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# Lebesgue number and total boundedness

Ajit Kumar Gupta, Saikat Mukherjee

**Abstract.** A generalization of the Lebesgue number lemma is obtained. For a metric space $X$ in the class of strongly metrizable spaces, sufficient conditions for each open cover of $X$ with a Lebesgue number has a finite subcover are obtained. It is proved that, if each countably infinite locally finite open cover of a chainable metric space $X$ has a Lebesgue number, then $X$ is totally bounded. A property for metric spaces which is a generalization of connectedness and Menger convexity is introduced. It is observed that Atsujiness and compactness are equivalent for a metric space with this introduced property as well as for a chainable metric space.

**Анотація.** В роботі отримано узагальнення леми Лебега. Для метричного простору $X$ з класу сильно метризованих просторів отримано достатні умови для того, щоб кожне відкрите покриття $X$ з числом Лебега мало скінченне підпокриття. Доведено, що якщо кожне зліченно нескінченне локально скінченне відкрите покриття ланцюгового метричного простору $X$ має число Лебега, то $X$ є цілком обмеженим. Введено властивість для метричних просторів, що узагальненює зв'язність та опуклість Менгера. Також показано, що для просторів з цією властивістю компактність еквівалентна тому, що кожне відкрите покриття має число Лебега.

## 1. Introduction

The Lebesgue number lemma ([10, p. 175]) says, if $X$ is a compact metric space, then each open cover of $X$ has a Lebesgue number. A metric space $X$ is said to be *Atsuji space* if each open cover of $X$ has a Lebesgue number. Therefore, the Atsuji spaces are weaker than compact metric spaces. Further, the Atsuji spaces are stronger than complete metric spaces, because the Atsuji spaces are such metric spaces in which each pseudo-Cauchy sequence with distinct terms has a limit point (see [3], p.402). In this manuscript, we obtain a generalization of the Lebesgue number lemma. Using this generalization, we immediately conclude that each locally finite open cover of a totally bounded subset $S$ in a complete metric space has a finite

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subcover for $S$. We discuss the class of strongly metrizable spaces and obtain sufficient conditions for a metric space $X$ in this class such that each open cover of $X$ with a Lebesgue number has a finite subcover.

Further, we find a sufficient condition under which a chainable metric space is totally bounded, and as a corollary to it, we observe that Atsujiness and compactness are equivalent for a chainable metric space. A property of a metric space which is weaker than connectedness is introduced, and it is found that Atsujiness and compactness are equivalent for a metric space having that introduced property.

## 2. Preliminaries

For a subset $A$ in a metric space $(X,d)$, we denote by $\mathrm{diam}(A)$ the diameter of $A$ and by $X\backslash A$ the complement of $A$ in $X$. An open ball centered at $a$ with radius $\varepsilon$ is denoted by $B_d(a,\varepsilon)$. Given an $\varepsilon > 0$, the collection of open balls $\{B_d(x,\varepsilon) \mid x \in X\}$ is denoted by $\mathcal{B}_d(\varepsilon)$. We will drop the subscript $d$ from the notations $B_d(a,\varepsilon)$ and $\mathcal{B}_d(\varepsilon)$, while dealing with a single metric in the space $X$.

Let $(X,d)$ be a metric space. For a given $\varepsilon > 0$, an *$\varepsilon$-chain of length $n$* between two points $x, y \in X$ is a finite sequence $a_0, a_1, \ldots, a_n$ in $X$ such that $a_0 = x$, $a_n = y$ and $d(a_{i-1}, a_i) \leqslant \varepsilon$ for all $i = 1, 2, \ldots, n$.

A metric space $X$ is said to be *finitely chainable* if for each $\varepsilon > 0$, there are finitely many points $p_1, p_2, \ldots, p_j \in X$ and $m \in \mathbb{N}$ such that each point of $X$ can be joined with some $p_i$, $1 \leqslant i \leqslant j$, by an $\varepsilon$-chain of length $m$.

A subset $S$ in a metric space $X$ is said to be *finitely chainable* if for each $\varepsilon > 0$ there exist finitely many points $p_1, p_2, \ldots, p_j$ in $X$ and $m \in \mathbb{N}$ such that each point of $S$ can be joined with some $p_i$, $1 \leqslant i \leqslant j$, by an $\varepsilon$-chain of length $m$. By a finitely chainable subspace $S$ in a metric space $(X,d)$, we mean the metric space $(S,d)$ is finitely chainable. A totally bounded subset in a metric space $X$ is a finitely chainable subspace in $X$.

A metric space is said to have an *Atsuji completion* if its completion is an Atsuji space.

An open cover $\mathcal{U}$ of a topological space is said to be *star finite* if each member of $\mathcal{U}$ intersects only finitely many members of $\mathcal{U}$. An open cover $\mathcal{U}$ of a topological space $X$ is said to be *locally finite* if each point in $X$ has a neighborhood which intersects only finitely many members of $\mathcal{U}$.

A topological space $X$ is said to be *paracompact* if for each open cover of $X$ there is a locally finite open refinement that covers $X$.

Given a subset $S$ in a topological space $X$, by a *locally finite open cover* $\mathcal{O}$ of $S$, we mean that $\mathcal{O}$ covers $S$, each member of $\mathcal{O}$ is open in $X$, and each point of $X$ has a neighborhood which intersects at most finitely many members of $\mathcal{O}$.

## 3. Lebesgue Number and Total Boundedness

Given an open cover $\mathcal{O}$ of a subset $S$ in a metric space $X$, if there is a number $\alpha > 0$ such that each $A \subset S$ with $\operatorname{diam}(A) < \alpha$ is contained in some $O \in \mathcal{O}$, then the number $\alpha$ is said to be a *Lebesgue number* for the cover $\mathcal{O}$.

Since, for a metric space $X$, each open cover of $X$ has a Lebesgue number if and only if each locally finite open cover of $X$ has a Lebesgue number, the Lebesgue number lemma ([10, p. 175]) can be equivalently restated as follows:

*Let $X$ be a complete metric space. If $X$ is totally bounded, then each locally finite open cover of $X$ has a Lebesgue number.*

However, each locally finite open cover of a totally bounded subset in a complete metric space need not have a Lebesgue number, for example, for the locally finite open cover $\{(0,1),(1,2)\}$ of the totally bounded subset $(0,1) \cup (1,2)$ in $\mathbb{R}$, there is no Lebesgue number, because the subset

$$(1-\varepsilon, 1+\varepsilon)\backslash\{1\} \subset (0,1) \cup (1,2)$$

is not contained in either $(0,1)$ or $(1,2)$ for any $\varepsilon \in (0,1]$. The following result provides a generalization of the Lebesgue number lemma.

**Theorem 3.1.** *Let $X$ be a complete metric space. If $S \subset X$ is totally bounded, then, for each locally finite open cover $\mathcal{O}$ of $S$ there is a number $\alpha > 0$ such that each subset $B \subset S$ with $\operatorname{diam}(B) < \alpha$ has a finite subcover from $\mathcal{O}$.*

*Proof.* For a locally finite open cover $\mathcal{O}$ of a totally bounded subset $S \subset X$, consider the open cover $\mathcal{A}$ of $S$ consisting of all finite unions of elements of $\mathcal{O}$. We prove, there is a Lebesgue number $\alpha > 0$ for the cover $\mathcal{A}$.

On the contrary, we assume, there is no Lebesgue number for the cover $\mathcal{A}$. Then, for each $n$, there is $C_n \subset S$ with $\operatorname{diam}(C_n) < 1/n$ such that $C_n \not\subset A$ for all $A \in \mathcal{A}$.

Consider an element $x_n \in C_n$. Since $S$ is totally bounded, the sequence $\{x_n\}$ has a Cauchy subsequence, say $\{x_{n_i}\}_{i=1}^{\infty}$, converging to some $a \in X$. If $a \in S'$, and since $\mathcal{O}$ is locally finite, there is an $\varepsilon > 0$ such that the ball $B(a,\varepsilon)$ will intersect at most finitely many members of $\mathcal{O}$. For $\varepsilon/2$, there is $i$ such that $\operatorname{diam}(C_{n_i}) < \varepsilon/2$, which implies $C_{n_i} \subset B(x_{n_i}, \varepsilon/2)$, and for $\varepsilon/2$, there is $j$ such that $x_{n_j} \in B(a, \varepsilon/2)$.

Taking $m = \max\{i,j\}$, we have $C_{n_m} \subset B(a,\varepsilon)$, and since $C_{n_m} \subset S$, it follows that $C_{n_m}$ is contained in a union of some finite number of elements of $\mathcal{O}$. On the other hand, if $x_{n_i} = a$ for all $i$, then $a$ is in some $O \in \mathcal{O}$, which implies $B(a,\varepsilon) \subset O$ for some $\varepsilon > 0$.

For the $\varepsilon > 0$, there is $m$ such that $\operatorname{diam}(C_{n_m}) < 1/n_m < \varepsilon$, this implies $C_{n_m} \subset B(x_{n_m}, \varepsilon) = B(a, \varepsilon) \subset O$. Thus, in either case, we reach a contradiction to the assumption. $\square$

It is to be noted that, if we take a collection $\mathcal{O}$ of open sets in a topological space $X$ that covers a subset $S$ in $X$ such that each point in the closure of $S$ has a neighborhood that intersects at most finitely many members of $\mathcal{O}$; and call the collection $\mathcal{O}$ as *$\bar{S}$-locally finite open cover* of $S$, then Theorem 3.1 holds for such open cover $\mathcal{O}$ of $S$ as well.

So if $S$ is totally bounded subset in a complete metric space $X$, then, for each $\bar{S}$-locally finite open cover $\mathcal{O}$ of $S$ there is a number $\alpha > 0$ such that each subset $B \subset S$ with $\operatorname{diam}(B) < \alpha$ has a finite subcover from $\mathcal{O}$.

**Theorem 3.2.** *Let $X$ be a complete metric space. If a subset $S$ in $X$ has an Atsuji completion, then, for each locally finite open cover $\mathcal{O}$ of $S$, there is an $\alpha > 0$ such that each subset $B \subset S$ with* $\operatorname{diam}(B) < \alpha$ *has a finite subcover from $\mathcal{O}$.*

*Proof.* Since $\mathcal{O}$ is a locally finite open cover of $S$, for each $x$ in the closure $\overline{S}$ of $S$, there is a neighborhood $U_x$ of $x$ that intersects finitely many members of $\mathcal{O}$. The collection $\mathcal{U} = \{U_x \mid x \in \overline{S}\}$ is an open cover of $\overline{S}$. By the hypothesis, $\overline{S}$ is an Atsuji space, and therefore $\mathcal{U}$ has a Lebesgue number, say $\alpha > 0$, for $\overline{S}$.

This implies that for each subset $B \subset S$ with $\operatorname{diam}(B) < \alpha$ is contained in some $U \in \mathcal{U}$. Since $U$ intersects only finitely many members of $\mathcal{O}$, the set $B$ is covered by finitely many members of $\mathcal{O}$. $\square$

In [7, Proposition 3.3], it is pointed out that in a finitely chainable metric space $(X, d)$ the set $\{x \in X \mid I(x) > \varepsilon\}$ is finite for all $\varepsilon > 0$, or equivalently, $I(x_n)$ converges to 0 for every sequence $\{x_n\} \subset X$ with distinct terms, where $I(x) = d(x, X \backslash \{x\})$. It is true for finitely chainable subsets in a metric space as well.

That is, if $S$ is a finitely chainable subset in a metric space $X$, then for all $\varepsilon > 0$ we have the set $T_\varepsilon = \{x \in S \mid I(x) > \varepsilon\}$ is finite. Indeed, suppose that the set $T_{\varepsilon'}$ is infinite for some $\varepsilon' > 0$. Consider a sequence $\{p_n\} \subset T_{\varepsilon'}$ of distinct terms. Define a function $f\colon X \to \mathbb{R}$ as follows: $f(x) = n$ if $x = p_n$ for some $n$, and $f(x) = 0$ if $x \neq p_n$ for all $n$. It can be verified that

$$|f(p) - f(q)| \leqslant d(p, q) \quad \text{for all} \quad p, q \in B(x, \varepsilon') \text{ for all } x \in X.$$

Hence, $f$ is a uniformly locally Lipschitz function and unbounded. But, by Theorem 2.1 in [8], $f(S)$ has to be bounded. Hence, we get a contradiction.

Following the technique used in the proof of Theorem 3.1 and using Theorem 3.8 in [6], we get the following:

**Corollary 3.3.** *Let $X$ be an Atsuji space. If $S \subset X$ is finitely chainable, then, for each locally finite open cover $\mathcal{O}$ of $S$ there is a number $\alpha > 0$ such that each subset $B \subset S$ with* $\operatorname{diam}(B) < \alpha$ *has a finite subcover from $\mathcal{O}$.*

Theorem 3.1 may not hold if $X$ is not complete. For instance,

**Example 3.4.** Consider the subspace $X = (0, 2) \subset \mathbb{R}$, and a totally bounded subset $S = \{1/n \mid n \in \mathbb{N}\}$ in $X$. Let $r_n = 1/n - 1/(n+1)$. Then, $\mathcal{O} = \{B(1/n, r_n) \mid n \in \mathbb{N}\}$ is a locally finite open cover of $S$, and for each $\varepsilon > 0$ the subset $\{1/n \mid 1/n < \varepsilon\} \subset S$ has no finite subcover from $\mathcal{O}$.

The converse of Theorem 3.1 is not true. For example,

**Example 3.5.** Consider the Banach space $X = (l_1, \|\cdot\|_1)$, and a subset $S = \{e_m/n \mid m, n \in \mathbb{N}\} \cup \{(0, 0, \ldots)\}$ of it, where $e_m$ denotes the element having the $m$-th element 1 and all others 0. Since $S$ is an Atsuji subset, each covering of $S$ by sets open in $X$ has a Lebesgue number. But, the set $S$ is not totally bounded.

For a metric space $X$, the following are equivalent ([7, Theorem 3.5]):

(i) $X$ is totally bounded;
(ii) each locally finite open cover of $X$ with a Lebesgue number has a finite subcover.

Being more general, we have the following as well.

**Theorem 3.6.** *For a subset $S$ in a metric space $X$, the following conditions are equivalent:*

(i) *$S$ is totally bounded;*
(ii) *each locally finite open cover of $S$ with a Lebesgue number has a finite subcover for $S$.*

*Proof.* (i)⇒(ii) Consider a totally bounded subset $S$, and a locally finite open cover $\mathcal{O}$ of $S$ with a Lebesgue number $\alpha$. By the total boundedness of $S$, for $\alpha/3$, there are finitely many points $x_1, x_2, \ldots, x_n \in S$ such that

$$S \subset \bigcup_{i=1}^{n} B(x_i, \alpha/3).$$

Hence

$$S \subset \bigcup_{i=1}^{n} [S \cap B(x_i, \alpha/3)].$$

Since, $\alpha$ is a Lebesgue number for $\mathcal{O}$, each set $S \cap B(x_i, \alpha/3)$ is contained in some $O \in \mathcal{O}$. Therefore, $\mathcal{O}$ has a finite subcover for $S$.

(ii)⇒(i) To prove the converse part, we shall use the technique used in proving [7, Theorem 3.5]. If possible, we assume $S$ is not totally bounded. Then, there is a sequence $\{x_n\} \subset S$ such that $d(x_m, x_n) > \varepsilon$ for some $\varepsilon > 0$, whenever $m \neq n$. If

$$S \not\subset \bigcup_{n=1}^{\infty} B(x_n, \varepsilon/4),$$

then

$$\mathcal{U} = \{B(x_n, \varepsilon/4) \mid n \in \mathbb{N}\} \cup \{X \backslash E\}$$

is an open cover of $S$, where $E = \{x_n \mid n \in \mathbb{N}\}$. It can be verified that, for all $x \in X$, and for all $\delta \in (0, \varepsilon/4)$, the ball $B(x, \delta)$ intersects at most one ball $B(x_n, \varepsilon/4)$, $n \in \mathbb{N}$. Hence, $\mathcal{U}$ is a locally finite open cover of $S$.

Now we prove that $\varepsilon/16$ is a Lebesgue number for $\mathcal{U}$. Consider a subset $R \subset S$ with $\operatorname{diam}(R) < \varepsilon/16$. Then, for a point $p \in R$, $R \subset B(p, \varepsilon/16)$. If $p \in B(x_n, \varepsilon/8)$ for some $n \in \mathbb{N}$, then $B(p, \varepsilon/16) \subset B(x_n, \varepsilon/4)$. If

$$p \in X \backslash B(x_n, \varepsilon/8) \quad \text{for all} \quad n \in \mathbb{N},$$

then $B(p, \varepsilon/16) \subset X \backslash E$.

Thus, $\mathcal{U}$ is locally finite open cover of $S$ with a Lebesgue number $\varepsilon/16$. But $\mathcal{U}$ has no finite subcover for $S$, which is a contradiction.

On the other side, if $S \subset \bigcup_{n=1}^{\infty} B(x_n, \varepsilon/4)$, then, as before, the collection

$$\mathcal{V} = \{B(x_n, \varepsilon/4) \mid n \in \mathbb{N}\}$$

is a locally finite open cover of $S$. We will prove now that $\varepsilon/2$ is a Lebesgue number for $\mathcal{V}$. If some element $p_1$ of a set $T \subset S$ with $\operatorname{diam}(T) < \varepsilon/2$ is contained in some $B(x_m, \varepsilon/4)$ and some element $p_2$ of $T$ is contained in some $B(x_n, \varepsilon/4)$, where $m \neq n$; then

$$d(x_m, x_n) \leqslant d(x_m, p_1) + d(p_1, p_2) + d(p_2, x_n) < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon.$$

And therefore, the set $T$ is contained in some $B(x_m, \varepsilon/4)$. But the cover $\mathcal{V}$ has no finite subcover for $S$, which is a contradiction. $\square$

So, in general, those locally finite open covers of a totally bounded subset which have the Lebesgue numbers have finite subcovers. A question arises:

*Can those locally finite open covers of a totally bounded subset which do not have any Lebesgue numbers have finite subcovers?*

**Theorem 3.7.** *Let $S$ be a subset in a complete metric space $X$. Then, $S$ is totally bounded if and only if each locally finite open cover of $S$ has a finite subcover for $S$.*

*Proof.* Let $\mathcal{O}$ be a locally finite open cover of $S$. By Theorem 3.1, there is an $\alpha > 0$ such that each subset $B \subset S$ with $\operatorname{diam}(B) < \alpha$ is covered by finitely many members of $\mathcal{O}$. Now, by the total boundedness of $S$, for the number $\alpha/3 > 0$, there are finitely many points $s_1, s_2, \ldots, s_n \in S$ such that

$$S = \bigcup_{i=1}^{n} [B(s_i, \alpha/3) \cap S].$$

And thus, the cover $\mathcal{O}$ has a finite subcover for $S$.

Conversely, for a given $\varepsilon > 0$, consider the open cover

$$\mathcal{U} = \{B(x, \varepsilon/3) \mid x \in X\}$$

of $X$. Since a metric space is paracompact, the cover $\mathcal{U}$ of $X$ has a locally finite open refinement $\mathcal{O}$ that covers $X$. And therefore, by the hypothesis, $\mathcal{O}$ has a finite subcover $\{O_1, O_2, \ldots, O_n\}$ for $S$, for some $n \in \mathbb{N}$.

Thus we eventually get, for each $\varepsilon > 0$ there are finitely many subsets $O_1, O_2, \ldots, O_n$ in $X$ such that $\mathrm{diam}(O_i) < \varepsilon$ for all $i$, and $S \subset \bigcup_{i=1}^{n} O_i$. Hence, $S$ is totally bounded. □

**Corollary 3.8.** *Let $S$ be a nonempty subset in an Atsuji space $X$. Then, $S$ is a finitely chainable subspace if and only if each locally finite open cover of $S$ has a finite subcover for $S$.*

*Proof.* By using [7, Theorem 3.5], we have that given a subset $S$ in an Atsuji space $X$, $S$ is totally bounded if and only if $S$ is finitely chainable subspace. □

Due to the paracompactness of a metric space, we get the following.

**Corollary 3.9.** *If $S$ is a totally bounded subset in a complete metric space $X$, then each open cover of $X$ has a finite subcover for $S$.*

A metrizable topological space $(X, \tau)$ is said to be *strongly metrizable* ([1]) if the union of countably many star finite open covers is a basis for its topology $\tau$. Let the class of strongly metrizable spaces be denoted by $\mathcal{N}$.

**Theorem 3.10.** *Let $X$ be a metric space in $\mathcal{N}$. If each star finite open cover of $X$, consisting of open balls, has a finite subcover, then each open cover of $X$ with a Lebesgue number has a finite subcover.*

*Proof.* Let $\mathcal{O}$ be an open cover of the metric space $(X, d)$ in $\mathcal{N}$, with a Lebesgue number $\alpha$. Since $\alpha$ is the Lebesgue number for the cover $\mathcal{O}$, for each $x \in X$ the open ball $B_d(x, \alpha)$ is contained in some $O \in \mathcal{O}$. Because the metric space $X \in \mathcal{N}$, by [1, Theorem 1.4], there is an admissible metric $d'$ on $X$ such that the collection of open balls $\mathcal{B}_{d'}(\varepsilon)$ is star finite for all $\varepsilon > 0$.

Also, for a fixed $\varepsilon_0 > 0$, for each ball $B_{d'}(x, \varepsilon_0)$, $x \in X$, there is $\delta_x > 0$ such that $B_d(x, \delta_x) \subset B_{d'}(x, \varepsilon_0)$. Let $m_x = \min\{\alpha, \delta_x\}$. Then, the collection $\mathcal{C}$ of open balls $B_d(x, m_x)$, $x \in X$, is a star finite collection and each member of this collection is in some $O \in \mathcal{O}$. By our hypothesis, $\mathcal{C}$ has a finite subcover for $X$, and hence the family $\mathcal{O}$ has a finite subcover for $X$. □

**Corollary 3.11.** *Consider a metric space $X$ in $\mathcal{N}$. If each star finite open cover of $X$, consisting of open balls, has a finite subcover, then $X$ is totally bounded.*

By the characterizations of strongly metrizable space in [1, Theorem 1.4], we consider a subclass $\mathcal{N}'$ of metric spaces $X$ in $\mathcal{N}$ as follows: $\mathcal{N}'$ is a class of metric spaces $X \in \mathcal{N}$ which admits a uniformly equivalent metric $d$ for which the collection $\mathcal{B}_d(\varepsilon)$ is star finite for all $\varepsilon > 0$.

**Theorem 3.12.** *Let $X$ be a metric space in $\mathcal{N}'$. Then, each star finite open cover $\mathcal{B}(\varepsilon), \varepsilon > 0,$ of $X$ has a finite subcover if and only if every open cover of $X$ with a Lebesgue number has a finite subcover.*

*Proof.* Let $(X, d)$ be a metric space in $\mathcal{N}'$ and $\mathcal{O}$ be an open cover of $X$ with a Lebesgue number $\alpha$. Then, each open ball $B_d(x, \alpha)$, $x \in X$, is contained in some $O \in \mathcal{O}$. Since $X$ is in $\mathcal{N}'$, there is a uniformly equivalent metric $d'$ on $X$ such that the family $\mathcal{B}_{d'}(\varepsilon)$ is star finite for all $\varepsilon > 0$. Because the metric $d'$ is uniformly equivalent to $d$, there are $\beta_1$, $\beta_2 > 0$ such that

$$B_d(x, \beta_1) \subset B_{d'}(x, \beta_2) \subset B_d(x, \alpha) \text{ for all } x \in X.$$

We notice that the collection $\mathcal{B}_d(\beta_1)$ is star finite. By our hypothesis, the star finite open cover $\mathcal{B}_d(\beta_1)$ has a finite subcover, and therefore the open cover $\mathcal{O}$ has a finite subcover.

The proof of converse part follows from the fact that, for all $\varepsilon > 0$ the open cover $\mathcal{B}_d(\varepsilon)$ has a Lebesgue number. □

The finite chainability for a metric space, which is a generalization of total boundedness, has been characterized by several statements in [7, Theorem 2.15]. A metric space $X$ is *finitely chainable* if and only if every star finite open cover of $X$ with a Lebesgue number has a finite subcover.

**Corollary 3.13.** *A metric space $X \in \mathcal{N}'$ is finitely chainable if and only if each star finite open cover $\mathcal{B}(\varepsilon), \varepsilon > 0,$ of $X$ has a finite subcover.*

We introduce now a property of a metric space $X$ which generalizes the connectedness as well as the Menger convexity of $X$. We will study a relation between Lebesgue number and total boundedness in the spaces which are more general than connected spaces. First let us look at some definitions and existing results.

A metric space $X$ is said to be *chainable* if for each $\varepsilon > 0$ each pair of points $x$, $y$ can be joined by an $\varepsilon$-chain. All connected metric spaces are chainable.

A metric space $X$ is said to be *metrically convex* if for any two distinct points $x, y \in X$ there is a $z \in X \backslash \{x, y\}$ such that $d(x, y) = d(x, z) + d(z, y)$.

A metric space $X$ is said to be *Menger convex* ([4]) if for each $x, y \in X$ and each $0 \leqslant r \leqslant d(x, y)$, the set

$$B[x, r] \cap B[y, d(x, y) - r] \neq \varnothing.$$

Clearly, a Menger convex metric space is always metrically convex; but the converse need not be true, for example the set of rationals $\mathbb{Q}$ is metrically convex but not Menger convex.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to have the *intermediate value property* (IVP) (see [5, p. 83]) if, whenever $x < y$ and $f(x) \neq f(y)$, then, for all $q$ intermediate to $f(x)$ and $f(y)$, there is $p \in (x, y)$ such that $f(p) = q$.

We now consider the following statement. A map $f\colon \mathbb{R} \to \mathbb{R}$ has IVP$'$ property if $\forall a \in \mathbb{R}$ and $\forall r$ with

$$f(a) < r < \sup_{x \in \mathbb{R}} f(x),$$

there exists $b \in \mathbb{R}$ such that $f(b) = r$. It may be noted that, IVP implies IVP$'$, but the converse is not true. We can replace the domain of $f$ in IVP$'$ with a metric space $X$, and call it the generalized intermediate value property (GIVP), that is, a function $f\colon X \to \mathbb{R}$ has the GIVP if, $\forall a \in X$ and $\forall r$ with

$$f(a) < r < \sup_{x \in X} f(x),$$

there is $b \in X$ such that $f(b) = r$. Using the GIVP, we introduce a property for a metric space, as follows.

**Definition 3.14.** A metric space $(X, d)$ is said to have the *property* $P$ if for all $x \in X$, the map $d(x, \cdot)\colon X \to \mathbb{R}$ satisfies the GIVP, i.e., for all $x \in X$ and for all $r$ with

$$0 \leqslant r < \sup\{d(x, y) \mid y \in X\},$$

there is a $z \in X$ such that $d(x, z) = r$.

Let the class of metric spaces with the property $P$ be denoted by $\mathcal{P}$. Then, $\mathcal{P}$ contains:

(i) **All connected metric spaces**. Given a pair of distinct points $x, y$ in a connected metric space $X$ and $0 < r < d(x, y)$, the set

$$\{z \in X \mid d(x, z) = r\}$$

can not be empty, otherwise the open ball $B(x, r)$ would be closed too. Singleton spaces obviously have the property $P$.

(ii) **Complete metrically convex spaces**, due to Menger's Theorem [9].

(iii) **Menger convex spaces**. Given a pair of points $x, y$ in a metric space $X$ and $0 \leqslant r \leqslant d(x, y)$,

$$B[x, r] \cap B[y, d(x, y) - r] = S[x, r] \cap S[y, d(x, y) - r],$$

where

$$S[x,r] = \{z \in X \mid d(x,z) = r\}.$$

We note that a subspace in $\mathbb{R}^2$ defined as the union of two distinct parallel lines has the property $P$ but it is neither chainable nor metrically convex. The subspace $\mathbb{Q}$ of $\mathbb{R}$ is metrically convex and chainable but it does not have the property $P$.

**Lemma 3.15.** *Consider a pair of distinct points $x$, $y$ in a chainable metric space $X$, and $0 < \alpha < \beta < d(x,y)$, $\varepsilon = \beta - \alpha$. Then, in each $\varepsilon$-chain between $x$ and $y$, there is a point $p$ such that $\alpha \leqslant d(x,p) < \beta$.*

*Proof.* Consider an $\varepsilon$-chain $C$ between $x$ and $y$. If possible, suppose, there is no point $p$ in $C$ with $\alpha \leqslant d(x,p) < \beta$, that is, $\forall p \in C$ we have $d(x,p) < \alpha$ or $d(x,p) \geqslant \beta$. And therefore,

$$C = (C \cap B(x,\alpha)) \cup (C \cap [X \backslash B(x,\beta)]).$$

Choose arbitrary points $c' \in C \cap B(x,\alpha)$ and $c'' \in C \cap [X \backslash B(x,\beta)]$. Then,

$$d(c',c'') \geqslant d(c'',x) - d(c',x) > \beta - \alpha = \varepsilon,$$

which implies $C$ is not an $\varepsilon$-chain. Hence, the proof follows. □

M. H. A. Newman, in [11, Theorem 5.1, p. 81], proved that if each open cover of a chainable metric space has a finite subcover, then it is connected. By considering more general hypotheses, in the following result, we show that the space is totally bounded.

**Theorem 3.16.** *Let $X$ be a chainable metric space. If every countably infinite locally finite open cover of $X$ has a Lebesgue number, then $X$ is totally bounded.*

*Proof.* If possible, suppose $(X,d)$ is not totally bounded. Then, there is a sequence $\{x_n\}$ for which there is some $\varepsilon > 0$ such that $d(x_m,x_n) > \varepsilon$ for all $m \neq n$.

Consider an open cover

$$\mathcal{O} = \{B(x_n, \varepsilon/4n) \mid n \in \mathbb{N}\} \cup \{X \backslash S\}$$

of $X$, where $S = \{x_n \mid n \in \mathbb{N}\}$. For each $x \in X$, the ball $B(x,\delta)$ intersects at most two members of $\mathcal{O}$, where $0 < \delta < \varepsilon/4$. Hence, $\mathcal{O}$ is locally finite.

Now, we prove $\mathcal{O}$ has no Lebesgue number. Let $0 < \gamma < \varepsilon$. Then, there is $s$ with $0 < s < \gamma$, and there is some $p \in \mathbb{N}$ such that $\varepsilon/4p < s$. Let $t = \gamma - s$. Then, since $X$ is chainable, by Lemma 3.15, each $t$-chain between $x_p$ and $x_m$, $p \neq m$, has a point $c$ with $s \leqslant d(x_p, c) < \gamma$.

Therefore, there is no element in $\mathcal{O}$ which can contain the open ball $B(x_p, \gamma)$. Hence, the proof. □

One of the main results in [2] by G. Beer is: *given a chainable space $X$, $X$ is compact if and only if it is uniformly locally compact and uniformly chainable.* Using this result we get the following

**Corollary 3.17.** *For a chainable metric space $X$, the following are equivalent:*

(i) *$X$ is Atsuji;*
(ii) *$X$ is compact;*
(iii) *$X$ is uniformly locally compact and uniformly chainable.*

By using the technique used in the proof of Theorem 3.16, we get the following.

**Theorem 3.18.** *Let $X$ be a metric space with the property $P$. If every countably infinite locally finite open cover of $X$ has a Lebesgue number, then $X$ is totally bounded.*

**Corollary 3.19.** *Let $X$ be a metric space with the property $P$. Then $X$ is Atsuji if and only if it is compact.*

We notice that, in a metric spaces $(X, d)$ that is either chainable or with the property $P$, the set $\{x \in X \mid d(x, X\backslash\{x\}) > 0\}$ is empty. And therefore [7, Theorem 3.2] also concludes Corollaries 3.17 and 3.19, that is, for a metric space that is either chainable or with the property $P$, Atsujiness and compactness are equivalent.

The converses of Theorems 3.16, 3.18 are not true. For instance:

**Example 3.20.** Consider the subspace $X = (0, 1] \subset \mathbb{R}$ which is connected. The space $X$ is totally bounded. Let $r_n = 1/n - 1/(n+1), n \in \mathbb{N}$. Then, the locally finite open cover $\mathcal{O} = \{B(1/n, r_n) \subset X \mid n \in \mathbb{N}\}$ of $X$ has no Lebesgue number.

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A. K. Gupta
Department of Applied Science and Humanities
Sankalchand Patel University, Visnagar 384315, Gujarat
India
*Email:* emailstoajit@gmail.com; ajitgupta.hm@spcevng.ac.in
*ORCID:* 0000-0003-2042-0558

S. Mukherjee
Department of Mathematics
National Institute of Technology Meghalaya, Shillong 793003, Meghalaya
India
*Email:* saikat.mukherjee@nitm.ac.in
*ORCID:* 0000-0002-3943-9235