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On geodesic mappings of threesymmetric spaces

V. Kiosak, O. Prishlyak, O. Gudyreva

Abstract. The paper is devoted to the study of properties of pseudo-Riemannian spaces admitting nontrivial geodesic mappings. Necessary and sufficient conditions are found for A-threesymmetric spaces to admit nontrivial geodesic mappings. The research is carried out locally, in tensor form without restrictions to the sign of the metric tensor and the signature of the space.

Анотація. Робота присвячена вивченню властивостей псевдоріманових просторів, які допускають нетривіальні геодезичні відображення. Знайдено необхідні і достатні умови того, щоб A-трисиметричні простори дозволяли нетрівіальні геодезичні відображення. Дослідження ведуться локально, в тензорній формі без обмежень на знак метричного тензора та сигнатуру простору.

Introduction

The symmetric spaces introduced by E. Cartan have always aroused scientific interest among geometers and physicists [1]. In the theory of geodesic mappings, the main results for these spaces were obtained by M. S. Sinyukov [13]. Later, it became clear that the question of covariant stability of not only the internal objects of pseudo-Riemannian spaces, but also of arbitrary tensors is of interest [3–5].

In particular, to study the possibility of reducing the metric tensor to a special form [2]. Following the way of increasing the number of derivatives, M. S. Sinyukov came to the study of geodesic mappings of semisymmetric spaces. More general results for arbitrary tensors with the property of semisymmetry were obtained by J. Mikesh [11,12]. A series of papers on geodesic mappings of generalized symmetric spaces belongs to V. S. Sobchuk [14–17].

This paper is devoted to the study of geodesic mappings of pseudo-Riemannian spaces containing a field of a bivalent tensor whose third covariant derivative is zero.

Keywords: symmetric spaces, A-threesymmetric spaces, geodesic mappings
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1. Threesymmetric spaces

A pseudo-Riemannian space V_n with a metric tensor g_{ij} is said to be *A-threesymmetric* if there exists a tensor A_{ij} for which the following conditions hold

$$A_{ij,klm} = 0, \qquad A_{ij,kl} \neq 0, \qquad A_{ij} \neq c g_{ij}, \\ g^{\alpha\beta} A_{\alpha\beta,i} \neq 0, \qquad g^{\alpha\beta} A_{\alpha i,\beta} \neq 0, \tag{1.1}$$

where, "," is the sign of the covariant derivative on the connection V_n and c is some constant.

A covariant derivative is defined by the formula

$$A_{ij,k} = \partial_k A_{ij} - A_{\alpha j}\Gamma^{\alpha}_{ik} - A_{i\alpha}\Gamma^{\alpha}_{jk},$$

where Γ^h_{ij} are the coefficients of connection V_n, the Einstein rule of summation applies to the repeated indices, and g^{ij} are the elements of the inverse matrix g_{ij} to the metric tensor V_n.

The tensor A_{ij} can be written as the sum of the symmetric and skew-symmetric tensors

$$A_{ij} = S_{ij} + K_{ij}. \tag{1.2}$$

Here S_{ij} is a symmetric and K_{ij} is a skew-symmetric tensor.

By differentiating (1.2) three times and taking into account (1.1), we obtain

$$S_{ij,klm} + K_{ij,klm} = 0. \tag{1.3}$$

Further, symmetrizing and alternating (1.3), we get that

$$S_{ij,klm} = 0 \quad \text{and} \quad K_{ij,klm} = 0.$$

Thus, if there exists a tensor A_{ij} in the space V_n satisfying (1.1), then there exist both a symmetric tensor and a skew-symmetric tensor, whose third covariant derivatives are zero. Of course, the tensor itself can be either skew-symmetric or symmetric.

Therefore, we will assume further that A_{ij} is a symmetric tensor, i.e.

$$A_{ij} = A_{ji}.$$

If the condition (1.1) is satisfied by the Ricci tensor of a pseudo-Riemannian space V_n, then such spaces are called *Ricci threesymmetric*. The Ricci tensor of a pseudo-Riemannian space is defined by the formula

$$R_{ij} = R^{\alpha}_{.ij\alpha}.$$

The tensor $R^h_{.ijk}$ is the Riemann tensor of a pseudo-Riemannian space V_n such that

$$R^h_{.ijk} = \partial_j \Gamma^h_{ik} + \Gamma^{\alpha}_{ik}\Gamma^h_{j\alpha} - \partial_k \Gamma^h_{ij} - \Gamma^{\alpha}_{ij}\Gamma^h_{k\alpha}.$$

The following theorem holds:

Theorem 1.1. *In A-threesymmetric pseudo-Riemannian spaces, the tensor A_{ij} satisfies the following conditions:*

$$A_{\alpha j} R^{\alpha}_{ilk,m} + A_{i\alpha} R^{\alpha}_{jlk,m} + A_{ij,\alpha} R^{\alpha}_{mlk} = 0. \tag{1.4}$$

Proof. For the tensor A_{ij}, the following conditions hold

$$A_{ij,kl} - A_{ij,lk} = A_{\alpha j} R^{\alpha}_{ikl} + A_{i\alpha} R^{\alpha}_{jkl}.$$

Covariantly differentiating the latter and taking into account (1.1), we obtain

$$A_{\alpha j,m} R^{\alpha}_{ikl} + A_{i\alpha,m} R^{\alpha}_{jkl} + A_{\alpha j} R^{\alpha}_{ikl,m} + A_{i\alpha} R^{\alpha}_{jkl,m} = 0. \tag{1.5}$$

Alternating (1.1) by the indices l and m, we get

$$A_{\alpha j,k} R^{\alpha}_{ilm} + A_{i\alpha,k} R^{\alpha}_{jlm} + A_{ij,\alpha} R^{\alpha}_{klm} = 0. \tag{1.6}$$

Redefine now the indices k and m in (1.6). Then (1.6) will be written as follows:

$$A_{\alpha j,m} R^{\alpha}_{ilk} + A_{i\alpha,m} R^{\alpha}_{jlk} + A_{ij,\alpha} R^{\alpha}_{mlk} = 0. \tag{1.7}$$

Adding equations (1.5) and (1.7), and taking into account the properties of the Riemann tensor, we get the desired identity (1.4). □

Pseudo-Riemannian spaces in which the conditions hold

$$R_{ij} = \frac{R}{n} g_{ij} \tag{1.8}$$

are called *Einstein spaces*, where $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature of V_n.

Contracting (1.4) by the indices k and m, we get

$$A_{ij,\alpha} R^{\alpha}_{l} = A_{\alpha j} R^{\alpha}_{il\beta,}{}^{\beta} + A_{\alpha i} R^{\alpha}_{jl\beta,}{}^{\beta}, \tag{1.9}$$

where

$$R^{h}_{ij\beta,}{}^{\beta} = R^{h}_{ij\alpha,\beta} g^{\alpha\beta}, \qquad R^{h}_{i} = g^{\alpha h} R_{\alpha i}.$$

Since for Einstein spaces we have that $R^{\alpha}_{ijk,\alpha} = 0$, the equation (1.9) reduces to the following one:

$$A_{ij,\alpha} R^{\alpha}_{l} = 0. \tag{1.10}$$

Substituting the equation (1.10) into (1.8), we see that the following corollary of Theorem 1.1 holds:

Corollary 1.2. *In A-threesymmetric Einstein spaces, the scalar curvature is zero.*

2. Geodesic mappings

A bijection between points of pseudo-Riemannian spaces V_n with metric tensor g_{ij} and $\bar{V}_n$ with metric tensor $\bar{g}_{ij}$, in which each geodesic line V_n is associated with a geodesic line $\bar{V}_n$ is called a *geodesic mapping*.

A geodesic mapping other than homothety is called non-trivial.

A necessary and sufficient condition for the existence of nontrivial geodesic mappings is the existence of nontrivial solutions of the linear form of the basic equations of the theory of geodesic mappings, with respect to the tensor $a_{ij} = a_{ji} \neq cg_{ij}$ and the gradient vector $\lambda_i \neq 0$.

The linear form of the basic equations of the theory of geodesic mappings is given by the following identity, see [13, p.121]:

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik}. \tag{2.1}$$

Also, the differential extensions of the linear forms of the basic equations of the theory of geodesic mappings are given by, [13]:

$$n\lambda_{i,j} = \mu g_{ij} + a_{\alpha i} R^{\alpha}_{j} - a_{\alpha\beta} R^{\alpha}_{.ij.}{}^{\beta}, \tag{2.2}$$

where $\mu = \lambda_{\alpha,\beta} g^{\alpha\beta}$, $R^i_j = R_{\alpha j} g^{\alpha i}$, $R^{h}{}_{ij}{}^{k} = R^h_{ij\alpha} g^{\alpha k}$, and g^{ij} are elements of the inverse matrix to g_{ij}.

From the latter we have:

$$(n-1)\mu_{,i} = 2(n+1)\lambda_\alpha R^{\alpha}_{i} + a_{\alpha\beta}(2R^{\alpha}_{.i,.}{}^{\beta} - R^{\alpha\beta}{}_{,i}), \tag{2.3}$$

where

$$R^{i}_{.j,.}{}^{k} = R_{\alpha j,\beta} g^{\alpha i} g^{\beta k}, \qquad R^{ij}{}_{,k} = R_{\alpha\beta,k} g^{\alpha i} g^{\alpha j}$$

respectively.

The system of equations (2.1), (2.2) and (2.3) gives a basic possibility to answer the question: whether a given pseudo-Riemannian space V_n admits a geodesic mapping to the pseudo-Riemannian space $\bar{V}_n$. The question is reduced to studying the conditions of integration of these equations and their differential extensions. The main achievements in the theory of geodesic mappings of pseudo-Riemannian spaces are related with the usage of the linear form of the basic equations [6–9].

Consider geodesic mappings of pseudo-Riemannian spaces $V_n (n > 2)$, in which the following condition holds

$$\varphi_\alpha R^{\alpha}_{ijk} = 0, \tag{2.4}$$

where $\varphi_i \neq 0$ is a vector.

The integrability conditions for equation (2.1) are given by

$$a_{\alpha i} R^{\alpha}_{jkl} + a_{\alpha j} R^{\alpha}_{ikl} = \lambda_{li} g_{jk} + \lambda_{lj} g_{ik} - \lambda_{ki} g_{jl} - \lambda_{kj} g_{il}, \tag{2.5}$$

where $\lambda_{ij} = \lambda_{i,j}$.

Multiply (2.5) by the vector φ^l such that

$$\varphi^l = \varphi_\alpha g^{\alpha l},$$

and wrap it by the index l. Taking into account (2.4), we obtain that

$$\varphi^\alpha \lambda_{li} g_{jk} + \varphi^\alpha \lambda_{lj} g_{ik} - \lambda_{ki}\varphi_j - \lambda_{kj}\varphi_i = 0. \tag{2.6}$$

Let us wrap it by indices j and k

$$n\varphi^\alpha \lambda_{\alpha i} - \mu\varphi_i = 0. \tag{2.7}$$

Then after substituting equation (2.7) into equation (2.6) we will get that

$$\varphi_i\Big(\lambda_{jk} - \frac{\mu}{n} g_{jk}\Big) + \varphi_j\Big(\lambda_{ik} - \frac{\mu}{n} g_{ik}\Big) = 0. \tag{2.8}$$

Alternate further (2.8) by indices j and k:

$$\varphi_j\Big(\lambda_{ik} - \frac{\mu}{n} g_{ik}\Big) - \varphi_k\Big(\lambda_{ij} - \frac{\mu}{n} g_{ij}\Big) = 0.$$

By redefining indices i and k respectively we will get

$$\varphi_j\Big(\lambda_{ki} - \frac{\mu}{n} g_{ki}\Big) - \varphi_i\Big(\lambda_{kj} - \frac{\mu}{n} g_{kj}\Big) = 0. \tag{2.9}$$

Summing up equations (2.9) and (2.8), we obtain the following equation:

$$\varphi_j\Big(\lambda_{ki} - \frac{\mu}{n} g_{ki}\Big) = 0.$$

Now, since $\varphi_j \neq 0$, we see that

$$\lambda_{ki} - \rho g_{ki} = 0, \tag{2.10}$$

where $\rho = \dfrac{\mu}{n}$.

Integrability conditions for (2.10) can be written as follows:

$$\lambda_\alpha R^\alpha_{ijk} = \rho_k g_{ij} - \rho_j g_{ik}. \tag{2.11}$$

Multiplying equation (2.11) by λ^i and wrapping it by index i we obtain

$$\rho_k \lambda_j - \rho_j \lambda_k = 0. \tag{2.12}$$

Since $\lambda_j \neq 0$, we can choose a vector ξ^j such that $\lambda_\alpha \xi^\alpha = 1$, and multiply equation (2.12) by the vector ξ^j. Wrapping further it by the index j we get

$$\rho_k = B\lambda_k. \tag{2.13}$$

Equation (2.13) can further be written as follows:

$$\lambda_\alpha R^\alpha_{ijk} = B\big(\lambda_k g_{ij} - \lambda_j g_{ik}\big). \tag{2.14}$$

Let us multiply equation (2.14) by the vector φ^k and wrap it by the index k. Taking into account equation (2.4), we then obtain that

$$B\big(\varphi^\alpha \lambda_\alpha g_{ij} - \lambda_j \varphi_i\big) = 0.$$

If $\varphi^\alpha \lambda_\alpha g_{ij} - \lambda_j \varphi_i = 0$, then $\varphi^\alpha \lambda_\alpha = 0$ and $\lambda_j \varphi_i = 0$ respectively.

However, this is impossible. Therefore, $B = 0$ and the equation (2.13) reduces to teh following one:

$$\rho_k = 0. \tag{2.15}$$

Thus, the following theorem holds

Theorem 2.1. *If the pseudo-Riemannian space $V_n (n > 2)$, in which the condition (2.4) holds true, admits non-trivial geodesic mappings, then the system of equations (2.1), (2.10), (2.15) has a solution in this space with respect to the tensor a_{ij}, the vector λ_i, and the invariant ρ.*

Consider two vectors v_i and w_i such that

$$v_i = g^{\alpha\beta} A_{\alpha\beta,i}, \tag{2.16}$$

$$w_i = g^{\alpha\beta} A_{\alpha i,\beta}, \tag{2.17}$$

where the tensor A_{ij} satisfies the condition (1.1). Differentiating (2.16) and (2.17) twice, we see that

$$v_{i,jk} = w_{i,jk} = 0.$$

Alternating further it, we get the following identity:

$$w_\alpha R^\alpha_{ijk} = w_\alpha R^\alpha_{ijk} = 0.$$

Therefore, we can conclude from Theorem 2.1:

Corollary 2.2. *If the A-symmetric pseudo-Riemannian space admits non-trivial geodesic mappings, then*

- *either (2.1), (2.10), (2.15) hold true for the tensor a_{ij}, the vector λ_i, and the invariant ρ from the linear form of the basic equations,*
- *or*

$$g^{\alpha\beta} A_{\alpha\beta,i} = g^{\alpha\beta} A_{\alpha i,\beta} = 0.$$

In particular, if the Ricci tensor satisfies the condition (1.1), then if the pseudo-Riemannian space has a non-constant scalar curvature

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

Therefore the conditions (2.1), (2.10), and (2.15) hold true, whenever it admits nontrivial geodesic mappings.

Differentiating (2.1) and substituting (2.10) we obtain

$$a_{ij,kl} = \rho(g_{li} g_{jk} + g_{lj} g_{ik}).$$

Taking further into account (2.15), we get

$$a_{ij,klm} = 0.$$

This means that the tensor a_{ij} from the linear form of the basic equations has a zero third covariant derivative.

Thus, we have the following:

Theorem 2.3. *An A-threesymmetric pseudo-Riemannian space admits non-trivial geodesic mappings, if and only if there exists a solution of the system of equations* (2.1), (2.10), (2.15) *with respect to the tensor* a_{ij}*, the vector* λ_i*, and the constant* ρ.

A pseudo-Riemannian space V_n with a metric tensor g_{ij} is called *equidistant* if there exists a vector field $\psi_i \neq 0$ satisfying equation

$$\psi_{i,j} = \tau g_{ij},$$

where τ is some invariant and the comma "," is the sign of the covariant derivative in V_n. When $\tau \neq 0$, this is the equidistant space of the main case, and when $\tau = 0$, the equidistant space of the special case [10].

Thus, we see that A-threesymmetric spaces satisfying (2.10) are equidistant.

If the vector ψ_i is isotropic, i.e.

$$\psi_\alpha \psi_\beta g^{\alpha\beta} = 0,$$

then the equidistant space belongs, by necessity, to the special type. Equidistant spaces of basic type are characterized by the fact that they have a special coordinate system, in which the metric tensor can be written in the form

$$ds_n^2 = d(x^1)^2 + f(x^1) ds_{n-1}^2(x^2, \dots, x^n),$$

where $f(x^1) \neq 0$ is some function and ds_{n-1}^2 is a metric of an $(n-1)$-dimensional pseudo-Riemannian space.

A-threesymmetric spaces will be of the basic type due to the restriction $A_{ij,kl} \neq 0$ on the tensor a_{ij}. These spaces always admit nontrivial geodesic mappings. The tensor a_{ij} can be constructed as follows

$$a_{ij} = c g_{ij} + \psi_i \psi_j.$$

Thus, by choosing the function $f(x^1)$ appropriately, we obtain classes of A-threesymmetric spaces admitting non-trivial geodesic mappings.

3. Conclusion

The requirement that in a pseudo-Riemannian space there exists a field of a bivalent tensor whose third derivative is zero turned out to be a rather strict condition. In particular, for Einstein spaces the A-threesymmetry condition leads to Ricci flat spaces.

However, such spaces do exist and certain classes of them admit non-trivial geodesic mappings.

Since in this paper we have found a form of a linear system of basic equations of geodesic mappings for these spaces, it will be possible further to classify them completely.

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V. Kiosak
Odesa State Academy of Civil Engineering and Architecture, Didrihson st., 4, Odesa, 65029, Ukraine
Email: kiosakv@ukr.net
ORCID: 0000-0002-7433-6709

O. Prishlyak
Taras Shevchenko National University of Kyiv, 4th Academician Glushkov avenue, Kyiv, 03127, Ukraine
Email: prishlyak@yahoo.com
ORCID: 0000-0002-7164-807X

O. Gudyreva
Kherson State Maritime Academy, Ushakova av., 20, Kherson, 73000, Ukraine
Email: gudyreva@ukr.net
ORCID: 0000-0003-1841-8555