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On partial preliminary group classification of some class of (1+3)-dimensional Monge–Ampère equations, II. Two-dimensional Lie algebras

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Abstract. We perform the partial preliminary group classification of some class of $(1+3)$-dimensional Monge–Ampère equations using non-equivalent functional bases of the first-order differential invariants of two-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$.

Анотація. В роботі проведено часткову попередню групову класифікацію деякого класу $(1+3)$-вимірних рівнянь Монжа–Ампера використовуючи нееквівалентні фунціональні базиси диференціальних інваріантів першого порядку двовимірних неспряжених підалгебр алгебри Лі групи Пуанкаре $P(1,4)$.

1. Introduction

In many cases, the mathematical modeling of the real processes of the nature is reduced to the construction and investigation of classes of differential equations. It is known that a group classification is one of the powerful tools for investigation classes of differential equations. The history of group classification methods goes back to Sophus Lie [25]. The modern formulation of the problem of group classification of differential equations was suggested by Lev Ovsiannikov [29].

Currently, many studies are devoted to group classification of different classes of differential equations. In those works we can also find various generalizations of the classical Lie–Ovsiannikov approach. Here, we only mention some of them [3, 4, 20, 24, 25, 28, 29, 30, 38] (see also the references therein).

A solution of many problems of the geometry, theoretical physics, astrophysics, differential equations, optimal mass transportation, one-dimensional gas dynamics and etc. is reduced to the investigation of classes of

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Monge–Ampère equations in the spaces of different dimensions and different types. At the present time, there are a lot of papers and books in which those classes have been studied by different methods. Here, we only present some of them [1, 2, 5, 6, 7, 8, 9, 16, 17, 21, 22, 23, 26, 27, 31, 32, 33, 34, 35, 36, 37, 39, 40] (see also the references therein).

In this paper we present some of the results concerning partial preliminary group classification of some class of (1+3)-dimensional Monge–Ampère equations.

To present the results obtained, we give some information about the Lie algebra of the Poincaré group $P(1,4)$ and its nonconjugate subalgebras.

2. Lie algebra of the Poincaré group P(1,4) and its nonconjugate subalgebras

The group $P(1,4)$ is a group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the smallest group, which contains, as subgroups, the extended Galilei group $\widetilde{G}(1,3)$ [19] (the symmetry group of classical physics) and the Poincaré group $P(1,3)$ (the symmetry group of relativistic physics).

The Lie algebra of the group $P(1,4)$ is generated by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu,\nu = 0,1,2,3,4)$ and P_μ $(\mu = 0,1,2,3,4)$, which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \qquad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

In this paper, we consider the following representation [16] of the Lie algebra of the group $P(1,4)$

$$P_0 = \frac{\partial}{\partial x_0}, \qquad P_1 = -\frac{\partial}{\partial x_1}, \qquad P_2 = -\frac{\partial}{\partial x_2}, \qquad P_3 = -\frac{\partial}{\partial x_3},$$
$$P_4 = -\frac{\partial}{\partial u}, \qquad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \qquad x_4 \equiv u.$$

In the following, we will use the next basis elements:

$$G = M_{04}, \qquad L_1 = M_{23}, \qquad L_2 = -M_{13}, \qquad L_3 = M_{12},$$
$$P_a = M_{a4} - M_{0a}, \qquad C_a = M_{a4} + M_{0a}, \ (a = 1,2,3),$$
$$X_0 = \frac{1}{2}\left(P_0 - P_4\right), \qquad X_4 = \frac{1}{2}\left(P_0 + P_4\right), \qquad X_k = P_k, \ (k = 1,2,3).$$

Nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ have been described in the papers [10, 11, 12, 18].

The Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ is generated by the following basis elements:

$$L_1,\ L_2,\ L_3,\ P_1,\ P_2,\ P_3,\ X_0,\ X_1,\ X_2,\ X_3,\ X_4.$$

The classification of all nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ of dimensions $\leqslant 3$ was performed in [14].

3. On partial preliminary group classification of some class of (1+3)-dimensional Monge–Ampère equations

Let us consider the following class of $(1+3)$-dimensional Monge–Ampère equations:

$$\det(u_{\mu\nu}) = F(x_0, x_1, x_2, x_3, u, u_0, u_1, u_2, u_3),$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u_\alpha \equiv \frac{\partial u}{\partial x_\alpha}$, $\mu, \nu, \alpha = 0, 1, 2, 3$.

Here, $M(1,3)$ is a four-dimensional Minkowski space, F is an arbitrary real smooth function.

For the group classification of this class we have used the classical Lie–Ovsiannikov approach [25, 29].

From the results obtained by Fushchich and Serov in the work [16] it follows that the transformations from the group $P(1,4)$ belong to the equivalence group of the class under investigation.

The results of the classification [14] of two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$ can be formulated as

Proposition 3.1. *The Lie algebra of the group $P(1,4)$ contains 49 two-dimensional nonconjugate subalgebras.*

At the present time, we have performed partial preliminary group classification of the class under consideration, using non-equivalent functional bases of the first-order differential invariants of two-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$. More details about non-equivalent functional bases can be found in [13, 15] (see also the references therein).

From the results obtained, it follows that there exist 49 subclasses which are invariant with respect to two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$. All those classes can be written in the following form:

$$\det(u_{\mu\nu}) = \Phi(J_1, J_2, \ldots, J_7),$$

where $\{J_1, J_2, \ldots, J_7\}$ are non-equivalent functional bases of the first-order differential invariants of two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$.

Therefore, below for all subclasses we only present the basis elements of two-dimensional nonconjugate subalgebras as well as corresponding non-equivalent functional bases of the first-order differential invariants.

The Abelian subalgebras.

(1) $\langle G, L_3\rangle$.
$J_1 = x_3, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = (x_0^2 - u^2)^{1/2},$
$J_4 = (x_0 + u)^2 \frac{u_0-1}{u_0+1}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_0^2-1}{u_3^2},$
$J_7 = \frac{u_1^2+u_2^2}{u_3^2};$

(2) $\langle G, X_1\rangle$.
$J_1 = x_2, \quad J_2 = x_3, \quad J_3 = (x_0^2 - u^2)^{1/2}, \quad J_4 = \frac{x_0+u}{u_0+1}u_1,$
$J_5 = \frac{u_0^2-1}{u_1^2}, \quad J_6 = \frac{u_1}{u_2}, \quad J_7 = \frac{u_1}{u_3};$

(3) $\langle L_3 + eG, X_3, e > 0\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_0^2 - u^2)^{1/2}, \quad J_3 = \frac{x_0+u}{u_0+1}u_3,$
$J_4 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_5 = \ln(x_0 + u) + e \arctan\frac{x_1}{x_2}, \quad J_6 = \frac{u_1^2+u_2^2}{u_3^2},$
$J_7 = \frac{u_0^2-1}{u_3^2};$

(4) $\langle P_3 + C_3, L_3\rangle$.
$J_1 = x_0, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = (x_3^2 + u^2)^{1/2}, \quad J_4 = \frac{u_3u+x_3}{u-x_3u_3},$
$J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_1^2+u_2^2}{u_0^2}, \quad J_7 = \frac{u_3^2+1}{u_0^2};$

(5) $\langle P_3 + C_3 + 2L_3, X_0 + X_4\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_3^2 + u^2)^{1/2}, \quad J_3 = \frac{x_1u-x_2x_3}{x_2u+x_1x_3},$
$J_4 = \frac{u_3u+x_3}{u-x_3u_3}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_1^2+u_2^2}{u_0^2},$
$J_7 = \frac{u_3^2+1}{u_0^2};$

(6) $\langle P_3 + C_3 + eL_3, X_0 + X_4, e > 2\rangle$.
$J_1 = (x_1^2+x_2^2)^{1/2}, \ J_2 = (x_3^2+u^2)^{1/2}, \ J_3 = 2\arctan\frac{x_1}{x_2} - e\arctan\frac{x_3}{u},$
$J_4 = \frac{u_3u+x_3}{u-x_3u_3}, \ J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \ J_6 = \frac{u_1^2+u_2^2}{u_0^2}, \ J_7 = \frac{u_3^2+1}{u_0^2};$

(7) $\langle G + aX_3, L_3, a < 0\rangle$.
$J_1 = x_3 - a\ln(x_0 + u), \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = (x_0^2 - u^2)^{1/2},$
$J_4 = (x_0 + u)^2 \frac{1-u_0}{u_0+1}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_3^2}{u_0^2-1},$
$J_7 = \frac{u_1^2+u_2^2}{u_3^2};$

(8) $\langle G, L_3 + dX_3, d < 0\rangle$.
$J_1 = x_3 + d\arctan\frac{x_1}{x_2}, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = (x_0^2 - u^2)^{1/2},$
$J_4 = (x_0 + u)^2\frac{1-u_0}{u_0+1}, \quad J_5 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_6 = \frac{u_3^2}{u_0^2-1},$
$J_7 = \frac{u_1^2+u_2^2}{u_3^2};$

(9) $\langle G + a_2X_2, X_1, a_2 < 0\rangle$.
$J_1 = x_3, \quad J_2 = (x_0^2 - u^2)^{1/2}, \quad J_3 = x_2 - a_2\ln(x_0 + u),$
$J_4 = (x_0 + u)\frac{u_1}{u_0+1}, \quad J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_1}{u_3}, \quad J_7 = \frac{u_0^2-1}{u_1^2};$

(10) $\langle P_3 + C_3, L_3 + d(X_0 + X_4), d < 0\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_3^2 + u^2)^{1/2}, \quad J_3 = x_0 + d\arctan\frac{x_2}{x_1},$
$J_4 = \frac{u_3u+x_3}{u-x_3u_3}, \; J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \; J_6 = \frac{u_1^2+u_2^2}{u_0^2}, \; J_7 = \frac{u_3^2+1}{u_0^2};$

(11) $\langle L_3 + \alpha(X_0 + X_4), P_3 + C_3 + \beta(X_0 + X_4), \alpha < 0, \beta < 0\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_3^2 + u^2)^{1/2}, \quad J_3 = \frac{u_3u+x_3}{u-x_3u_3},$
$J_4 = 2x_0 + \beta\arctan u_3 - 2\alpha\arctan\frac{x_1}{x_2}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2},$
$J_6 = \frac{u_1^2+u_2^2}{u_3^2+1}, \quad J_7 = \frac{u_3^2+1}{u_0^2};$

(12) $\langle G + \alpha X_3, L_3 + \beta X_3, \alpha < 0, \beta < 0\rangle$.
$J_1 = x_3 - \alpha\ln(x_0 + u) - \beta\arctan\frac{x_2}{x_1}, \quad J_2 = (x_1^2 + x_2^2)^{1/2},$
$J_3 = (x_0^2 - u^2)^{1/2}, \quad J_4 = (x_0 + u)^2\frac{1-u_0}{u_0+1}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2},$
$J_6 = \frac{u_3^2}{u_0^2-1}, \quad J_7 = \frac{u_1^2+u_2^2}{u_3^2}.$

Note that the next nonconjugate subalgebras belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$.

(13) $\langle P_1, P_2\rangle$.
$J_1 = x_3, \quad J_2 = x_0 + u, \quad J_3 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2},$
$J_4 = \frac{x_1}{x_0+u} + \frac{u_1}{u_0+1}, \quad J_5 = \frac{x_2}{x_0+u} + \frac{u_2}{u_0+1}, \quad J_6 = \frac{u_3}{u_0+1},$
$J_7 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(14) $\langle L_3, P_3\rangle$.
$J_1 = x_0 + u, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = (x_0^2 - x_3^2 - u^2)^{1/2},$
$J_4 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_1^2+u_2^2}{(u_0+1)^2},$
$J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(15) $\langle P_3, X_1\rangle$.
$J_1 = x_2, \quad J_2 = x_0 + u, \quad J_3 = (x_0^2 - x_3^2 - u^2)^{1/2},$

$J_4 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}$, $J_5 = \frac{u_1}{u_2}$, $J_6 = \frac{u_1}{u_0+1}$,
$J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1}$;

(16) $\langle P_3, X_4 \rangle$.
$J_1 = x_1$, $J_2 = x_2$, $J_3 = x_0 + u$, $J_4 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}$,
$J_5 = \frac{u_1}{u_2}$, $J_6 = \frac{u_1}{u_0+1}$, $J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1}$;

(17) $\langle L_3 - P_3, X_4 \rangle$.
$J_1 = x_0 + u$, $J_2 = (x_1^2 + x_2^2)^{1/2}$, $J_3 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}$,
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}$, $J_5 = \arctan\frac{x_1}{x_2} + \frac{x_3}{x_0+u}$, $J_6 = \frac{u_1^2+u_2^2}{(u_0+1)^2}$,
$J_7 = \frac{u_3^2+2(u_0+1)}{(u_0+1)^2}$;

(18) $\langle L_3, X_4 \rangle$.
$J_1 = x_3$, $J_2 = x_0 + u$, $J_3 = (x_1^2 + x_2^2)^{1/2}$, $J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}$,
$J_5 = u_0$, $J_6 = u_3$, $J_7 = u_1^2 + u_2^2$;

(19) $\langle L_3, X_0 + X_4 \rangle$.
$J_1 = x_3$, $J_2 = (x_1^2 + x_2^2)^{1/2}$, $J_3 = u$, $J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}$,
$J_5 = u_0$, $J_6 = u_3$, $J_7 = u_1^2 + u_2^2$;

(20) $\langle L_3, X_0 - X_4 \rangle$.
$J_1 = x_0$, $J_2 = x_3$, $J_3 = (x_1^2 + x_2^2)^{1/2}$, $J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}$,
$J_5 = u_0$, $J_6 = u_3$, $J_7 = u_1^2 + u_2^2$;

(21) $\langle X_0 + X_4, X_0 - X_4 \rangle$.
$J_1 = x_1$, $J_2 = x_2$, $J_3 = x_3$, $J_4 = u_0$, $J_5 = u_1$,
$J_6 = u_2$, $J_7 = u_3$;

(22) $\langle X_1, X_4 \rangle$.
$J_1 = x_0 + u$, $J_2 = x_2$, $J_3 = x_3$, $J_4 = u_0$, $J_5 = u_1$,
$J_6 = u_2$, $J_7 = u_3$;

(23) $\langle X_1, X_0 - X_4 \rangle$.
$J_1 = x_0$, $J_2 = x_2$, $J_3 = x_3$, $J_4 = u_0$, $J_5 = u_1$, $J_6 = u_2$,
$J_7 = u_3$;

(24) $\langle L_3 - X_4, P_3 + hX_0, h > 0 \rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}$, $J_2 = (x_0 + u)^2 - 2hx_3$,
$J_3 = h \arctan\frac{x_1}{x_2} + h(x_0 - u) + \frac{2}{3h}(x_0 + u)^3 - 2x_3(x_0 + u)$,

$J_4 = x_0 + u + h\frac{u_3}{u_0+1}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_1^2+u_2^2}{(u_0+1)^2},$
$J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(25) $\langle L_3 - X_4, P_3\rangle$.
$J_1 = x_0 + u, \quad J_2 = (x_1^2 + x_2^2)^{1/2},$
$J_3 = x_0^2 - x_3^2 - u^2 + (x_0 + u)\arctan\frac{x_1}{x_2},$
$J_4 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2},$
$J_6 = \frac{u_1^2+u_2^2}{(u_0+1)^2}, \quad J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(26) $\langle L_3, P_3 + X_0\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_0 + u)^2 - 2x_3,$
$J_3 = x_0 - u + \frac{2}{3}(x_0 + u)^3 - 2x_3(x_0 + u), \quad J_4 = x_0 + u + \frac{u_3}{u_0+1},$
$J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_1^2+u_2^2}{(u_0+1)^2}, \quad J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(27) $\langle P_3 + X_0, X_1\rangle$.
$J_1 = x_2, \quad J_2 = (x_0 + u)^2 - 2x_3,$
$J_3 = x_0 - u + \frac{2}{3}(x_0 + u)^3 - 2x_3(x_0 + u), \quad J_4 = x_0 + u + \frac{u_3}{u_0+1},$
$J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_1}{u_0+1}, \quad J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(28) $\langle P_3 + X_2, X_1\rangle$.
$J_1 = x_0 + u, \quad J_2 = (x_0^2 - x_3^2 - u^2)^{1/2}, \quad J_3 = x_2 + \frac{x_3}{x_0+u},$
$J_4 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}, \quad J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_1}{u_0+1},$
$J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(29) $\langle P_3 + X_0, X_4\rangle$.
$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = (x_0 + u)^2 - 2x_3, \quad J_4 = x_0 + u + \frac{u_3}{u_0+1},$
$J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_1}{u_0+1}, \quad J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(30) $\langle P_3 + X_1, X_4\rangle$.
$J_1 = x_2, \quad J_2 = x_0 + u, \quad J_3 = x_1 + \frac{x_3}{x_0+u},$
$J_4 = \frac{x_3}{x_0+u} + \frac{u_3}{u_0+1}, \quad J_5 = \frac{u_1}{u_2}, \quad J_6 = \frac{u_2}{u_0+1},$
$J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(31) $\langle L_3 - P_3 + \alpha_0X_0, X_4, \alpha_0 < 0\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = x_0 + u - \alpha_0\arctan\frac{u_1}{u_2},$
$J_3 = (x_0 + u)^2 + 2\alpha_0x_3, \quad J_4 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2},$

$J_5 = x_0 + u - \alpha_0 \frac{u_3}{u_0+1}, \quad J_6 = \frac{u_1^2+u_2^2}{(u_0+1)^2},$
$J_7 = \frac{u_3^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(32) $\langle L_3 - X_4, X_3 \rangle$.
$J_1 = x_0 + u, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = x_0 - u - \arctan \frac{u_2}{u_1},$
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_1^2 + u_2^2;$

(33) $\langle L_3 + d_3X_3, X_0 + X_4, d_3 < 0 \rangle$.
$J_1 = u, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = x_3 + d_3 \arctan \frac{x_1}{x_2},$
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_1^2 + u_2^2;$

(34) $\langle L_3 + d_3X_3, X_0 - X_4, d_3 < 0 \rangle$.
$J_1 = x_0, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = x_3 + d_3 \arctan \frac{u_1}{u_2},$
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_1^2 + u_2^2;$

(35) $\langle L_3 + d_4X_4, X_0 - X_4, d_4 < 0 \rangle$.
$J_1 = x_3, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = 2x_0 - d_4 \arctan \frac{u_1}{u_2},$
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_1^2 + u_2^2;$

(36) $\langle L_3 + \alpha(X_0 + X_4), X_4, \alpha < 0 \rangle$.
$J_1 = x_3, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = x_0 + u - \alpha \arctan \frac{x_1}{x_2},$
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_1^2 + u_2^2;$

(37) $\langle L_3 + a_3X_3, X_4, \alpha < 0 \rangle$.
$J_1 = x_0 + u, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = x_3 + a_3 \arctan \frac{u_1}{u_2},$
$J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0, \quad J_6 = u_3, \quad J_7 = u_1^2 + u_2^2;$

(38) $\langle P_1 + X_3, P_2 \rangle$.
$J_1 = x_0 + u, \quad J_2 = (x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2}, \quad J_3 = x_3 + \frac{x_1}{x_0+u},$
$J_4 = \frac{x_1}{x_0+u} + \frac{u_1}{u_0+1}, \quad J_5 = \frac{x_2}{x_0+u} + \frac{u_2}{u_0+1}, \quad J_6 = \frac{u_3}{u_0+1},$
$J_7 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(39) $\langle P_1 + X_3, P_2 + \gamma X_2 + \beta X_3, \gamma > 0, \beta > 0 \rangle$.
$J_1 = x_0 + u, \quad J_2 = \frac{x_2}{x_0+u-\gamma} + \frac{u_2}{u_0+1},$
$J_3 = \frac{\beta x_2}{x_0+u-\gamma} + \frac{x_1}{x_0+u} + x_3, \quad J_4 = \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u-\gamma} + 2u,$
$J_5 = \frac{x_1}{x_0+u} + \frac{u_1}{u_0+1}, \quad J_6 = \frac{u_3}{u_0+1}, \quad J_7 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(40) $\langle P_1 + X_3, P_2 + \gamma X_2, \gamma > 0 \rangle$.
$J_1 = x_0 + u, \quad J_2 = \frac{x_2}{x_0+u-\gamma} + \frac{u_2}{u_0+1}, \quad J_3 = x_1 + x_3(x_0 + u),$

$J_4 = \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u-\gamma} + 2u, \quad J_5 = x_1 + u_1\frac{x_0+u}{u_0+1},$
$J_6 = \frac{u_3}{u_0+1}, \quad J_7 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(41) $\langle P_1, P_2 + X_2 + \beta X_3, \beta > 0\rangle$.
$J_1 = x_0 + u, \quad J_2 = x_3 + \frac{\beta x_2}{x_0+u-1},$
$J_3 = \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u-1} + 2u, \quad J_4 = \frac{x_1}{x_0+u} + \frac{u_1}{u_0+1},$
$J_5 = \frac{x_2}{x_0+u-1} + \frac{u_2}{u_0+1}, \quad J_6 = \frac{u_3}{u_0+1},$
$J_7 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1};$

(42) $\langle P_1, P_2 + X_2\rangle$.
$J_1 = x_3, \quad J_2 = x_0 + u, \quad J_3 = \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u-1} + 2u,$
$J_4 = \frac{x_1}{x_0+u} + \frac{u_1}{u_0+1}, \quad J_5 = \frac{x_2}{x_0+u-1} + \frac{u_2}{u_0+1}, \quad J_6 = \frac{u_3}{u_0+1},$
$J_7 = \frac{u_1^2+u_2^2}{(u_0+1)^2} + \frac{2}{u_0+1}.$

The non-Abelian subalgebras.

(1) $\langle G, P_3\rangle$.
$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = (x_0^2 - x_3^2 - u^2)^{1/2}, \quad J_4 = \frac{x_0+u}{u_0+1}u_1,$
$J_5 = x_3 + \frac{x_0+u}{u_0+1}u_3, \quad J_6 = \frac{u_1}{u_2}, \quad J_7 = \frac{u_0^2-u_3^2-1}{u_1^2};$

(2) $\langle L_3 + eG, P_3, e > 0\rangle$.
$J_1 = (x_1^2 + x_2^2)^{1/2}, \quad J_2 = (x_0^2 - x_3^2 - u^2)^{1/2}, \quad J_3 = x_3 + \frac{x_0+u}{u_0+1}u_3,$
$J_4 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_5 = (u_1^2 + u_2^2)\left(\frac{x_0+u}{u_0+1}\right)^2,$
$J_6 = e\arctan\frac{u_1}{u_2} + \ln(x_0 + u), \quad J_7 = \frac{u_0^2-u_3^2-1}{u_1^2+u_2^2};$

(3) $\langle G, X_4\rangle$.
$J_1 = x_1, \quad J_2 = x_2, \quad J_3 = x_3, \quad J_4 = \frac{u_1^2}{u_0^2-1}, \quad J_5 = \frac{x_0+u}{u_0+1}u_1,$
$J_6 = \frac{u_1}{u_2}, \quad J_7 = \frac{u_1}{u_3};$

(4) $\langle L_3 + eG, X_4, e > 0\rangle$.
$J_1 = x_3, \quad J_2 = (x_1^2 + x_2^2)^{1/2}, \quad J_3 = \ln(x_0 + u) + e\arctan\frac{x_1}{x_2},$
$J_4 = \frac{x_0+u}{u_0+1}u_3, \quad J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad J_6 = \frac{u_0^2-1}{u_3^2},$
$J_7 = \frac{u_1^2+u_2^2}{u_3^2};$

(5) $\langle G + aX_1, P_3, a < 0\rangle$.
$J_1 = x_1 - a\ln(x_0 + u), \quad J_2 = x_2, \quad J_3 = (x_0^2 - x_3^2 - u^2)^{1/2},$

$J_4 = (x_0+u)\frac{u_1}{u_0+1}$, $J_5 = x_3 + (x_0+u)\frac{u_3}{u_0+1}$, $J_6 = \frac{u_1}{u_2}$,
$J_7 = \frac{u_0^2-u_3^2-1}{u_1^2}$;

(6) $\langle G + cX_1, X_4, c<0\rangle$.
$J_1 = x_2$, $J_2 = x_3$, $J_3 = x_1 - c\ln(x_0+u)$,
$J_4 = (x_0+u)\frac{u_2}{u_0+1}$, $J_5 = \frac{u_1}{u_2}$, $J_6 = \frac{u_1}{u_3}$, $J_7 = \frac{u_3^2}{u_0^2-1}$;

(7) $\langle L_3 + eG + \kappa_3 X_3, X_4, e>0, \kappa_3<0\rangle$.
$J_1 = (x_1^2+x_2^2)^{1/2}$, $J_2 = x_3 + \kappa_3 \arctan\frac{x_1}{x_2}$,
$J_3 = \ln(x_0+u) + e\arctan\frac{x_1}{x_2}$, $J_4 = (x_0+u)\frac{u_0-1}{u_3}$,
$J_5 = \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}$, $J_6 = \frac{u_0^2-1}{u_3^2}$, $J_7 = \frac{u_1^2+u_2^2}{u_3^2}$.

4. Conclusions

We perform the partial preliminary group classification of the class under consideration, using non-equivalent functional bases of the first-order differential invariants of two-dimensional nonconjugate subalgebras of the Lie algebra of the Poincaré group $P(1,4)$. From the results obtained it follows that

- there exist 49 subclasses which are invariant with respect to two-dimensional nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$;
- among those subclasses, there exist 42 ones, which are invariant with respect to two-dimensional Abelian nonconjugate subalgebras and 7 ones, which are invariant with respect to two-dimensional non-Abelian nonconjugate subalgebras of the Lie algebra of the group $P(1,4)$;
- among those subclasses, there exist 30 ones, which are invariant with respect to two-dimensional nonconjugate subalgebras which belong to the Lie algebra of the extended Galilei group $\widetilde{G}(1,3) \subset P(1,4)$;
- symmetry properties of the obtained subclasses can be, in particular, used for the symmetry reduction of them to ones with a less number of independent variables.

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