Conformal recurrent Kähler spaces

A. G. Savchenko, T. I. Shevchenko, S. I. Hedulian

Abstract. In this paper we study pseudo-Riemannian spaces with recurrent tensor of conformal curvature, which admit a Kähler structure. It is proved that Kähler conformally recurrent spaces other than recurrent spaces do not exist, if their dimension is four. Recurrent Kähler spaces are divided into two types. For each type, the internal necessary characteristic is given. Some properties of four-dimensional Kähler conformally recurrent Kähler spaces are studied.

Анотація. В роботі вивчаються псевдоріманові простори з рекурентним тензором конформної кривини, які допускають келерову структуру. Доведено, що келерових конформно рекурентних просторів відмінних від рекурентних просторів не існує, якщо їх розмірність дорівнює чотирьох. Рекурентні келерові простори поділені на два типи. Для кожного з типів наведена внутрішня необхідна ознака. Вивчені деякі властивості чотирьоххірних конформно рекурентних келерових просторів.

INTRODUCTION

The theory of Kähler spaces has gone through a century of development and has always attracted researchers [7, 22]. The study of the possibility of special pseudo-Riemannian spaces to admit a Kähler structure is one of the main research topics in this theory. Therefore, it does not lose its relevance [6, 19].

Many authors have studied pseudo-Riemannian spaces with restrictions imposed on internal objects and the metric tensor itself [3, 17]. In the theory of Kähler spaces, there are analogues of tensors that play a certain role in the theory of pseudo-Riemannian spaces. The internal geometry of Kähler spaces includes not only objects defined by a metric tensor, but also an affine covariantly constant structure. The requirement of the structure sometimes becomes an obstacle to the existence of certain spaces [4, 10, 13, 17].

In this paper, we study conformally recurrent Kähler spaces, i.e. spaces in which the conformal curvature tensor is recurrent.

Keywords: geodesic mappings, pseudo-Riemannian spaces, Kähler spaces, conformal mappings, conformally recurrent Kähler spaces.

DOI: http://dx.doi.org/10.15673/pigc.v17i1.2752
The notion of tensor recurrence generalizes the notion of symmetry. Other ways of generalization proposed in [8, 9, 15, 21] allow application in the theory of recurrence tensors in Kähler spaces.

1. CONFORMAL MAPPINGS

Let $V_n$ ($n > 2$) be a pseudo-Riemannian space with metric tensor $g_{ij}(x)$ and $\bar{V}_n$ be a pseudo-Riemannian space with metric tensor $\bar{g}_{ij}(x)$. A conformal mapping is a mutually unambiguous correspondence between points of $V_n$ and $\bar{V}_n$ such that

$$\bar{g}_{ij}(x) = e^{2\sigma(x)}g_{ij}(x),$$

(1.1)

here $\sigma$ is some function [11].

If $\sigma$ is constant, then the mapping is called homothetic. In what follows, unless otherwise specified, we will limit ourselves to considering mappings other than homothetic.

From (1.1) we get that

$$\bar{g}_{ij} = e^{-2\sigma}g^{ij}.$$  

Here $g^{ij}$ are elements of a matrix inverse to $g_{ij}$. There are following formulas for Christoffel symbols:

- let $\Gamma^h_{ij} = g^{\alpha h}\Gamma_{i j \alpha}$ and $\Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$, where $\delta_i^h$ is Kronecker symbols, then

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta_i^h \sigma_j + \delta_j^h \sigma_i - \sigma^h_{ij};$$

(1.2)

- for the Riemannian tensor such that

$$\bar{R}^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\alpha_{ik} \Gamma^h_{j \alpha} - \Gamma^\alpha_{ij} \Gamma^h_{k \alpha},$$

the following conditions are met:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta_i^h \sigma_j - \delta_j^h \sigma_i + g^{h\alpha}(\sigma_{ah} g_{ij} - \sigma_{ai} g_{jk}) + \Delta_1 \sigma(\delta_i^h g_{jk} - \delta_j^h g_{ik});$$

- for Ricci tensor $R_{ij} = R^\alpha_{ij \alpha}$:

$$\bar{R}_{ij} = R_{ij} + (n - 2)\sigma_{ij} + (\Delta_2 \sigma + (n - 2)\Delta_1 \sigma)g_{ij};$$

(1.3)

- for the scalar curvature $R = R^{\alpha \beta}g^{\alpha \beta}$ we have:

$$\bar{R} = e^{-2\sigma} (R + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma).$$

Here and further in the paper

$$\sigma_i \equiv \frac{\partial \sigma}{\partial x^i} \equiv \sigma, i, \quad \sigma^h = \sigma_{\alpha} g^{\alpha h}, \quad \sigma_{ij} = \sigma_{ij} - \sigma_{i} \sigma_{,j},$$

while $\Delta_1 \sigma$ and $\Delta_2 \sigma$ denote the first and second Beltrami characters defined by

$$\Delta_1 \sigma = g^{\alpha \beta} \sigma_{, \alpha} \sigma_{, \beta}, \quad \Delta_2 \sigma = g^{\alpha \beta} \sigma_{, \alpha \beta},$$
and comma ”,” is the sign of the covariant derivative of the connection $V_n$. The objects of the space $\bar{V}_n$ conformally corresponding to $V_n$ are denoted by a dash.

Substituting the expression for $\Delta_2 \sigma$ into (1.3), we obtain for $n > 2$

$$\bar{P}_{ij} = P_{ij} + \sigma_{ij} + \frac{1}{2} \Delta_1 \sigma g_{ij},$$

where

$$P_{ij} = \frac{1}{n-2} \left( R_{ij} - \frac{1}{2(n-1)} R g_{ij} \right).$$

(1.5)

Similarly, $\bar{V}_n$ is defined via $\bar{P}_{ij}$. Excluding the tensor $\sigma_{ij}$ from (1.2) using (1.4), we obtain that

$$C_{ijk}^h(x) = C_{ijk}^h(x),$$

(1.6)

where

$$C_{ijk}^h(x) = R_{ijk}^h + \delta_j^h P_{ik} - \delta_k^h P_{ij} + P_j^h g_{ik} - P_k^h g_{ij}$$

(1.7)

and, similarly, the tensor $\bar{C}_{ijk}^h$ is defined in $\bar{V}_n$.

The tensor $C_{ijk}^h$ is called the tensor of the conformal curvature of the space $V_n$. Conditions of (1.6) show that the conformal curvature tensor is invariant with respect to conformal mappings [11,16].

In the case of a tensor field $S$ of type $(p_q)$, the covariant derivative of the connection of $V_n$, which we will denote by a comma in each coordinate system $x^1, x^2, \ldots, x^n$, is defined as follows:

$$S_{j_1j_2\ldots j_q,k}^{i_1i_2\ldots i_p}(x) = \partial_k S_{j_1j_2\ldots j_q}^{i_1i_2\ldots i_p}(x) + \Gamma_{ka}^{i_1} S_{j_1j_2\ldots j_q}^{\alpha i_2\ldots i_p}(x) + \ldots$$

$$+ \Gamma_{ka}^{i_p} S_{j_1j_2\ldots j_q}^{i_1i_2\ldots i_{p-1}\alpha}(x) - \Gamma_{kj_1}^{\beta} S_{j_1j_2\ldots j_q}^{i_1i_2\ldots i_p}(x) - \ldots$$

$$- \Gamma_{kj_q}^{\beta} S_{j_1j_2\ldots j_{q-1}\beta}(x),$$

where $i_1, \ldots, i_p, j_1, \ldots, j_q, k = 1, 2, \ldots, n$.

For the conformal curvature tensor, we obtain

$$C_{ijk,l}^h = \partial_l C_{ijk}^h - C_{\alpha jk}^h \Gamma_{il}^\alpha - C_{i\alpha jk}^h \Gamma_{tl}^\alpha - C_{ijk\alpha}^h \Gamma_{kl} + C_{ijk}^\alpha \Gamma_{l\alpha}^h.$$

(1.8)

Pseudo-Riemannian spaces $V_n$, in which the conformal curvature tensor $C_{ijk}^h$ satisfies the condition

$$C_{ijk,l}^h = \rho_l C_{ijk}^h,$$

(1.9)

with some vector $\rho_l$, are called conformally recurrent.
2. KÄHLER SPACES

A Kähler space $K_n (n = 2N)$ is a pseudo-Riemannian space with a metric tensor $g_{ij}(x)$, in which there exists a structure $F^h_i(x)$, satisfying the relations [20]:

$$F^h_i F^\alpha_i = -\delta^h_i, \quad F_{ij} = 0, \quad F^h_{ij} = 0,$$  \hspace{1cm} (2.1)

where $F_{ij} \equiv g_{i\alpha} F^\alpha_j$, the comma is the sign of the covariant derivative on the connection $K_n$.

For the convenience, we introduce the conjugation operation in $K_n$:

$$A_{\cdots i} \equiv A_{\alpha \cdots i} F^\alpha_{i}; \quad B_{\cdots i} \equiv B_{\alpha \cdots i} F^\alpha_{i}.$$  \hspace{1cm} (2.2)

Here $A$ and $B$ are arbitrary tensors of any valence. By virtue of (2.1) and (2.2), the following properties hold:

$$A_{\bar{i}} = -A_i, \quad A_{\bar{i}} B^\alpha = A_{i} B^\bar{\alpha}, \quad (A_{\bar{i}}), j = A_{i}, j;$$

$$B_{\bar{i}} = -B_i, \quad A_{\bar{i}} B^\bar{\alpha} = -A_{i} B^{\alpha}, \quad (B_{\bar{i}}), j = B_{i}, j.$$  

The metric tensor and Kronecker symbols satisfy the relations:

$$g_{i\bar{j}} = g_{ij}, \quad g_{ij} = -g_{\bar{i}j}, \quad \delta^h_i = \delta^h_{\bar{i}} = F^h_i, \quad \delta^h_{\bar{i}} = -\delta^h_i.$$  

In addition to the known identities, the Riemannian and Ricci tensors satisfy the following properties:

$$R_{\bar{h}i\bar{j}k} = R_{hi\bar{j}k}, \quad R_{\alpha\bar{i}jk} = 2R_{\bar{i}jk}, \quad R_{\bar{i}\bar{j}} = R_{ij}.$$  \hspace{1cm} (2.3)

For the tensor $P_{ij}$, defined by equation (1.5), the following conditions are satisfied:

$$P_{ij} = P_{i\bar{j}}, \quad P_{ij} + P_{\bar{i}j} = 0.$$  

The internal objects of $K_n$ include the objects defined from the metric tensor $g_{ij}$ and the structure $F^h_i$.

Let us consider conformally recurrent Kähler spaces. Taking into account (1.7) and (1.8), the equation (1.9) can be written as follows:

$$(R_{hij,k} - \rho_l R_{hijk}) = (P_{hk,l} - \rho_l P_{hk}) g_{ij} - (P_{hk,l} - \rho_l P_{hj}) g_{ik} +$$

$$+ (P_{ij,l} - \rho_l P_{ij}) g_{hk} - (P_{ik,l} - \rho_l P_{ik}) g_{hj}. \quad (2.4)$$

Now, perform the conjugation operation on the indices $j$ and $k$ and subtract the result from (2.4):

$$(P_{hk,l} - \rho_l P_{hk}) g_{ij} - (P_{h,i,l} - \rho_l P_{hj}) g_{ik} + (P_{ij,l} - \rho_l P_{ij}) g_{hk} -$$

$$- (P_{ik,l} - \rho_l P_{ik}) g_{hj} = (P_{h,i,l} - \rho_l P_{hj}) g_{ij} + (P_{ij,l} - \rho_l P_{ij}) g_{ik} -$$

$$- (P_{ij,l} - \rho_l P_{ij}) g_{hk} + (P_{ik,l} - \rho_l P_{ik}) g_{hj} = 0.$$
Multiply the latter equation by $g^{ij}$ and wrap by indices $i$ and $j$:

$$(n - 4) (P_{hk,l} - ρ_l P_{hk}) - (P_{l} - ρ_l P) g_{hk} = 0. \tag{2.5}$$

Multiplying further (2.5) by $g^{hk}$ and wrapping by indices $h$ and $k$, we get

$$2(n - 2) (P_{l} - ρ_l P) = 0. \tag{2.6}$$

It then follows from (2.5) and (2.6) that

$$(n - 4) (P_{hk,l} - ρ_l P_{hk}) = 0.$$ 

**Theorem 2.1.** Kähler conformally recurrent spaces are recurrent or their dimension is four.

### 3. Recurrent Kähler Spaces

Let $n \neq 4$. Then

$$R_{hijk,l} = ρ_l R_{hijk}.$$ 

Cycling by the indices $j$, $k$, $l$ and taking to account the properties of the Riemannian tensor we obtain that

$$ρ_l R_{hijk} + ρ_j R_{hikl} + ρ_k R_{hilj} = 0. \tag{3.1}$$

Performing the conjugation operation on the indices $l$, $j$, $k$, we see that the vector $ρ_l$ also satisfies the following condition:

$$ρ_l R_{hijk} + ρ_j R_{hikl} + ρ_k R_{hilj} = 0. \tag{3.2}$$

Note that the vectors $ρ_l$ and $ρ_i$ are mutually orthogonal. If the vector $ρ_i$ is nonisotropic, then the vector $ρ_i$ is also nonisotropic, due to the skew-symmetry of the tensor $F_{ij}$. Further, let us consider recurrent Kähler spaces in which the recurrence vector is nonisotropic.

The following theorem holds

**Theorem 3.1.** The Riemannian and Ricci tensors and the scalar curvature of a Kähler recurrence space with a non-isotropic recurrence vector satisfy the following identity:

$$\frac{R}{2} R_{hijk} = R_{hk} R_{ij} - R_{hj} R_{ik}. \tag{3.3}$$

**Proof.** Multiply (3.1) by $g^{lh}$ and wrap by indices $l$ and $h$. Then we get that

$$ρ_α R_α^{i j k} + ρ_j R_{i k} - ρ_k R_{i j} = 0. \tag{3.4}$$

Multiplying further the latter equation by $g^{ij}$ and wrapping by indices $i$ and $j$, we obtain that

$$ρ_α R_α^k = \frac{R}{2} ρ_k. \tag{3.5}$$
Now, multiply (3.1) by the vector $\rho^l$ and wrap by the index $l$:

$$\rho_\alpha \rho^\alpha R_{hijk} + \rho_j \rho_\alpha R_{khi}^\alpha + \rho_k \rho_\alpha R_{jhi}^\alpha = 0. \quad (3.6)$$

Taking into account the equation (3.4) we see that

$$\rho_\alpha \rho^\alpha R_{hijk} = \rho_j \rho_i R_{kh} - \rho_j \rho_h R_{ki} + \rho_k \rho_h R_{ij} - \rho_k \rho_i R_{jh}. \quad (3.7)$$

Similarly, we get from the equation (3.2) for the vector $\rho_i$ that

$$\rho_\alpha \rho^\alpha R_{hijk} = \rho_j \rho_i R_{kh} - \rho_j \rho_\ell R_{ki} + \rho_\ell \rho_\ell R_{ij} - \rho_\ell \rho_i R_{jh}. \quad (3.8)$$

Apply the conjugation operation on the indices $j$ and $k$ to the equation (3.4):

$$\rho_\alpha \rho^\alpha \bar{R}_{hijk} + \rho_j \rho_i \bar{R}_{kh} - \rho_j \rho_h \bar{R}_{ki} + \rho_k \rho_h \bar{R}_{ij} - \rho_k \rho_i \bar{R}_{jh} = 0. \quad (3.9)$$

Subtracting further (3.9) from (3.4) we obtain that

$$\rho_j R_{ik} - \rho_k R_{ij} - \rho_j \bar{R}_{ik} + \rho_\ell \bar{R}_{ij} = 0. \quad (3.10)$$

Multiply now (3.10) by the vector $\rho^j$ and roll up by $j$. Then, taking into account (3.5), we get that

$$\rho_\alpha \rho^\alpha R_{ik} = \frac{R}{2} (\rho_k \rho_i + \rho_\ell \rho_i). \quad (3.11)$$

Adding (3.8) and (3.7) we obtain

$$2 \rho_\alpha \rho^\alpha R_{hijk} = (\rho_j \rho_i + \rho_\ell \rho_j) R_{kh} - (\rho_j \rho_\ell + \rho_j \rho_\ell) R_{ki} + (\rho_k \rho_h + \rho_\ell \rho_h) R_{ij} - (\rho_k \rho_i + \rho_\ell \rho_\ell) R_{jh}. \quad (3.12)$$

Multiplying (3.12) by the expression and taking into account (3.11) we get the desired identity (3.3):

$$RR_{hijk} = 2 (R_{hk} R_{ij} - R_{hj} R_{ik}). \quad \Box$$

Let us prove the following statement.

**Lemma 3.2.** In recurrent Kähler spaces, the following conditions hold

$$R_{ij,k}^\alpha = \rho_\alpha R_{kji}^\alpha. \quad (3.13)$$

**Proof.** Note that recurrence condition implies that

$$R_{ijk,\alpha}^\alpha = \rho_\alpha R_{ij,k}^\alpha. \quad (3.14)$$

From the identity of Bianchi we also get that

$$R_{ijk,l}^h + R_{ikl,j}^h + R_{i,j,l}^h = 0.$$

Wrapping this identity by indices $h$ and $l$, we obtain that

$$R_{ijk,\alpha}^\alpha = R_{ij,k}^\alpha - R_{ik,j}^\alpha. \quad (3.15)$$
Now (3.15) and (3.14) imply that
\[ R_{ij,k} - R_{ik,j} = \rho_\alpha R^\alpha_{ijk}. \]  
(3.16)

Applying the conjugation operation on the indices \( i \) and \( j \) we get that
\[ R_{ij,k} - R_{ik,j} = \rho_\alpha R^\alpha_{ijk}. \]  
(3.17)

Symmetrize further (3.17) by indices \( i \) and \( k \):
\[ R_{ij,k} + R_{kj,i} = \rho_\alpha R^\alpha_{ijk} + \rho_\alpha R^\alpha_{jik}, \]  
(3.18)

and reciprocally rename the indices \( i \) and \( j \) in (3.18):
\[ R_{ij,k} + R_{ik,j} = \rho_\alpha R^\alpha_{jik} + \rho_\alpha R^\alpha_{kij}. \]  
(3.19)

Adding up (3.19) and (3.16) we get
\[ 2R_{ij,k} = \rho_\alpha R^\alpha_{ijk} + \rho_\alpha R^\alpha_{jik} + \rho_\alpha R^\alpha_{kij}. \]  
(3.20)

Applying now the conjugation operation by indices \( j \) and \( k \) we see that
\[ 2R_{i\bar{j},k} = \rho_\alpha R^\alpha_{i\bar{j}k} - \rho_\alpha R^\alpha_{j\bar{i}k} - \rho_\alpha R^\alpha_{k\bar{i}j}. \]  
(3.21)

Finally, taking into account the algebraic properties of the Riemannian tensor, we see that (3.13) hold. This proves the lemma. \( \square \)

Note that the equations (3.14) coincide with the integrability conditions for the equation defining \( \varphi(\text{Ric}) \)-vector fields and Ricci solitons [1, 2, 5, 12, 14, 18].

Consider now recurrent Kähler spaces with an isotropic recurrence vector, i.e.
\[ \rho_\alpha \rho^\alpha = 0. \]  
(3.20)

Then, from (3.6), taking into account (3.20), we obtain that
\[ \rho_j \rho_\alpha R^\alpha_{kih} + \rho_k \rho_\alpha R^\alpha_{jhi} = 0. \]  
(3.21)

Applying Lemma 3.2 to (3.21) we get that
\[ \rho_j R_{hi,k} + \rho_k R_{ih,j} = 0. \]  
(3.22)

If the space is recurrent, then the Ricci tensor is also recurrent, i.e.
\[ R_{ij,k} = \rho_k R_{ij}. \]  
(3.23)

Hence, substituting (3.23) into (3.22) and grouping the corresponding summands we will obtain that
\[ (\rho_k \rho_j - \rho_j \rho_k) R_{hi} = 0. \]

Since
\[ (\rho_k \rho_j - \rho_j \rho_k) \neq 0, \]

the following identity holds:
\[ R_{ij} = 0. \]  
(3.24)
Pseudo-Riemannian spaces in which the conditions (3.24) are satisfied for the Ricci tensor are called *Ricci flat*. Hence we get the following

**Theorem 3.3.** Recurrent Kähler spaces with an isotropic recurrence vector are Ricci flat spaces.

Thus, for \( n \neq 4 \), conformally recurrent Kähler spaces are recurrent Kähler spaces, which, in turn, are divided into two types depending on the type of recurrence vector. If the recurrence vector is non-isotropic, then the formula (3.3) holds, and if it is isotropic, then the equation (3.24) holds.

4. CONFORMALLY RECURRENT SPACES

We now turn to the consideration of four-dimensional conformally recurrent Kähler spaces.

**Theorem 4.1.** In conformally recurrent Kähler spaces \( K_n (n = 4) \), the following conditions hold

\[
2\rho_{\alpha} R_{i\bar{j}l}^{\alpha} = \tau_{l} g_{ij} + \rho_{i} R_{ij} + R_{i\bar{j}l}. \tag{4.1}
\]

**Proof.** The equations (1.5), (1.7), (1.9) for \( n = 4 \) can be written as follows:

\[
P_{ij} = R_{ij} - \frac{1}{6} R g_{ij},
\]

\[
C_{hijk} = R_{hijk} + \frac{1}{2} (g_{hj} R_{ik} - g_{hk} R_{ij} + R_{hj} g_{ik} - R_{hk} g_{ij}) - \frac{1}{6} R (g_{hj} g_{ik} - g_{hk} g_{ij}),
\]

\[
2 (R_{hijk,l} - \rho_{l} R_{hijk}) = g_{hk} (R_{ij,l} - \rho_{l} R_{ij}) + g_{ij} (R_{hk,l} - \rho_{l} R_{hk}) - g_{hj} (R_{ik,l} - \rho_{l} R_{ik}) - g_{ik} (R_{hj,l} - \rho_{l} R_{hj}).
\]

Cycling by indices \( j, k, l \), we obtain

\[
2 (\rho_{l} R_{ihjk} + \rho_{j} R_{ihkl} + \rho_{k} R_{ihlj}) = g_{ij} A_{hkl} + g_{ik} A_{hlj} + g_{il} A_{hjk} + g_{hk} A_{ijl} + g_{hl} A_{ikj} + g_{hj} A_{ilk}, \tag{4.2}
\]

where

\[
A_{ijk} = R_{ij,k} - \rho_{k} R_{ij} - R_{ik,j} + \rho_{j} R_{ik}.
\]

Multiply the equation (4.2) by \( g^{hk} \) and wrap by indices \( h \) and \( k \). Then after summing the similar summands we get that

\[
2\rho_{\alpha} R_{i\bar{j}l}^{\alpha} = \tau_{l} g_{ij} - \tau_{j} g_{il} + R_{ij,l} - R_{il,j} + \rho_{l} R_{ij} - \rho_{j} R_{il}, \tag{4.3}
\]

where

\[
\tau_{i} = \rho_{\alpha} R_{i}^{\alpha} - \frac{1}{2} R_{i}. \]
Applying to the expression (4.3) the conjugation operation on the indices \( i \) and \( j \) we obtain that

\[
2\rho_{\alpha}R_{ijl}^{\alpha} = \tau_lg_{ij} - \tau_jg_{il} + R_{ij,l} - R_{il,j} + \rho_lR_{ij} - \rho_jR_{il}. \tag{4.4}
\]

Symmetrize further (4.4) by indices \( i \) and \( l \):

\[
2\left( \rho_{\alpha}R_{ijl}^{\alpha} + \rho_{\alpha}R_{lij}^{\alpha} \right) = \tau_lg_{ij} + \tau_jg_{li} + R_{ij,l} + R_{li,j} + \rho_lR_{ij} + \rho_iR_{lj}, \tag{4.5}
\]

and exchange notations for indices \( i \) and \( j \) in (4.5):

\[
2\left( \rho_{\alpha}R_{jil}^{\alpha} + \rho_{\alpha}R_{lij}^{\alpha} \right) = \tau_lg_{ji} + \tau_jg_{li} + R_{ji,l} + R_{li,j} + \rho_lR_{ji} + \rho_jR_{li}. \tag{4.6}
\]

Adding now (4.6) and (4.3) we obtain that

\[
\rho_{\alpha}R_{ijl}^{\alpha} + \rho_{\alpha}R_{jil}^{\alpha} + \rho_{\alpha}R_{lij}^{\alpha} = \tau_lg_{ij} + R_{ij,l} + \rho_lR_{ij}. \tag{4.7}
\]

Applying to (4.7) the conjugation operation on the indices \( j \) and \( l \), and taking into account the properties of the Riemannian tensor, we will get the desired equation (4.1).

\[\square\]

5. Conclusion

Thus, it has been proved that there are no conformally recurrent Kähler spaces other than recurrent ones, if the dimension of the space is not four. The fact that conformally recurrent Kähler spaces are recurrent allows us to distinguish between two types of these spaces depending on the properties of the recurrence vector. These types can be distinguished in the classification proposed in the current paper. Namely, if the recurrence vector is not isotropic, then the scalar curvature of such spaces is distinct from zero, and the Riemannian tensor satisfies the condition

\[
\frac{R}{2}R_{hijk} = R_{hk}R_{ij} - R_{hj}R_{ik}.
\]

Thus, recurrent Kähler spaces with an isotropic recurrence vector are Ricci flat, and therefore they have zero scalar curvature. Different from recurrent conformally recurrent Kähler spaces may exist only when the dimension of the space is 4. We found that tensor condition is satisfied by four-dimensional conformally recurrent Kähler spaces, which may allow us to construct examples of such spaces. The construction of an example of conformally recurrent Kähler spaces other than recurrent ones remains an unsolved problem.
REFERENCES


Received: February 23, 2024, accepted: March 26, 2024.

A. G. Savchenko
Kherson State University, University Street 27, Kherson, 73000, Ukraine
Email: savchenko.o.g@ukr.net
ORCID: 0000-0003-4687-5542

T. I. Shevchenko
Odesa State Academy of Civil Engineering and Architecture, Didrihson St., 4, 65029, Ukraine
Email: tashev1@ukr.net
ORCID: 0000-0002-7304-1706

S. I. Hedulian
Odesa State Academy of Civil Engineering and Architecture, Didrihson St., 4, 65029, Ukraine
Email: shedulian@ogasa.org.ua
ORCID: 0000-0001-5732-6042