & \text{if } 0 \leqslant x \leqslant x_1 = \Delta^{Q_2}_{(001)}, \\ \delta_1(x), & \text{if } x_1 \leqslant x \leqslant 1, \end{cases}$$

and a number $x_* = \Delta^{Q_2}_{00(1)} \in (x_1; x_0)$. Then we have that

$$f_2(f_1(x_*)) = f_2(\omega(\Delta^{Q_2}_{00(1)})) = f_2(\Delta^{Q_2}_{0(1)}) = \delta_1(\Delta^{Q_2}_{0(1)}) = \Delta^{Q_2}_{10(1)},$$

$$f_1(f_2(x_*)) = f_1(\delta_1(\Delta^{Q_2}_{00(1)})) = f_1(\Delta^{Q_2}_{100(1)}) = \delta_1(\Delta^{Q_2}_{100(1)}) = \Delta^{Q_2}_{1100(1)}.$$

So, $f_2(f_1(x_*)) \neq f_1(f_2(x_*))$, and therefore $f_2 \circ f_1 \neq f_1 \circ f_2$.

Let us construct a continuum class of transformations $\varphi$ that depends on an infinite number of parameters.

By Lemma 2.2, the points

$$x_{2n-1} = \Delta^{Q_2}_{(\underbrace{0\ldots010}_{2^n})} \quad \text{and} \quad x_{2n} = \Delta^{Q_2}_{(\underbrace{0\ldots001}_{2^n})}$$

are invariant under the mapping $y = \omega^{2n}(x)$. At the same time $x_{2n} < x_{2n-1}$.

For the transformation $\varphi$, we put $\varphi(x) = x$ if $x_{2n-1} \leqslant x \leqslant x_{2n-2}$ for any $n \in N$, where $x_0 = 1$.

It remains to define $\varphi$ on the intervals $(x_{2n}; x_{2n-1})$. To this end, taking into account Theorem 2.4, we introduce the following functions: $r_{1n}(x) = x$, $\forall n \in N$,

$$r_{01}(x) = \begin{cases} \omega^2(x), & \text{if } x_2 \leqslant x \leqslant u_2 \equiv \Delta^{Q_2}_{(0110)}, \\ \delta_{10}(x), & \text{if } u_2 \leqslant x \leqslant x_1, \end{cases}$$

$$r_{21}(x) = \begin{cases} \delta_{01}(x), & \text{if } x_2 \leqslant x \leqslant u_1 \equiv \Delta^{Q_2}_{(1001)}, \\ \omega^2(x), & \text{if } u_1 \leqslant x \leqslant x_1, \end{cases}$$

$$r_{02}(x) = \begin{cases} \omega^4(x), & \text{if } x_4 \leqslant x \leqslant u_4 \equiv \Delta^{Q_2}_{(00010001)}, \\ \delta_{0010}(x), & \text{if } u_4 \leqslant x \leqslant x_3, \end{cases}$$

$$r_{22}(x) = \begin{cases} \delta_{0001}(x), & \text{if } x_4 \leqslant x \leqslant u_3 \equiv \Delta^{Q_2}_{(00100001)}, \\ \omega^4(x), & \text{if } u_3 \leqslant x \leqslant x_3, \end{cases}$$

$$r_{0n}(x) = \begin{cases} \omega^{2n}(x), & \text{if } x_{2n} \leqslant x \leqslant u_{2n} \equiv \Delta^{Q_2}_{(\underbrace{0\ldots01}_{2n}\underbrace{0\ldots010}_{2n})}, \\ \delta_{\underbrace{0\ldots00110}_{2n+2}}(x), & \text{if } u_{2n} \leqslant x \leqslant u_{2n-1}, \end{cases}$$

$$r_{2n}(x) = \begin{cases} \delta_{\underbrace{0\ldots01001}_{2n+2}}(x), & \text{if } x_{2n} \leqslant x \leqslant u_{2n-1} \equiv \Delta^{Q_2}_{(\underbrace{0\ldots010}_{2n}\underbrace{0\ldots01}_{2n})}, \\ \omega^{2n}(x), & \text{if } u_{2n-1} \leqslant x \leqslant x_{2n-1}. \end{cases}$$

For an arbitrarily chosen sequence $(j_n)$, where $j_n \in \{0, 1, 2\}$, we define the function $\varphi_{(j_n)}(x)$ by putting $\varphi_{(j_n)}(x) = r_{j_n n}(x)$ on the segment $[x_{2n}; x_{2n-1}]$. At zero, the function is defined by continuity: $\varphi_{(j_n)}(0) \equiv 0$. The set of such functions is denoted by $V$.

We can establish a one-to-one correspondence from the set of all numbers of the segment $[0; 1]$ to the subclass $V_0$ of the constructed class $V$ of continuous transformations of unit segment preserving tails of $Q_2$-representation by the following rule. Write the number $c$ in the ternary numeral system

$$c = \sum_{n=1}^{\infty} c_n 3^{-n} \equiv \Delta^3_{c_1 c_2 \ldots c_n \ldots},$$

so we get its ternary code $(c_n)$, i.e., the sequence of elements of the alphabet $\{0, 1, 2\}$, which corresponds to the transformation $\varphi_{(c_n)} \in V$. It is clear that two different representations of the ternary-rational numbers correspond to two different functions of class $V$. Hence, $V_0$ is a continuum set. Since the set of ternary-rational numbers (that have two representations) is countable, then the set $V$ is continuum. Therefore, $W$ is uncountable.

Every continuous transformation of the segment $[0; 1]$ preserving tails of $Q_2$-representation of numbers is a transformation preserving frequencies of the digits of the representation, since frequencies of the digits in the representation of a number do not depend on any finite number of its first digits. Since all transformations preserving frequencies of digits form a group, $W$ is a subgroup of this group. The group $W$ is in a similar relationship to the group of transformations preserving the fractal Hausdorff-Besicovitch dimension [1]. □

The structure of the group $W$ requires a special study. Remark that the set $\{f_1, f_3, e\}$, where $e(x) = x$, and

$$f_3(x) = \begin{cases} \delta_0(x), & \text{if } 0 \leqslant x \leqslant x_3 \equiv \Delta^{Q_2}_{(10)}, \\ \omega(x), & \text{if } x_3 \leqslant x \leqslant 1; \end{cases}$$

is a commutative subgroup of $W$, and there are many such subgroups of order three. The transformations $f_1$, $f_2$, $f_3$ are shown in Figure 4 a) and b).

## 4. Concluding remarks

Since $q_0$ is an arbitrary number from the interval $(0; 1)$, there exists a continuum family of $Q_2$-representations of numbers, and hence there exists a continuum family of groups of continuous transformations of the segment $[0; 1]$ that preserve tails of $Q_2$-representation of numbers.

We do not know whether there are continuous transformations of the segment $[0; 1]$ that preserve the tails of the $Q_2$-representation of numbers other than piecewise-linear transformations.

The main result of this paper is a constructive proof of the existence of a continuum subset $V$ of the group $W$, which is actually a justification of its uncountability. Each of these continuous transformations belongs to the group of transformations that preserve the frequencies of digits and to the group of transformations of the segment $[0;1]$ that preserve the fractal Hausdorff-Besicovitch dimension [1].

Groups of continuous transformations of the segment that preserve the tails of other representations were considered in papers [3,7] that only stated these groups are infinite.

In our opinion, the construction of the above transformations can be used when studying objects related to time scales [4].

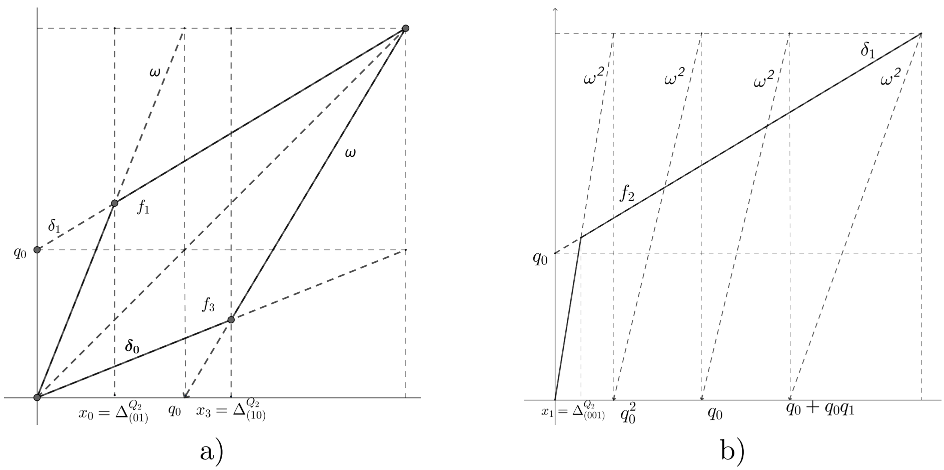


Figure 4.1. Transformations $f_1$, $f_2$ and $f_3$

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