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Conformal recurrent Kähler spaces

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Abstract. In this paper we study pseudo-Riemannian spaces with recurrent tensor of conformal curvature, which admit a Kähler structure. It is proved that Kähler conformally recurrent spaces other than recurrent spaces do not exist, if their dimension is four. Recurrent Kähler spaces are divided into two types. For each type, the internal necessary characteristic is given. Some properties of four-dimensional Kähler conformally recurrent Kähler spaces are studied.

Анотація. В роботі вивчаються псевдорімановi простори з рекурентним тензором конформної кривини, які допускають келерову структуру. Доведено, що келерових конформно рекурентних просторів відмінних від рекурентних просторів не існує, якщо їх розмірність дорівнює чотирьом. Рекурентні келерові простори поділені на два типи. Для кожного з типів наведена внутрішня необхідна ознака. Вивчені деякі властивості чотирьохмірних конформно рекурентних келерових просторів.

Introduction

The theory of Kähler spaces has gone through a century of development and has always attracted researchers [7, 22]. The study of the possibility of special pseudo-Riemannian spaces to admit a Kähler structure is one of the main research topics in this theory. Therefore, it does not lose its relevance [6, 19].

Many authors have studied pseudo-Riemannian spaces with restrictions imposed on internal objects and the metric tensor itself [3,17]. In the theory of Kähler spaces, there are analogues of tensors that play a certain role in the theory of pseudo-Riemannian spaces. The internal geometry of Kähler spaces includes not only objects defined by a metric tensor, but also an affine covariantly constant structure. The requirement of the structure sometimes becomes an obstacle to the existence of certain spaces [4, 10, 13, 17].

In this paper, we study conformally recurrent Kähler spaces, i.e. spaces in which the conformal curvature tensor is recurrent.

Keywords: geodesic mappings, pseudo-Riemannian spaces, Kähler spaces, conformal mappings, conformally recurrent Kähler spaces.

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The notion of tensor recurrence generalizes the notion of symmetry. Other ways of generalization proposed in [8, 9, 15, 21] allow application in the theory of recurrence tensors in Kähler spaces.

1. Conformal mappings

Let V_n $(n > 2)$ be a pseudo-Riemannian space with metric tensor $g_{ij}(x)$ and $\bar{V}_n$ be a pseudo-Riemannian space with metric tensor $\bar{g}_{ij}(x)$. *A conformal mapping* is a mutually unambiguous correspondence between points of V_n and $\bar{V}_n$ such that

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \tag{1.1}$$

here σ is some function [11].

If σ is constant, then the mapping is called *homothetic*. In what follows, unless otherwise specified, we will limit ourselves to considering mappings other than homothetic.

From (1.1) we get that

$$\bar{g}^{ij} = e^{-2\sigma} g^{ij}.$$

Here g^{ij} are elements of a matrix inverse to g_{ij}. There are following formulas for Christoffel symbols:

- let $\Gamma^h_{ij} = g^{\alpha h}\Gamma_{ij\alpha}$ and $\Gamma_{ijk} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)$, where δ^h_i is Kronecker symbols, then

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}; \tag{1.2}$$

- for the Riemannian tensor such that

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\alpha_{ik}\Gamma^h_{j\alpha} - \Gamma^\alpha_{ij}\Gamma^h_{k\alpha},$$

  the following conditions are met:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{h\alpha}(\sigma_{\alpha h} g_{ij} - \sigma_{\alpha j} g_{ik}) + \Delta_1\sigma(\delta^h_k g_{ij} - \delta^h_j g_{ik});$$

- for Ricci tensor $R_{ij} = R^\alpha_{ij\alpha}$:

$$\bar{R}_{ij} = R_{ij} + (n-2)\sigma_{ij} + (\Delta_2\sigma + (n-2)\Delta_1\sigma) g_{ij}; \tag{1.3}$$

- for the scalar curvature $R = R_{\alpha\beta} g^{\alpha\beta}$ we have:

$$\bar{R} = e^{-2\sigma}(R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma).$$

Here and further in the paper

$$\sigma_i \equiv \frac{\partial\sigma}{\partial x^i} \equiv \sigma_{,i}, \qquad \sigma^h = \sigma_\alpha g^{\alpha h}, \qquad \sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j},$$

while $\Delta_1\sigma$ and $\Delta_2\sigma$ denote the first and second Beltrami characters defined by

$$\Delta_1\sigma = g^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}, \qquad \Delta_2\sigma = g^{\alpha\beta}\sigma_{,\alpha\beta},$$

and comma "," is the sign of the covariant derivative of the connection V_n. The objects of the space $\bar{V}_n$ conformally corresponding to V_n are denoted by a dash.

Substituting the expression for $\Delta_2\sigma$ into (1.3), we obtain for $n > 2$

$$\overline{P}_{ij} = P_{ij} + \sigma_{ij} + \frac{1}{2}\Delta_1\sigma g_{ij}, \tag{1.4}$$

where

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{1}{2(n-1)}Rg_{ij}\right). \tag{1.5}$$

Similarly, $\overline{V}_n$ is defined via $\overline{P}_{ij}$. Excluding the tensor σ_{ij} from (1.2) using (1.4), we obtain that

$$\overline{C}^h_{ijk}(x) = C^h_{ijk}(x), \tag{1.6}$$

where

$$C^h_{ijk}(x) = R^h_{.ijk} + \delta^h_j P_{ik} - \delta^h_k P_{ij} + P^h_j g_{ik} - P^h_k g_{ij} \tag{1.7}$$

and, similarly, the tensor $\overline{C}^h_{ijk}$ is defined in $\overline{V}_n$.

The tensor C^h_{ijk} is called the *tensor of the conformal curvature* of the space V_n. Conditions of (1.6) show that the conformal curvature tensor is invariant with respect to conformal mappings [11, 16].

In the case of a tensor field S of type $\binom{p}{q}$, the covariant derivative of the connection of V_n, which we will denote by a comma in each coordinate system $x^1, x^2, \ldots, x^n$, is defined as follows:

$$\begin{aligned} S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q, k}(x) = \partial_k S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \Gamma^{i_1}_{k\alpha}(x) S^{\alpha i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) + \ldots \\ + \Gamma^{i_p}_{k\alpha}(x) S^{i_1 i_2 \ldots i_{p-1}\alpha}_{j_1 j_2 \ldots j_q}(x) - \Gamma^{\beta}_{kj_1}(x) S^{i_1 i_2 \ldots i_p}_{\beta j_2 \ldots j_q}(x) - \ldots \\ - \Gamma^{\beta}_{kj_q}(x) S^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_{q-1}\beta}(x), \end{aligned}$$

where $i_1, \ldots, i_p, j_1, \ldots j_q, k = 1, 2, \ldots, n$.

For the conformal curvature tensor, we obtain

$$C^h_{ijk,l} = \partial_l C^h_{ijk} - C^h_{\alpha jk}\Gamma^\alpha_{il} - C^h_{i\alpha k}\Gamma^\alpha_{jl} - C^h_{ij\alpha}\Gamma^\alpha_{kl} + C^\alpha_{ijk}\Gamma^h_{\alpha l}. \tag{1.8}$$

Pseudo-Riemannian spaces V_n, in which the conformal curvature tensor C^h_{ijk} satisfies the condition

$$C^h_{ijk,l} = \rho_l C^h_{ijk}, \tag{1.9}$$

with some vector ρ_l, are called *conformally recurrent*,

2. Kähler spaces

A Kähler space K_n $(n = 2N)$ is a pseudo-Riemannian space with a metric tensor $g_{ij}(x)$, in which there exists a structure $F_i^h(x)$, satisfying the relations [20]:

$$F_\alpha^h F_i^\alpha = -\delta_i^h, \qquad F_{(ij)} = 0, \qquad F_{i,j}^h = 0, \tag{2.1}$$

where $F_{ij} \equiv g_{i\alpha} F_j^\alpha$, the comma is the sign of the covariant derivative on the connection K_n.

For the convenience, we introduce the conjugation operation in K_n:

$$A_{\bar{i}\ldots}^{\cdots} \equiv A_{\alpha\ldots}^{\cdots} F_i^\alpha; \qquad B_{\cdots}^{\bar{i}\ldots} \equiv B_{\cdots}^{\alpha\ldots} F_\alpha^i. \tag{2.2}$$

Here A and B are arbitrary tensors of any valence. By virtue of (2.1) and (2.2), the following properties hold:

$$\begin{aligned} &A_{\bar{\bar{i}}} = -A_i, && A_{\bar{\alpha}} B^\alpha = A_\alpha B^{\bar{\alpha}}, && (A_{\bar{i}})_{,\,j} = A_{\bar{i},\,j}, \\ &B^{\bar{\bar{i}}} = -B^i, && A_{\bar{\alpha}} B^{\bar{\alpha}} = -A_\alpha B^\alpha, && (B^{\bar{i}})_{,\,j} = B^{\bar{i}}_{,\,j}. \end{aligned}$$

The metric tensor and Kronecker symbols satisfy the relations:

$$g_{\bar{i}\,\bar{j}} = g_{ij}, \qquad g_{\bar{i}j} = -g_{i\bar{j}}, \qquad \delta_{\bar{i}}^h = \delta_i^{\bar{h}} = F_i^h, \qquad \delta_{\bar{i}}^{\bar{h}} = -\delta_i^h.$$

In addition to the known identities, the Riemannian and Ricci tensors satisfy the following properties:

$$R_{\bar{h}\bar{i}jk} = R_{hijk}, \qquad R^\alpha_{\bar{\alpha}jk} = 2R_{j\bar{k}}, \qquad R_{\bar{i}\,\bar{j}} = R_{ij}. \tag{2.3}$$

For the tensor P_{ij}, defined by equation (1.5), the following conditions are satisfied:

$$P_{ij} = P_{\bar{i}\,\bar{j}}, \qquad P_{\bar{i}j} + P_{i\bar{j}} = 0.$$

The internal objects of K_n include the objects defined from the metric tensor g_{ij} and the structure F_i^h.

Let us consider conformally recurrent Kähler spaces. Taking into account (1.7) and (1.8), the equation (1.9) can be written as follows:

$$\begin{aligned} (R_{hijk,l} - \rho_l R_{hijk}) = (P_{hk,l} - \rho_l P_{hk})\, g_{ij} - (P_{hj,l} - \rho_l P_{hj})\, g_{ik} + \\ + (P_{ij,l} - \rho_l P_{ij})\, g_{hk} - (P_{ik,l} - \rho_l P_{ik})\, g_{hj}. \end{aligned} \tag{2.4}$$

Now, perform the conjugation operation on the indices j and k and subtract the result from (2.4):

$$\begin{aligned} &(P_{hk,l} - \rho_l P_{hk})\, g_{ij} - (P_{hj,l} - \rho_l P_{hj})\, g_{ik} + (P_{ij,l} - \rho_l P_{ij})\, g_{hk} - \\ &\quad - (P_{ik,l} - \rho_l P_{ik})\, g_{hj} - \left(P_{h\bar{k},l} - \rho_l P_{h\bar{k}}\right) g_{i\bar{j}} + \left(P_{h\bar{j},l} - \rho_l P_{h\bar{j}}\right) g_{i\bar{k}} - \\ &\qquad - \left(P_{i\bar{j},l} - \rho_l P_{i\bar{j}}\right) g_{h\bar{k}} + \left(P_{i\bar{k},l} - \rho_l P_{i\bar{k}}\right) g_{h\bar{j}} = 0. \end{aligned}$$

Multiply the latter equation by g^{ij} and wrap by indices i and j:

$$(n-4)\left(P_{hk,l}-\rho_l P_{hk}\right)-\left(P_{,l}-\rho_l P\right)g_{hk}=0. \tag{2.5}$$

Multiplying further (2.5) by g^{hk} and wrapping by indices h and k, we get

$$2(n-2)\left(P_{,l}-\rho_l P\right)=0. \tag{2.6}$$

It then follows from (2.5) and (2.6) that

$$(n-4)\left(P_{hk,l}-\rho_l P_{hk}\right)=0.$$

Theorem 2.1. *Kähler conformally recurrent spaces are recurrent or their dimension is four.*

3. Recurrent Kähler spaces

Let $n \neq 4$. Then

$$R_{hijk,l}=\rho_l R_{hijk}.$$

Cycling by the indices j, k, l and taking to account the properties of the Riemannian tensor we obtain that

$$\rho_l R_{hijk}+\rho_j R_{hikl}+\rho_k R_{hilj}=0. \tag{3.1}$$

Performing the conjugation operation on the indices l, j, k, we see that the vector $\rho_{\bar{l}}$ also satisfies the following condition:

$$\rho_{\bar{l}} R_{hijk}+\rho_{\bar{j}} R_{hikl}+\rho_{\bar{k}} R_{hilj}=0. \tag{3.2}$$

Note that the vectors ρ_l and $\rho_{\bar{l}}$ are mutually orthogonal. If the vector ρ_i is nonisotropic, then the vector $\rho_{\bar{i}}$ is also nonisotropic, due to the skew-symmetry of the tensor F_{ij}. Further, let us consider recurrent Kähler spaces in which the recurrence vector is nonisotropic.

The following theorem holds

Theorem 3.1. *The Riemannian and Ricci tensors and the scalar curvature of a Kähler recurrence space with a non-isotropic recurrence vector satisfy the following identity:*

$$\frac{R}{2} R_{hijk}=R_{hk}R_{ij}-R_{hj}R_{ik}. \tag{3.3}$$

Proof. Multiply (3.1) by g^{lh} and wrap by indices l and h. Then we get that

$$\rho_\alpha R^\alpha_{ijk}+\rho_j R_{ik}-\rho_k R_{ij}=0. \tag{3.4}$$

Multiplying further the latter equation by g^{ij} and wrapping by indices i and j, we obtain that

$$\rho_\alpha R^\alpha_k=\frac{R}{2}\rho_k. \tag{3.5}$$

Now, multiply (3.1) by the vector ρ^l and wrap by the index l:

$$\rho_\alpha\rho^\alpha R_{hijk} + \rho_j\rho_\alpha R^\alpha_{kih} + \rho_k\rho_\alpha R^\alpha_{jhi} = 0. \tag{3.6}$$

Taking into account the equation (3.4) we see that

$$\rho_\alpha\rho^\alpha R_{hijk} = \rho_j\rho_i R_{kh} - \rho_j\rho_h R_{ki} + \rho_k\rho_h R_{ij} - \rho_k\rho_i R_{jh}. \tag{3.7}$$

Similarly, we get from the equation (3.2) for the vector $\rho_{\bar{i}}$ that

$$\rho_\alpha\rho^\alpha R_{hijk} = \rho_{\bar{j}}\rho_{\bar{i}} R_{kh} - \rho_{\bar{j}}\rho_{\bar{h}} R_{ki} + \rho_{\bar{k}}\rho_{\bar{h}} R_{ij} - \rho_{\bar{k}}\rho_{\bar{i}} R_{jh}. \tag{3.8}$$

Apply the conjugation operation on the indices j and k to the equation (3.4):

$$\rho_\alpha R^\alpha_{ijk} + \rho_{\bar{j}} R_{i\bar{k}} - \rho_{\bar{k}} R_{i\bar{j}} = 0. \tag{3.9}$$

Subtracting further (3.9) from (3.4) we obtain that

$$\rho_j R_{ik} - \rho_k R_{ij} - \rho_{\bar{j}} R_{i\bar{k}} + \rho_{\bar{k}} R_{i\bar{j}} = 0. \tag{3.10}$$

Multiply now (3.10) by the vector ρ^j and roll up by j. Then, taking into account (3.5), we get that

$$\rho_\alpha\rho^\alpha R_{ik} = \frac{R}{2}\left(\rho_k\rho_i + \rho_{\bar{k}}\rho_{\bar{i}}\right). \tag{3.11}$$

Adding (3.8) and (3.7) we obtain

$$\begin{aligned} 2\rho_\alpha\rho^\alpha R_{hijk} = {} & \left(\rho_j\rho_i + \rho_{\bar{i}}\rho_{\bar{j}}\right) R_{kh} - \left(\rho_j\rho_h + \rho_{\bar{j}}\rho_{\bar{h}}\right) R_{ki} + \\ & + \left(\rho_k\rho_h + \rho_{\bar{k}}\rho_{\bar{h}}\right) R_{ij} - \left(\rho_k\rho_i + \rho_{\bar{k}}\rho_{\bar{i}}\right) R_{hj}. \end{aligned} \tag{3.12}$$

Multiplying (3.12) by the expression and taking into account (3.11) we get the desired identity (3.3):

$$RR_{hijk} = 2\left(R_{hk}R_{ij} - R_{hj}R_{ik}\right). \qquad \square$$

Let us prove the following statement.

Lemma 3.2. *In recurrent Kähler spaces, the following conditions hold*

$$R_{i\bar{j},\bar{k}} = \rho_\alpha R^\alpha_{kji}. \tag{3.13}$$

Proof. Note that recurrence condition implies that

$$R^\alpha_{ijk,\alpha} = \rho_\alpha R^\alpha_{ijk}. \tag{3.14}$$

From the identity of Bianchi we also get that

$$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0.$$

Wrapping this identity by indices h and l, we obtain that

$$R^\alpha_{ijk,\alpha} = R_{ij,k} - R_{ik,j}. \tag{3.15}$$

Now (3.15) and (3.14) imply that

$$R_{ij,k} - R_{ik,j} = \rho_\alpha R^\alpha_{ijk}. \tag{3.16}$$

Applying the conjugation operation on the indices i and j we get that

$$R_{ij,k} - R_{\bar{i}k,\bar{j}} = \rho_\alpha R^\alpha_{\bar{i}\bar{j}k}. \tag{3.17}$$

Symmetrize further (3.17) by indices i and k:

$$R_{ij,k} + R_{kj,i} = \rho_\alpha R^\alpha_{\bar{i}\bar{j}k} + \rho_\alpha R^\alpha_{\bar{k}\bar{j}i}, \tag{3.18}$$

and reciprocally rename the indices i and j in (3.18):

$$R_{ij,k} + R_{ik,j} = \rho_\alpha R^\alpha_{\bar{j}\bar{i}k} + \rho_\alpha R^\alpha_{\bar{k}\bar{i}j}. \tag{3.19}$$

Adding up (3.19) and (3.16) we get

$$2R_{ij,k} = \rho_\alpha R^\alpha_{ijk} + \rho_\alpha R^\alpha_{\bar{j}\bar{i}k} + \rho_\alpha R^\alpha_{\bar{k}\bar{i}j}.$$

Applying now the conjugation operation by indices j and k we see that

$$2R_{i\bar{j},\bar{k}} = \rho_\alpha R^\alpha_{ijk} - \rho_\alpha R^\alpha_{jik} - \rho_\alpha R^\alpha_{kij}.$$

Finally, taking into account the algebraic properties of the Riemannian tensor, we see that (3.13) hold. This proves the lemma. □

Note that the equations (3.14) coincide with the integrability conditions for the equation defining $\varphi(Ric)$-vector fields and Ricci solitons [1, 2, 5, 12, 14, 18].

Consider now recurrent Kähler spaces with an isotropic recurrence vector, i.e.

$$\rho_\alpha \rho^\alpha = 0. \tag{3.20}$$

Then, from (3.6), taking into account (3.20), we obtain that

$$\rho_j \rho_\alpha R^\alpha_{kih} + \rho_k \rho_\alpha R^\alpha_{jhi} = 0. \tag{3.21}$$

Applying Lemma 3.2 to (3.21) we get that

$$\rho_j R_{h\bar{i},\bar{k}} + \rho_k R_{i\bar{h},\bar{j}} = 0. \tag{3.22}$$

If the space is recurrent, then the Ricci tensor is also recurrent, i.e.

$$R_{ij,k} = \rho_k R_{ij}. \tag{3.23}$$

Hence, substituting (3.23) into (3.22) and grouping the corresponding summands we will obtain that

$$\left(\rho_k \rho_{\bar{j}} - \rho_{\bar{k}} \rho_j\right) R_{hi} = 0.$$

Since

$$\left(\rho_k \rho_{\bar{j}} - \rho_{\bar{k}} \rho_j\right) \neq 0,$$

the following identity holds:

$$R_{ij} = 0. \tag{3.24}$$

Pseudo-Riemannian spaces in which the conditions (3.24) are satisfied for the Ricci tensor are called *Ricci flat*. Hence we get the following

Theorem 3.3. *Recurrent Kähler spaces with an isotropic recurrence vector are Ricci flat spaces.*

Thus, for $n \neq 4$, conformally recurrent Kähler spaces are recurrent Kähler spaces, which, in turn, are divided into two types depending on the type of recurrence vector. If the recurrence vector is non-isotropic, then the formula (3.3) holds, and if it is isotropic, then the equation (3.24) holds.

4. Conformally recurrent spaces

We now turn to the consideration of four-dimensional conformally recurrent Kähler spaces.

Theorem 4.1. *In conformally recurrent Kähler spaces K_n $(n = 4)$, the following conditions hold*

$$2\rho_\alpha R^\alpha_{lji} = \tau_{\bar{l}} g_{i\bar{j}} + \rho_{\bar{l}} R_{i\bar{j}} + R_{i\bar{j},\bar{l}}. \tag{4.1}$$

Proof. The equations (1.5), (1.7), (1.9) for $n = 4$ can be written as follows:

$$2P_{ij} = R_{ij} - \tfrac{1}{6} R g_{ij},$$

$$C_{hijk} = R_{hijk} + \tfrac{1}{2}\left(g_{hj}R_{ik} - g_{hk}R_{ij} + R_{hj}g_{ik} - R_{hk}g_{ij}\right) - \\ - \tfrac{1}{6} R\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right),$$

$$2\left(R_{hijk,l} - \rho_l R_{hijk}\right) = g_{hk}\left(R_{ij,l} - \rho_l R_{ij}\right) + g_{ij}\left(R_{hk,l} - \rho_l R_{hk}\right) - \\ - g_{hj}\left(R_{ik,l} - \rho_l R_{ik}\right) - g_{ik}\left(R_{hj,l} - \rho_l R_{hj}\right).$$

Cycling by indices j, k, l, we obtain

$$2\left(\rho_l R_{ihjk} + \rho_j R_{ihkl} + \rho_k R_{ihlj}\right) = \\ = g_{ij}A_{hkl} + g_{ik}A_{hlj} + g_{il}A_{hjk} + g_{hk}A_{ijl} + g_{hl}A_{ikj} + g_{hj}A_{ilk}, \tag{4.2}$$

where

$$A_{ijk} = R_{ij,k} - \rho_k R_{ij} - R_{ik,j} + \rho_j R_{ik}.$$

Multiply the equation (4.2) by g^{hk} and wrap by indices h and k. Then after summing the similar summands we get that

$$2\rho_\alpha R^\alpha_{ijl} = \tau_l g_{ij} - \tau_j g_{il} + R_{ij,l} - R_{il,j} + \rho_l R_{ij} - \rho_j R_{il}, \tag{4.3}$$

where

$$\tau_i = \rho_\alpha R^\alpha_i - \frac{1}{2} R_{,i}.$$

Applying to the expression (4.3) the conjugation operation on the indices i and j we obtain that

$$2\rho_\alpha R^\alpha_{\bar{i}\bar{j}l} = \tau_l g_{ij} - \tau_{\bar{j}} g_{\bar{i}l} + R_{ij,l} - R_{\bar{i}l,\bar{j}} + \rho_l R_{ij} - \rho_{\bar{j}} R_{\bar{i}l}. \tag{4.4}$$

Symmetrize further (4.4) by indices i and l:

$$2\left(\rho_\alpha R^\alpha_{\bar{i}\bar{j}l} + \rho_\alpha R^\alpha_{\bar{l}\bar{j}i}\right) = \tau_l g_{ij} + \tau_i g_{lj} + R_{ij,l} + R_{lj,i} + \rho_l R_{ij} + \rho_i R_{lj}, \tag{4.5}$$

and exchange notations for indices i and j in (4.5):

$$2\left(\rho_\alpha R^\alpha_{\bar{j}\bar{i}l} + \rho_\alpha R^\alpha_{\bar{l}\bar{i}j}\right) = \tau_l g_{ji} + \tau_j g_{li} + R_{ji,l} + R_{li,j} + \rho_l R_{ji} + \rho_j R_{li}. \tag{4.6}$$

Adding now (4.6) and (4.3) we obtain that

$$\rho_\alpha R^\alpha_{ijl} + \rho_\alpha R^\alpha_{\bar{j}\bar{i}l} + \rho_\alpha R^\alpha_{\bar{l}\bar{i}j} = \tau_l g_{ij} + R_{ij,l} + \rho_l R_{ij}. \tag{4.7}$$

Applying to (4.7) the conjugation operation on the indices j and l, and taking into account the properties of the Riemannian tensor, we will get the desired equation (4.1). □

5. Conclusion

Thus, it has been proved that there are no conformally recurrent Kähler spaces other than recurrent ones, if the dimension of the space is not four. The fact that conformally recurrent Kähler spaces are recurrent allows us to distinguish between two types of these spaces depending on the properties of the recurrence vector. These types can be distinguished in the classification proposed in the current paper. Namely, if the recurrence vector is not isotropic, then the scalar curvature of such spaces is distinct from zero, and the Riemannian tensor satisfies the condition

$$\frac{R}{2} R_{hijk} = R_{hk} R_{ij} - R_{hj} R_{ik}.$$

Thus, recurrent Kähler spaces with an isotropic recurrence vector are Ricci flat, and therefore they have zero scalar curvature. Different from recurrent conformally recurrent Kähler spaces may exist only when the dimension of the space is 4. We found that tensor condition is satisfied by four-dimensional conformally recurrent Kähler spaces, which may allow us to construct examples of such spaces. The construction of an example of conformally recurrent Kähler spaces other than recurrent ones remains an unsolved problem.

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