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# A certain method of construction of Thiele-Hermite continued fraction at a point

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**Abstract.** The problem of interpolation of the function of a complex variable at a point of a compact set by the Thiele-Hermite continued fraction is investigated. Formulas for calculating the coefficients of the continued fraction based on values of the function and its derivates at a point are obtained. Several examples of computations are provided.

**Анотація.** Досліджено задачу інтерполяції функції комплексної змінної в точці компакту ланцюговим дробом Тіле-Ерміта. Отримано формули для обчислення коефіцієнтів ланцюгового дробу за значеннями функції та її похідних в точці. Наведено приклади.

## 1. Introduction. The main result

The article is related to research conducted by the authors in [3]. An infinite continued fraction of the form

$$D = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}}$$

will be briefly written as follows

$$D = b_0 + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots .$$

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*Ключові слова*: Ланцюговий дріб, раціональна інтерполяція Ерміта-Тіле, рекурентні формули

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Similarly, a finite continued fraction, analogous to the partial sum of series

$$D_n = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}}, \qquad D_0 = b_0,$$

and called the *n-th approximant* of the continued fraction $D$, will be written as follows

$$D_n = b_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1} \frac{}{+} \frac{a_2}{b_2} \frac{}{+} \cdots \frac{}{+} \frac{a_n}{b_n}.$$

We will consider the following problem:

*Let $f$ be a function defined on a compact set $\mathcal{Z} \subset \mathbb{C}$ and $z_0 \in \mathcal{Z}$ be a point. It is necessary to find the coefficients of the continued fraction*

$$D_n(z) = \frac{P_n(z)}{Q_n(z)} = b_0 + \mathop{\mathrm{K}}_{k=1}^{n} \frac{z - z_0}{b_k}, \qquad b_k \in \mathbb{C}\backslash\{0\}, \quad k = \overline{0, n}, \tag{1.1}$$

*so that the following conditions are satisfied at the point $z_0$*

$$D_n(z_0) = f(z_0), \qquad D_n^{(m)}(z_0) = \left(\frac{P_n(z)}{Q_n(z)}\right)^{(m)}\Bigg|_{z=z_0} = w_m, \tag{1.2}$$

*where $w_m = f^{(m)}(z_0)$, $m = \overline{1, n}$.*

A continued fraction constructed in this way will correspond to the formal power series of the function expansion at the point $z_0$. The coefficients of the continued fraction (1.1) are derived from the interpolation conditions (1.2). We will call (1.1) the *Thiele-Hermite continued fraction* (THCF).

Two methods of finding coefficients $b_k$, $k = \overline{0, n}$, of the THCF via the values of the function and its derivatives at the point $z_0$ were proposed in [3]. One of them involves calculation of $m$-multiple sums. These sums are composed of currently known coefficients of THCF. According to that approach, the non-zero coefficients of THCF (1.1) are computed via the following recurrence formulas:

$$b_0 = w_0, \quad b_1 = \frac{1}{w_1}, \quad b_2 = \frac{-2w_1}{w_2 b_1}, \quad b_k = \frac{-\sum\limits_{i=1}^{[k/2]} (k)_i w_{k-i} \mathbf{B}_1^{[k-2,i-1]}}{w_k \mathbf{B}_1^{[k-1,0]} + \sum\limits_{i=1}^{[k/2]} (k)_i w_{k-i} \mathbf{B}_1^{[k,i]}},$$

where $k = \overline{3, n}$,

$$\mathbf{B}_1^{[p,l]} = \sum_{i_1=1}^{p+1-2l} B^1_{i_1-1} \sum_{i_2=i_1+2}^{p+3-2l} B^{i_1+2}_{i_2-1} \cdots \sum_{i_{l-1}=i_{l-2}+2}^{p-3} B^{i_{l-2}+2}_{i_{l-1}-1} \sum_{i_l=i_{l-1}+2}^{p-1} B^{i_{l-1}+2}_{i_l-1} B^{i_l+2}_p,$$

$$B^l_p = \prod_{j=l}^{p} b_j, \qquad B^l_{l-1} = 1, \tag{1.3}$$

and $(k)_i = k(k-1)(k-2)\cdots(k-i+1)$ is the Pochhammer symbol.

The second method of finding coefficients of THCF is related to Thiele's approximation formula [4, 8], and is an analogue of Taylor's formula in the theory of continued fraction.

Let

$$H_0^{(m)}(z_0) = 1, \qquad H_k^{(m)}(z_0) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2} \end{vmatrix} \neq 0,$$

$$c_m = \begin{cases} \frac{f^{(m)}(z_0)}{m!}, & \text{if } m \geqslant 0, \\ 0, & \text{if } m < 0, \end{cases}$$

be the Hankel determinants. Then the coefficients of THCF are determined as follows [4, 5]:

$$b_0(z_0) = c_0, \qquad b_1(z_0) = 1/c_1, \qquad b_{2k}(z_0) = \frac{-(H_k^{(1)}(z_0))^2}{H_k^{(2)}(z_0)\, H_{k-1}^{(2)}(z_0)},$$

$$b_{2k+1}(z_0) = \frac{(H_k^{(2)}(z_0))^2}{H_k^{(1)}(z_0)\, H_{k+1}^{(1)}(z_0)}, \qquad k = \overline{1, [n/2]}.$$

Another method of finding coefficients of THCF is substantiate in this article. The main result of the paper is the following

**Theorem 1.1.** *Suppose that* $f^{(k)}(z_0) \neq 0$, $k = \overline{0, n}$, *at the point* $z_0 \in \mathcal{Z}$. *Then the non-zero coefficients of THCF* (1.1) *can be computed via the following recurrent formulas:*

$$b_0 = w_0, \qquad b_1 = \frac{1}{w_1}, \qquad b_2 = \frac{-1}{b_1^2 \frac{w_2}{2!}}, \qquad b_3 = \frac{1}{b_1^2 b_2^2 \left(\frac{w_3}{3!} - \frac{1}{b_1^3 b_2^2}\right)}, \tag{1.4}$$

$$b_k = 1 \Big/ \prod_{j=1}^{k-1} b_j^2 \Big( \frac{(-1)^{k-1} w_k}{k!} - \Big( \frac{1}{b_1^3 b_2^2} \sum_{i_3=1}^{2} \frac{1}{b_{i_3} b_{i_3+1}} \sum_{i_4=1}^{i_3+1} \frac{1}{b_{i_4} b_{i_4+1}} \cdots$$

$$\cdots \sum_{i_{k-1}=1}^{i_{k-2}} \frac{1}{b_{i_{k-1}} b_{i_{k-1}+1}} + \frac{1}{b_1^2 b_2^2 b_3} \sum_{i_3=1}^{2} \frac{1}{b_{i_3} b_{i_3+1}} \sum_{i_4=1}^{i_3+1} \frac{1}{b_{i_4} b_{i_4+1}} \cdots$$

$$\cdots \sum_{i_{k-1}=1}^{i_{k-2}} \frac{1}{b_{i_{k-1}} b_{i_{k-1}+1}} + \frac{1}{b_4 \prod\limits_{j=1}^{3} b_j^2} \sum_{i_4=1}^{3} \frac{1}{b_{i_4} b_{i_4+1}} \sum_{i_5=1}^{i_4+1} \frac{1}{b_{i_5} b_{i_5+1}} \cdots$$

$$\cdots \sum_{i_{k-1}=1}^{i_{k-2}} \frac{1}{b_{i_{k-1}} b_{i_{k-1}+1}} + \cdots + \frac{1}{b_{k-1} \prod\limits_{j=1}^{k-2} b_j^2} \sum_{i_{k-1}=1}^{k-2} \frac{1}{b_{i_{k-1}} b_{i_{k-1}+1}} \Big)\Big), \tag{1.5}$$

*where* $k = \overline{4, n}$.

The article has the following structure. The second section contains the Euler-Minding formula for canonical polynomials of the numerator $P_n(z)$ and the denominator $Q_n(z)$ of the continued fraction (1.1). A statement analogous to Leibniz's formula for finding the derivative of the $m$-th order by the ratio of two differentiated functions is given in this section as well. The proof of Theorem 1.1 is contained in the third section. Examples of usage of the formulas (1.4)-(1.5) for finding the coefficients of THCF of some functions are given in the fourth section.

## 2. Euler-Minding formula. An analogue of Leibniz's formula for the $m$-th order derivative of the ratio of two functions

Let $\{b_i \colon b_i \neq 0, i = \overline{0, n}\}$ be the set of coefficients of THCF. Consider $p$-multiple sums of the form

$$E_p^{l,n} = \sum_{i_1=l}^{n+1-2p} \frac{1}{b_{i_1} b_{i_1+1}} \sum_{i_2=i_1+2}^{n+3-2p} \frac{1}{b_{i_2} b_{i_2+1}} \cdots \sum_{i_p=i_{p-1}+2}^{n-1} \frac{1}{b_{i_p} b_{i_p+1}}, \tag{2.1}$$

where $1 \leqslant p \leqslant [(n+1-l)/2]$, $l \geqslant 0$. We assume that $E_0^{l,n} = 1$ and $E_p^{l,n} = 0$, when $l > n+1-2p$.

Notice that $E_p^{l,n}$ satisfies the following recurrence relation:

$$E_p^{l,n} = \sum_{i=l}^{n+1-2p} \frac{1}{b_i b_{i+1}} E_{p-1}^{i+2,n}.$$

It is easy to see that formulas

$$E_p^{l,n} = \sum_{i=l}^{s} \frac{1}{b_i b_{i+1}} E_{p-1}^{i+2,n} + E_p^{s+1,n}, \quad s \geqslant l, \tag{2.2}$$

$$E_p^{l,n} - E_p^{l+k,n} = \sum_{i=l}^{l+k-1} \frac{1}{b_i b_{i+1}} E_{p-1}^{i+2,n} \tag{2.3}$$

follow from (2.1).

The canonical numerator $P_n(z)$ and denominator $Q_n(z)$ of the continued fraction (1.1) are polynomials. These polynomials are determined by the coefficients of continued fraction using the Euler-Minding formula [5, 7]:

$$P_n(z) = B_n^0 \sum_{i=0}^{r_1} E_i^{0,n} (z - z_0)^i, \qquad r_1 = [(n+1)/2], \tag{2.4}$$

$$Q_n(z) = B_n^1 \sum_{i=1}^{r_2} E_i^{1,n} (z - z_0)^i, \qquad r_2 = [n/2], \tag{2.5}$$

where $B_n^0$ and $B_n^1$ are defined in (1.3). It directly follows from (2.4) and (2.5) that:

$$\big(P_n(z_0)\big)^{(m)} = \begin{cases} m! B_n^0 E_m^{0,n}, & \textit{if } m \leqslant \left[\frac{n+1}{2}\right], \\ 0, & \textit{if } m > \left[\frac{n+1}{2}\right], \end{cases} \tag{2.6}$$

$$\big(Q_n(z_0)\big)^{(m)} = \begin{cases} m! B_n^1 E_m^{1,n}, & \textit{if } m \leqslant \left[\frac{n}{2}\right], \\ 0, & \textit{if } m > \left[\frac{n}{2}\right]. \end{cases} \tag{2.7}$$

An analogue of Leibniz's formula for finding the derivative of the $m$-th order of the ratio of two functions is the formula proved by Gerrish [1]. Suppose that $u$ and $v$ are functions in the domain $\mathcal{Z} \subset \mathbb{C}$ differentiable up to the $m$-order inclusively and $v(z) \neq 0$, $z \in \mathcal{Z}$. Then the following formula holds:

$$\left(\frac{u}{v}\right)^{(m)} = \frac{m!}{v^{m+1}} \begin{vmatrix} v & 0 & 0 & \cdots & 0 & 0 & u \\ v' & v & 0 & \cdots & 0 & 0 & u' \\ \frac{v''}{2!} & v' & v & \cdots & 0 & 0 & \frac{u''}{2!} \\ \frac{v'''}{3!} & \frac{v''}{2!} & v' & \cdots & 0 & 0 & \frac{u'''}{3!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{v^{(m-1)}}{(m-1)!} & \frac{v^{(m-2)}}{(m-2)!} & \frac{v^{(m-3)}}{(m-3)!} & \cdots & v' & v & \frac{u^{(m-1)}}{(m-1)!} \\ \frac{v^{(m)}}{m!} & \frac{v^{(m-1)}}{(m-1)!} & \frac{v^{(m-2)}}{(m-2)!} & \cdots & \frac{v''}{2!} & v' & \frac{u^{(m)}}{m!} \end{vmatrix}. \tag{2.8}$$

## 3. Proving the main result of the paper

*Proof of Theorem 1.1.* We need to prove the formulas (1.4)-(1.5). Substituting $z = z_0$ into (1.2) we get from (2.6) and (2.7) that

$$P_n(z_0) = B_n^0, \qquad\qquad Q_n(z_0) = B_n^1.$$

Since $w_0 = P_n/Q_n = b_0\, B_n^1/B_n^1$, we see that $b_0 = w_0$.

**1)** If $m = 1$, then (2.8) implies that

$$w_1 = \frac{1}{Q_n^2}\begin{vmatrix} Q_n & P_n \\ Q_n' & P_n' \end{vmatrix}.$$

Now it follows from (2.3), (2.6) and (2.7) that

$$w_1 = \frac{1}{(B_n^1)^2}\begin{vmatrix} B_n^1 & B_n^0 \\ B_n^1 E_1^{1,n} & B_n^0 E_1^{0,n} \end{vmatrix} = \frac{B_n^0 B_n^1}{(B_n^1)^2}\begin{vmatrix} 1 & 1 \\ E_1^{1,n} & E_1^{0,n} \end{vmatrix} = b_0 \begin{vmatrix} 1 & 0 \\ E_1^{1,n} & \dfrac{1}{b_0 b_1} \end{vmatrix} = \frac{1}{b_1}.$$

Hence, $b_1 = 1/w_1$.

**2)** Let $m = 2$. Then, similarly, we have that

$$w_2 = \frac{2!}{(Q_n)^3}\begin{vmatrix} Q_n & 0 & P_n \\ Q_n' & Q_n & P_n'(z_0) \\ \frac{Q_n''}{2} & Q_n' & \frac{P_n''}{2} \end{vmatrix} = \frac{2!}{(B_n^1)^3}\begin{vmatrix} B_n^1 & 0 & B_n^0 \\ B_n^1 E_1^{1,n} & B_n^1 & B_n^0 E_1^{0,n} \\ B_n^1 E_2^{1,n} & B_n^1 E_1^{1,n} & B_n^0 E_2^{0,n} \end{vmatrix} =$$

$$= \frac{2! b_0 (B_n^1)^3}{(B_n^1)^3}\begin{vmatrix} 1 & 0 & 1 \\ E_1^{1,n} & 1 & E_1^{0,n} \\ E_2^{1,n} & E_1^{1,n} & E_2^{0,n} \end{vmatrix} = 2! b_0 \begin{vmatrix} 1 & 0 & 0 \\ E_1^{1,n} & 1 & \frac{1}{b_0 b_1} E_0^{2,n} \\ E_2^{1,n} & E_1^{1,n} & \frac{1}{b_0 b_1} E_1^{2,n} \end{vmatrix} =$$

$$= \frac{2!}{b_0 b_1}\begin{vmatrix} 1 & 1 \\ E_1^{1,n} & E_1^{2,n} \end{vmatrix} = \frac{2!}{b_1}\begin{vmatrix} 1 & 0 \\ E_1^{1,n} & \frac{-1}{b_1 b_2} \end{vmatrix} = \frac{-2!}{b_1^2 b_2}.$$

Therefore, $b_2 = -1/\big(b_1^2 \frac{w_2}{2!}\big)$. Thus, formulas (1.4) are proved for $k = 0, 1, 2$.

**3)** Suppose that $3 \leqslant m \leqslant n$. We will consider two cases: $3 \leqslant m \leqslant \big[n/2\big]$ and $m > \big[n/2\big]$ and prove the relation

$$\frac{w_m}{m!} = \frac{(-1)^{m-1}}{b_1^2 b_2} \sum_{i_2=1}^{2} \frac{1}{b_{i_2} b_{i_2+1}} \sum_{i_3=1}^{i_2+1} \frac{1}{b_{i_3} b_{i_3+1}} \cdots \sum_{i_{m-1}=1}^{i_{m-2}+1} \frac{1}{b_{i_{m-1}} b_{i_{m-1}+1}}. \tag{3.1}$$

**a)** Let $3 \leqslant m \leqslant \left[n/2\right]$. Then we get from (2.6), (2.7) and (2.8) that

$$w_m = \frac{m!}{(B_n^1)^{m+1}} \begin{vmatrix} B_n^1 & 0 & 0 & \cdots & 0 & B_n^0 \\ B_n^1 E_1^{1,n} & B_n^1 & 0 & \cdots & 0 & B_n^0 E_1^{0,n} \\ B_n^1 E_2^{1,n} & B_n^1 E_1^{1,n} & B_n^1 & \cdots & 0 & B_n^0 E_2^{0,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_n^1 E_{m-1}^{1,n} & B_n^1 E_{m-2}^{1,n} & B_n^1 E_{m-3}^{1,n} & \cdots & B_n^1 & B_n^0 E_{m-1}^{0,n} \\ B_n^1 E_m^{1,n} & B_n^1 E_{m-1}^{1,n} & B_n^1 E_{m-2}^{1,n} & \cdots & B_n^1 E_1^{1,n} & B_n^0 E_m^{0,n} \end{vmatrix}.$$

Since all elements of the first $m$ columns of the determinant have a common factor $B_n^1$ and all elements of the $(m+1)$-th column have a factor $B_n^0$, we obtain that

$$\frac{w_m}{m!} = \frac{b_0 (B_n^1)^{m+1}}{(B_n^1)^{m+1}} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ E_1^{1,n} & 1 & 0 & \cdots & 0 & E_1^{0,n} \\ E_2^{1,n} & E_1^{1,n} & 1 & \cdots & 0 & E_2^{0,n} \\ E_3^{1,n} & E_2^{1,n} & E_1^{1,n} & \cdots & 0 & E_3^{0,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ E_{m-1}^{1,n} & E_{m-2}^{1,n} & E_{m-3}^{1,n} & \cdots & 1 & E_{m-1}^{0,n} \\ E_m^{1,n} & E_{m-1}^{1,n} & E_{m-2}^{1,n} & \cdots & E_1^{1,n} & E_m^{0,n} \end{vmatrix} = b_0 K_{m+1}. \quad (3.2)$$

Let us reduce the determinant $K_{m+1}$ to the triangular form. We will perform all transformations only on the last column of determinant, while other columns of the determinant will be unchanged. Similarly to [9] we will denote the $i$-th column of the determinant $K_{m+1}$ by $\kappa_i$, that is

$$\begin{aligned} \kappa_1 &= \left[1, E_1^{1,n}, \ldots, E_m^{1,n}\right]^T, \\ \kappa_i &= [\underbrace{0, \ldots, 0}_{i-1}, 1, E_1^{1,n}, \ldots, E_{m+1-i}^{1,n}]^T, \ i = \overline{2, m}, \\ \kappa_{m+1} &= \left[1, E_1^{0,n}, E_2^{0,n}, \ldots, E_m^{0,n}\right]^T. \end{aligned}$$

Then the determinant $K_{m+1}$ will be briefly written $K_{m+1} = |\kappa_1 \kappa_2 \ldots \kappa_{m+1}|$.

Subtract the elements of the first column from the elements of $(m+1)$-th column of the determinant $K_{m+1}$. Using (2.3) and factoring out $1/(b_0 b_1)$ from all elements of the column we obtain that

$$\kappa_{m+1} = \frac{1}{b_0 b_1}\left[0, 1, E_1^{2,n}, E_2^{2,n}, \ldots, E_{m-1}^{2,n}\right]^T.$$

Consider the column of the following form:

$$\mathcal{I}^{p}_{r,q}=\big[\underbrace{0,\dots,0}_{r},1,E^{p,n}_{1},\dots,E^{p,n}_{q-r}\big]^{T},\quad r=\overline{1,q},\ p\geqslant 2. \tag{3.3}$$

Taking into account (3.3) we can write the column $\kappa_{m+1}$ as follows:

$$\kappa_{m+1}=\frac{1}{b_0b_1}\mathcal{I}^{2}_{1,m}. \tag{3.4}$$

Similarly, subtracting the corresponding elements of the column $\kappa_{l+1}$ from all elements of the column $\mathcal{I}^{k}_{l,m}$ and using (2.3) we obtain the following recurrence relation:

$$\mathcal{I}^{k}_{l,m}=\sum_{i=1}^{k-1}\frac{-1}{b_ib_{i+1}}\mathcal{I}^{i+2}_{l+1,m}. \tag{3.5}$$

Substituting (3.5) successively, we get that

$$\begin{aligned}\mathcal{I}^{k}_{l,m}&=\sum_{i_1=1}^{k-1}\frac{-1}{b_{i_1}b_{i_1+1}}\mathcal{I}^{i_1+2}_{l+1,m}=\sum_{i_1=1}^{k-1}\frac{-1}{b_{i_1}b_{i_1+1}}\sum_{i_2=1}^{i_1+1}\frac{-1}{b_{i_2}b_{i_2+1}}\mathcal{I}^{i_2+2}_{l+2,m}=\dots=\\&=\sum_{i_1=1}^{k-1}\frac{(-1)^{m-l}}{b_{i_1}b_{i_1+1}}\sum_{i_2=1}^{i_1+1}\frac{1}{b_{i_2}b_{i_2+1}}\cdots\sum_{i_{m-l}=1}^{i_{m-l-1}+1}\frac{1}{b_{i_{m-l}}b_{i_{m-l}+1}}\big[\underbrace{0,\dots,0}_{m},1\big]^{T}.\end{aligned} \tag{3.6}$$

Due to (3.4) the column $\kappa_{m+1}$ can be written in the form:

$$\kappa_{m+1}=\frac{(-1)^{m-1}}{b_0b_1^2b_2}\sum_{i_2=1}^{i_1+1}\frac{1}{b_{i_2}b_{i_2+1}}\cdots\sum_{i_{m-1}=1}^{i_{m-2}+1}\frac{1}{b_{i_{m-1}}b_{i_{m-1}+1}}\big[\underbrace{0,\dots,0}_{m},1\big]^{T}.$$

Now the determinant $K_{m+1}$ is reduced to a triangular form, and therefore (3.2) will be written as follows:

$$\frac{w_m}{m!}=\frac{(-1)^{m-1}}{b_1^2b_2}\sum_{i_2=1}^{2}\frac{1}{b_{i_2}b_{i_2+1}}\cdots\sum_{i_{m-1}=1}^{i_{m-2}+1}\frac{1}{b_{i_{m-1}}b_{i_{m-1}+1}}.$$

Thus, the formula (3.1) holds for $3\leqslant m\leqslant[n/2]$.

**b)** Suppose now $[(n+1)/2]\leqslant m\leqslant n$. Then there are two additional cases: $n=2t-1$ and $n=2t$.

**b-1)** If $n=2t-1$, then $m=\overline{t,2t-1}$.

- Let $m=t$. Then, according to (2.6) and (2.7),

$$\big(P_n(z_0)\big)^{(t)}!=t!B^{0}_{n}E^{0,n}_{t},\qquad\qquad \big(Q_n(z_0)\big)^{(t)}=0.$$

Substituting these values into (2.8) we get

$$w_t = \frac{t!}{(B_n^1)^{t+1}} \begin{vmatrix} B_n^1 & 0 & 0 & \cdots & 0 & B_n^0 \\ B_n^1 E_1^{1,n} & B_n^1 & 0 & \cdots & 0 & B_n^0 E_1^{0,n} \\ B_n^1 E_2^{1,n} & B_n^1 E_1^{1,n} & B_n^1 & \cdots & 0 & B_n^0 E_2^{0,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ B_n^1 E_{t-1}^{1,n} & B_n^1 E_{t-2}^{1,n} & B_n^1 E_{t-3}^{1,n} & \cdots & B_n^1 & B_n^0 E_{t-1}^{0,n} \\ 0 & B_n^1 E_{t-1}^{1,n} & B_n^1 E_{t-2}^{1,n} & \cdots & B_n^1 E_1^{1,n} & B_n^0 E_t^{0,n} \end{vmatrix}.$$

Factoring out the multiplier $B_n^1$ from the elements of the first $t$ columns and the multiplier $B_n^0$ from the last column we get that

$$\frac{w_t}{t!} = b_0 \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ E_1^{1,n} & 1 & 0 & \cdots & 0 & E_1^{0,n} \\ E_2^{1,n} & E_1^{1,n} & 1 & \cdots & 0 & E_2^{0,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ E_{t-1}^{1,n} & E_{t-2}^{1,n} & E_{t-3}^{1,n} & \cdots & 1 & E_{t-1}^{0,n} \\ 0 & E_{t-1}^{1,n} & E_{t-2}^{1,n} & \cdots & E_1^{1,n} & E_t^{0,n} \end{vmatrix} = b_0 L_{t+1}. \tag{3.7}$$

As above, denote the columns of the determinant $L_{t+1}$ by

$$\begin{aligned} \lambda_1 &= \left[1, E_1^{1,n}, E_2^{1,n}, \dots, E_{t-1}^{1,n}, 0\right]^T, \\ \lambda_i &= [\underbrace{0, 0, \dots, 0}_{i-1}, 1, E_1^{1,n}, E_2^{1,n}, \dots, E_{t+1-i}^{1,n}]^T, \qquad i = \overline{2, t} \\ \lambda_{t+1} &= \left[1, E_1^{0,n}, E_1^{0,n}, \dots, E_t^{0,n}\right]^T, \end{aligned}$$

and all the determinant $L_{t+1}$ by $L_{t+1} = |\lambda_1 \lambda_2 \dots \lambda_{t+1}|$.

Subtract the elements of $\lambda_1$ from the elements of $\lambda_{t+1}$ and use (2.3). Further, write $E_t^{0,n}$ according to formula (2.2) if $s = 0$ and factor out the factor $1/(b_0 b_1)$ from elements of the column. Then we will get that

$$\lambda_{t+1} = \frac{1}{b_0 b_1} \left[0, 1, E_1^{2,n}, E_1^{2,n}, \dots, E_{t-1}^{2,n}\right]^T = \frac{1}{b_0 b_1} \mathcal{I}_{1,t}^2,$$

where $\mathcal{I}_{1,t}^2$ is defined in (3.3).

It now follows from (3.5) and (3.6) that

$$\lambda_{t+1} = \frac{(-1)^{t-1}}{b_0 b_1^2 b_2} \sum_{i_2=1}^{2} \frac{1}{b_{i_2} b_{i_2+1}} \cdots \sum_{i_{t-1}=1}^{i_{t-2}+1} \frac{1}{b_{i_{t-1}} b_{i_{t-1}+1}} [\underbrace{0, \dots, 0}_{t}, 1]^T.$$

Now the determinant $L_{t+1}$ is reduced to a triangular form, and we get from (3.7) that

$$\frac{w_t}{t!} = \frac{(-1)^{t-1}}{b_1^2 b_2} \sum_{i_2=1}^{2} \frac{1}{b_{i_2} b_{i_2+1}} \sum_{i_3=1}^{i_2+1} \frac{1}{b_{i_3} b_{i_3+1}} \cdots \sum_{i_{t-1}=1}^{i_{t-2}+1} \frac{1}{b_{i_{t-1}} b_{i_{t-1}+1}}.$$

Thus, formula (3.1) holds for $m = t$.

- Suppose $m = t + l$, where $1 \leqslant l \leqslant t - 1$. Then (2.8) will have the form

$$w_{t+l} = \frac{(t+l)!}{(B_n^1)^{t+l+1}} \begin{vmatrix} B_n^1 & 0 & \cdots & 0 & \cdots & 0 & B_n^0 \\ B_n^1 E_1^{1,n} & B_n^1 & \cdots & 0 & \cdots & 0 & B_n^0 E_1^{0,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ B_n^1 E_{t-1}^{1,n} & B_n^1 E_{t-2}^{1,n} & \cdots & B_n^1 & \cdots & 0 & B_n^0 E_{t-1}^{0,n} \\ 0 & B_n^1 E_{t-1}^{1,n} & \cdots & B_n^1 E_1^{1,n} & \cdots & 0 & B_n^0 E_t^{0,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_n^1 E_{t-2}^{1,n} & \cdots & B_n^1 & 0 \\ 0 & 0 & \cdots & B_n^1 E_{t-1}^{1,n} & \cdots & B_n^1 E_1^{1,n} & 0 \end{vmatrix}.$$

The multiplier $B_n^1$ factors out from the elements of the first $(t+l)$ columns of the determinant and the multiplier $B_n^0$ factors out from the last column, hence

$$\frac{w_{t+l}}{(t+l)!} = b_0 \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \\ E_1^{1,n} & 1 & 0 & \cdots & 0 & \cdots & 0 & E_1^{0,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ E_{t-1}^{1,n} & E_{t-2}^{1,n} & E_{t-3}^{1,n} & \cdots & 0 & \cdots & 0 & E_{t-1}^{0,n} \\ 0 & E_{t-1}^{1,n} & E_{t-2}^{1,n} & \cdots & 1 & \cdots & 0 & E_t^{0,n} \\ 0 & 0 & E_{t-1}^{1,n} & \cdots & E_1^{1,n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E_{t-2}^{1,n} & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & E_{t-1}^{1,n} & \cdots & E_1^{1,n} & 0 \end{vmatrix} = b_0 M_{t+l+1}. \quad (3.8)$$

Denote the columns of the determinant $M_{t+l+1}$ by

$$\mu_i = [\underbrace{0, \ldots, 0}_{i-1}, 1, E_1^{1,n}, \ldots, E_{t-1}^{1,n}, \underbrace{0, \ldots, 0}_{l+2-i}]^T, \qquad i = \overline{1, l+2},$$

$$\mu_i = \big[\underbrace{0,\dots,0}_{i-1},1,E_1^{1,n},\dots,E_{t+l+1-i}^{1,n}\big]^T, \qquad i=\overline{l+3,t+l},$$

$$\mu_{t+l+1} = \big[1,E_1^{0,n},\dots,E_t^{0,n},\underbrace{0,\dots,0}_{l}\big]^T,$$

and all the determinant $M_{t+l+1}$ as $M_{t+l+1} = |\mu_1,\mu_2,\dots,\mu_{t+l+1}|$.

Subtract the first column $\mu_1$ from the last column $\mu_{t+l+1}$ and use (2.2) and (2.3). The multiplier $1/(b_0b_1)$ factor out from the elements of column, and we obtain that

$$\mu_{t+l+1} = \frac{1}{b_0b_1}\mathcal{J}_1^2, \qquad \mathcal{J}_1^2 = \big[0,1,E_1^{2,n},\dots,E_{t-1}^{2,n},\underbrace{0,\dots,0}_{l}\big]^T. \tag{3.9}$$

Subtracting further the column $\mu_2$ from the obtained column $\mathcal{J}_1^2$, using (2.3), and factoring out from elements the multiplier $(-1/b_1b_2)$ we get that

$$\mathcal{J}_1^2 = \frac{-1}{b_1b_2}\mathcal{J}_2^3, \qquad \mathcal{J}_2^3 = \big[0,0,1,E_1^{3,n},\dots,E_{t-2}^{3,n},\underbrace{0,\dots,0}_{l}\big]^T. \tag{3.10}$$

Similarly, subtracting the column $\mu_3$ from the column $\mathcal{J}_2^3$, and using (2.3) and (2.2) if $s=2$, we obtain

$$\mathcal{J}_2^3 = \sum_{i_2=1}^{2}\frac{-1}{b_{i_2}b_{i_2+1}}\mathcal{J}_3^{i_2+2},$$

$$\mathcal{J}_3^{i_2+2} = \big[0,0,0,1,E_1^{i_2+2,n},\dots,E_{t-2}^{i_2+2,n},\underbrace{0,\dots,0}_{l-1}\big]^T.$$

Consider the column of the following form:

$$\mathcal{J}_k^m = \big[\underbrace{0,\dots,0}_{k},1,E_1^{m,n},\dots,E_{t-2}^{m,n},\underbrace{0,\dots,0}_{l+2-k}\big]^T, \quad k=\overline{2,l+1},\ m\geqslant 2.$$

Subtracting column $\mu_{k+1}$ from the column $\mathcal{J}_k^m$, and using (2.3) and (2.2) if $s=m-1$, we get

$$\mathcal{J}_k^m = \sum_{i_k=1}^{m-1}\frac{-1}{b_{i_k}b_{i_k+1}}\mathcal{J}_{k+1}^{i_k+2}. \tag{3.11}$$

If $k=l+1$, then (3.11) reduces to the following form:

$$\mathcal{J}_{l+1}^m = \sum_{i_{l+1}=1}^{m-1}\frac{-1}{b_{i_{l+1}}b_{i_{l+1}+1}}\big[\underbrace{0,\dots,0}_{l+2},1,E_1^{i_{l+1}+2,n},\dots,E_{t-2}^{i_{l+1}+2,n}\big]^T =$$

$$= \sum_{i_{l+1}=1}^{m-1} \frac{-1}{b_{i_{l+1}} b_{i_{l+1}+1}} \mathcal{I}_{l+2,t+l}^{i_{l+1}+2}. \tag{3.12}$$

It now follows from (3.6) that

$$\mathcal{I}_{l+2,t+l}^{i_{l+1}+2} = \sum_{i_{l+2}=1}^{i_{l+1}+1} \frac{(-1)^{t-2}}{b_{i_{l+2}} b_{i_{l+2}+1}} \cdots \sum_{i_{t+l-l}=1}^{i_{t+l-2}+1} \frac{1}{b_{i_{t+l-l}} b_{i_{t+l-1}+1}} \big[\underbrace{0, \ldots, 0}_{t+l}, 1\big]^T. \tag{3.13}$$

We can successively substitute (3.13) into (3.12), then into (3.11) for $k = \overline{2, l+1}$, then into (3.10), and finally into (3.9). As a result, we get

$$\mu_{t+l+1} = \frac{1}{b_0 b_1} \frac{(-1)^{t+l-1}}{b_1 b_2} \sum_{i_2=1}^{2} \frac{1}{b_{i_2} b_{i_2+1}} \sum_{i_3=1}^{i_2+1} \frac{1}{b_{i_3} b_{i_3+1}} \cdots \sum_{i_{l+1}=1}^{i_l+1} \frac{1}{b_{i_{l+1}} b_{i_{l+1}+1}} \times$$

$$\times \sum_{i_{l+2}=1}^{i_{l+1}+1} \frac{1}{b_{i_{l+2}} b_{i_{l+2}+1}} \cdots \sum_{i_{t+l-l}=1}^{i_{t+l-2}+1} \frac{1}{b_{i_{t+l-1}} b_{i_{t+l1-1}+1}} \big[\underbrace{0, \ldots, 0}_{t+l}, 1\big]^T.$$

Notice that the determinant $L_{t+l+1}$ is reduced now to a triangular form. Then it follows from (3.8) that

$$\frac{w_{t+l}}{(t+l)!} = \frac{(-1)^{t+l-1}}{b_1^2 b_2} \sum_{i_2=1}^{2} \frac{1}{b_{i_2} b_{i_2+1}} \sum_{i_3=1}^{i_2+1} \frac{1}{b_{i_3} b_{i_3+1}} \cdots \sum_{i_{t+l-1}=1}^{i_{t+l-2}+1} \frac{1}{b_{i_{t+l-1}} b_{i_{t+l-1}+1}}.$$

Thus, the formula (3.1) holds for $m = t + l, l = \overline{1, t-1}$.

**b-2)** Let $n = 2t$. Then $m = \overline{t, 2t}$.

- First suppose, $m = t$. Then we get from (2.6) and (2.7) that

$$\big(P_n(z_0)\big)^{(t)} = t! B_n^0 E_t^{0,n}, \qquad \big(Q_n(z_0)\big)^{(t)} = t! B_n^1 E_t^{1,n},$$

whence the relation (2.8) can be written in the form

$$w_t = \frac{t!}{(B_n^1)^{t+1}} \begin{vmatrix} B_n^1 & 0 & \cdots & 0 & B_n^0 \\ B_n^1 E_1^{1,n} & B_n^1 & \cdots & 0 & B_n^0 E_1^{0,n} \\ B_n^1 E_2^{1,n} & B_n^1 E_1^{1,n} & \cdots & 0 & B_n^0 E_2^{0,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_n^1 E_t^{1,n} & B_n^1 E_{t-1}^{1,n} & \cdots & B_n^1 & B_n^0 E_t^{0,n} \end{vmatrix}.$$

Similarly, as in the previous case, the multiplier $B_n^1$ factors out from the first $t$ columns of the determinant and the multiplier $B_n^0$ factors out from the last column. Then we get (3.2), which proves (3.1) for this case.

- Now, let $m = t + l$, $l = \overline{1,t}$. Then the relation (2.8) can be written as follows:

$$w_{t+l} = \frac{(t+l)!}{(B_n^1)^{t+l+1}} \begin{vmatrix} B_n^1 & 0 & \cdots & 0 & \cdots & 0 & B_n^0 \\ B_n^1E_1^{1,n} & B_n^1 & \cdots & 0 & \cdots & 0 & B_n^0E_1^{0,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ B_n^1E_t^{1,n} & B_n^1E_{t-1}^{1,n} & \cdots & B_n^1E_1^{1,n} & \cdots & 0 & B_n^0E_t^{0,n} \\ 0 & B_n^1E_t^{1,n} & \cdots & B_n^1E_2^{1,n} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_n^1E_{t-1}^{1,n} & \cdots & B_n^1 & 0 \\ 0 & 0 & \cdots & B_n^1E_t^{1,n} & \cdots & B_n^1E_1^{1,n} & 0 \end{vmatrix}.$$

Similarly, the multiplier $B_n^1$ factor out from first $(t+l)$ columns of the determinant and the multiplier $B_n^0$ factor out from the last column, so

$$\frac{w_{t+l}}{(t+l)!} = b_0 \begin{vmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 1 \\ E_1^{1,n} & 1 & \cdots & 0 & \cdots & 0 & E_1^{0,n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ E_t^{1,n} & E_{t-1}^{1,n} & \cdots & E_1^{1,n} & \cdots & 0 & E_t^{0,n} \\ 0 & E_t^{1,n} & \cdots & E_2^{1,n} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & E_{t-1}^{1,n} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & E_t^{1,n} & \cdots & E_1^{1,n} & 0 \end{vmatrix} = b_0 N_{t+l+1}. \qquad (3.14)$$

Again denote the columns of the determinant $N_{t+l+1}$ by

$$\nu_i = [\underbrace{0, \cdots, 0}_{i-1}, 1, E_1^{1,n}, E_2^{1,n}, \cdots, E_t^{1,n}, \underbrace{0, \ldots, 0}_{l+1-i}]^T, \qquad i = \overline{1,l},$$

$$\nu_i = [\underbrace{0, \cdots, 0}_{i-1}, 1, E_1^{1,n}, E_2^{1,n}, \cdots, E_{t+l+1-i}^{1,n}]^T, \qquad i = \overline{l+1, t+l},$$

$$\nu_{t+l+1} = [1, E_1^{0,n}, E_2^{0,n}, \cdots, E_t^{0,n}, \underbrace{0, \ldots, 0}_{l}]^T,$$

and the determinant $N_{t+l+1}$ by $N_{t+l+1} = |\nu_1 \nu_2 \ldots \nu_{t+l+1}|$.

We will now reduce the determinant $N_{t+l+1}$ to a triangular form. Subtracting the first column $\nu_1$ of the determinant from the last column $\nu_{t+l+1}$

and using (2.3) we get

$$\nu_{t+l+1} = \frac{1}{b_0 b_1}\mathcal{K}_1^2, \qquad \mathcal{K}_1^2 = [0, 1, E_1^{2,n}, \cdots, E_{t-1}^{2,n}, \underbrace{0, \ldots, 0}_{l}]^T. \tag{3.15}$$

Subtracting further the second column $\nu_2$ from the obtained column $\mathcal{K}_1^2$ and using (2.2) and (2.3) we obtain

$$\mathcal{K}_1^2 = \frac{-1}{b_1 b_2}\mathcal{K}_2^3, \qquad \mathcal{K}_2^3 = [0, 0, 1, E_1^{3,n}, \ldots, E_{t-1}^{3,n}, \underbrace{0, \ldots, 0}_{l-1}]^T. \tag{3.16}$$

Similarly, subtracting column $\nu_3$ from column $\mathcal{K}_2^3$ and using the same formulas we get

$$\mathcal{K}_2^3 = \sum_{i_2=1}^{2} \frac{-1}{b_{i_2} b_{i_2+1}}\mathcal{K}_3^{i_2+2}, \tag{3.17}$$

where

$$\mathcal{K}_3^{i_2+2} = [0, 0, 0, 1, E_1^{i_2+2,n}, \ldots, E_{t-1}^{i_2+2,n}, \underbrace{0, \ldots, 0}_{l-2}]^T.$$

Consider the column of the following form:

$$\mathcal{K}_k^m = [\underbrace{0, \ldots, 0}_{k}, 1, E_1^{m,n}, \ldots, E_{t-1}^{m,n}, \underbrace{0, \ldots, 0}_{l+1-k}]^T, \quad k = \overline{2, l}.$$

Subtracting the column $\nu_{k+1}$, $k = \overline{4, l+1}$, from the column $\mathcal{K}_k^m$ and using (2.2) and (2.3), we get

$$\mathcal{K}_k^m = \sum_{i_k=1}^{m-1} \frac{-1}{b_{i_k} b_{i_k+1}}\mathcal{K}_{k+1}^{i_k+2}. \tag{3.18}$$

Finally, subtracting the column $\nu_{l+2}$ from the obtained column $\mathcal{K}_{l+1}^m$ we will obtain that

$$\mathcal{K}_{l+1}^m = \sum_{i_{l+1}=1}^{m-1} \frac{-1}{b_{l+1} b_{l+1} + 1}\mathcal{I}_{l+2,t+l}^{i_{l+1}+2},$$

so the column $\mathcal{I}_{l+2,t+l}^{i_{l+1}+2}$ satisfies relation (3.13). By successively substituting (3.13) into (3.18), (3.17), (3.16) and (3.15), we finally have

$$\nu_{t+l+1} = \frac{1}{b_0 b_1}\frac{(-1)^{t+l-1}}{b_1 b_2}\sum_{i_2=1}^{2}\frac{1}{b_{i_2} b_{i_2+1}}\sum_{i_3=1}^{i_2+1}\frac{1}{b_{i_3} b_{i_3+1}} \cdots \sum_{i_{l+1}=1}^{i_l+1}\frac{1}{b_{i_{l+1}} b_{i_{l+1}+1}} \times$$
$$\times \sum_{i_{l+2}=1}^{i_{l+1}+1}\frac{1}{b_{i_{l+2}} b_{i_{l+2}+1}} \cdots \sum_{i_{t+l-l}=1}^{i_{t+l-2}+1}\frac{1}{b_{i_{t+l-l}} b_{i_{t+l-1}+1}}[\underbrace{0, \ldots, 0}_{t+l}, 1]^T.$$

Thus, the determinant $N_{t+l+1}$ is reduced to a triangular form. Then it follows from (3.14) that

$$\frac{w_{t+l}}{(t+l)!}=\frac{(-1)^{t+l-1}}{b_1^2 b_2}\sum_{i_2=1}^{2}\frac{1}{b_{i_2}b_{i_2+1}}\sum_{i_3=1}^{i_2+1}\frac{1}{b_{i_3}b_{i_3+1}}\cdots\sum_{i_{t+l-1}=1}^{i_{t+l-2}+1}\frac{1}{b_{i_{t+l-1}}b_{i_{t+l-1}+1}},$$

so the formula (3.1) holds in this case too.

Let us prove the formulas (1.4) and (1.5) for computation of the coefficients of the continued fraction (1.1). Recall that formulas (1.4) have already been proven for $k=0,1,2$. Assume that for each value $k=\overline{3,n}$ the coefficients $b_1, b_2, \ldots, b_{k-1}$ are known. Consider the last term of the relation (3.1) which contains the unknown coefficient $b_k$:

$$\frac{w_{k+1}}{(k+1)!}=\frac{(-1)^{k}}{b_1^2 b_2}\sum_{i_2=1}^{2}\frac{1}{b_{i_2}b_{i_2+1}}\sum_{i_3=1}^{i_2+1}\frac{1}{b_{i_3}b_{i_3+1}}\cdots\sum_{i_k=1}^{i_{k-1}+1}\frac{1}{b_{i_k}b_{i_k+1}}.$$

If $k=3$ then (3.1) will be written in the form

$$\frac{w_3}{3!}=\frac{1}{b_1^2 b_2}\Big(\frac{1}{b_1 b_2}+\frac{1}{b_2 b_3}\Big)=\frac{1}{b_1^3 b_2^2}+\frac{1}{b_1 b_2^2 b_3},$$

whence $b_3 = 1\big/b_1^2 b_2^2\big(\frac{w_3}{3!}-\frac{1}{b_1^3 b_2^2}\big)$. Thus, the formulas (1.4) hold for $k=3$.

Let $4 \leqslant k \leqslant n$. We perform the transformation on the right side of the relation (3.1) to choose the term containing the coefficients $b_k$.

$$\frac{w_k}{k!}=\frac{(-1)^{k-1}}{b_1^2 b_2}\sum_{i_2=1}^{2}\frac{1}{b_{i_2}b_{i_2+1}}\sum_{i_3=1}^{i_2+1}\frac{1}{b_{i_3}b_{i_3+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}=$$

$$=\frac{(-1)^{k-1}}{b_1^2 b_2}\Big(\frac{1}{b_1 b_2}\sum_{i_3=1}^{2}\frac{1}{b_{i_3}b_{i_3+1}}\sum_{i_4=1}^{i_3+1}\frac{1}{b_{i_4}b_{i_4+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+$$

$$+\frac{1}{b_2 b_3}\sum_{i_3=1}^{3}\frac{1}{b_{i_3}b_{i_3+1}}\sum_{i_4=1}^{i_3+1}\frac{1}{b_{i_4}b_{i_4+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}\Big)=$$

$$=\frac{(-1)^{k-1}}{b_1^2 b_2}\Big(\frac{1}{b_1 b_2}\sum_{i_3=1}^{2}\frac{1}{b_{i_3}b_{i_3+1}}\sum_{i_4=1}^{i_3+1}\frac{1}{b_{i_4}b_{i_4+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+$$

$$+\frac{1}{b_2 b_3}\sum_{i_3=1}^{2}\frac{1}{b_{i_3}b_{i_3+1}}\sum_{i_4=1}^{i_3+1}\frac{1}{b_{i_4}b_{i_4+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+$$

$$+\frac{1}{b_2 b_3^2 b_4}\sum_{i_4=1}^{4}\frac{1}{b_{i_4}b_{i_4+1}}\sum_{i_5=1}^{i_4+1}\frac{1}{b_{i_5}b_{i_5+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}\Big)=\cdots=$$

$$\begin{aligned}
&= (-1)^{k-1}\Bigg(\frac{1}{b_1^3 b_2^2}\sum_{i_3=1}^{2}\frac{1}{b_{i_3}b_{i_3+1}}\sum_{i_4=1}^{i_3+1}\frac{1}{b_{i_4}b_{i_4+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+\\
&\quad+\frac{1}{b_1^2 b_2^2 b_3}\sum_{i_3=1}^{2}\frac{1}{b_{i_3}b_{i_3+1}}\sum_{i_4=1}^{i_3+1}\frac{1}{b_{i_4}b_{i_4+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+\\
&\quad+\frac{1}{b_4\prod\limits_{j=1}^{3}b_j^2}\sum_{i_4=1}^{3}\frac{1}{b_{i_4}b_{i_4+1}}\sum_{i_5=1}^{i_4+1}\frac{1}{b_{i_5}b_{i_5+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+\\
&\quad+\frac{1}{b_5\prod\limits_{j=1}^{4}b_j^2}\sum_{i_5=1}^{4}\frac{1}{b_{i_5}b_{i_5+1}}\sum_{i_6=1}^{i_5+1}\frac{1}{b_{i_6}b_{i_6+1}}\cdots\sum_{i_{k-1}=1}^{i_{k-2}+1}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+\cdots+\\
&\quad+\frac{1}{b_{k-1}\prod\limits_{j=1}^{k-2}b_j^2}\sum_{i_{k-1}=1}^{k-2}\frac{1}{b_{i_{k-1}}b_{i_{k-1}+1}}+\frac{1}{b_k\prod\limits_{j=1}^{k-1}b_j^2}\Bigg).
\end{aligned}$$

From here, we get the formula (1.5). ☐

## 4. EXAMPLES OF FUNCTIONS EXPANSION INTO A THIELE-HERMITE CONTINUED FRACTION

The resulting formulas (1.4)-(1.5) can easily be implemented in a computer algebra system, such as **maxima**, or in the algorithmic language, such as **gfortran**, which are free software tools in the operating system **Linux**.

It is well known fact that the function $e^z$ has a power series expansion $e^z = \sum\limits_{k=0}^{\infty}\frac{z^k}{k!}$. Using the formulas (1.4)-(1.5), we find the coefficients of continued fraction $b_0 = 1$, $b_{2k-1} = (-1)^{k-1}(2k-1)$, $b_{2k} = (-1)^k 2$. The obtaining continued fraction coincides with the corresponding continued fraction of function [2].

We have the expansion of the function $\sqrt{1+z}$ into a power series

$$\sqrt{1+z} = 1 + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-3)!!}{(2k)!!}z^k.$$

The coefficients of continued fraction that calculated using the formulas (1.4)-(1.5) are $b_0 = 1$, $b_n = 2$, $n \in \mathbb{N}$. The resulting continued fraction coincides with the Thiele continued fraction corresponding to the power series [5].

The general formulas of the expansion coefficients of the functions $\sin z$ and $\cos z$ into the continued Thiele fraction not established. The expansion of the functions $\sin z, \cos z, \sinh z, \cosh z$ into Thiele type functional continued fractions is obtained in [5, 6].

The $\sin z$ power series expansion in the neighborhood of $z_0 = \frac{\pi}{4}$ has the form $\sin z = \frac{\sqrt{2}}{2} \sum\limits_{k=0}^{\infty} \frac{(z-z_0)^k}{k!}$. We can find the required number of THCF coefficients. The first eight coefficients are

$$
\begin{aligned}
b_0 &= \frac{\sqrt{2}}{2}, & b_1 &= \sqrt{2}, & b_2 &= \sqrt{2},\\
b_3 &= \frac{-\,3\sqrt{2}}{5}, & b_4 &= \frac{-\,25\sqrt{2}}{7}, & b_5 &= \frac{-\,49\sqrt{2}}{55},\\
b_6 &= \frac{-\,3025\sqrt{2}}{623}, & b_7 &= \frac{55447\sqrt{2}}{171665}, & b_8 &= \frac{24011568734507558765\sqrt{2}}{1448213983480872329}.
\end{aligned}
$$

Similarly, we have the expansion of the function $\cos z$ in the neighborhood the point $z_0 = \pi/3$.

$$
\cos z = 1 - \frac{\sqrt{3}}{2 \cdot 1!}(z-z_0) - \frac{1}{2 \cdot 2!}(z-z_0)^2 + \frac{\sqrt{3}}{2 \cdot 3!}(z-z_0)^3 + \frac{1}{2 \cdot 4!}(z-z_0)^4 - \cdots .
$$

We find the first nine coefficients of continued fraction

$$
b_0 = \frac{1}{2}, \quad b_1 = \frac{-2}{\sqrt{3}}, \quad b_2 = 3, \quad b_3 = \frac{2}{3\sqrt{3}}, \quad b_4 = \frac{-27}{5}, \quad b_5 = \frac{250}{111\sqrt{3}},
$$

$$
b_6 = \frac{-4107}{295}, \quad b_7 = \frac{-48734}{137751\sqrt{3}}, \quad b_8 = \frac{189873}{7729}, \quad b_9 = \frac{-34322}{39729\sqrt{3}}.
$$

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