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Probability currents for the Relativistic Schrödinger equation

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Abstract. In this work, we start from the problem of quantizing the relativistic Hamiltonian of a free massive particle (rest mass different from 0), a problem exceptionally difficult in the standard approaches to quantum mechanics. In fact, in tempered distribution state space, we find the natural way to define the relativistic Hamiltonian operator and its associated Schrödinger equation. The existence of a linear continuous Hermitian operator associated with the Einstein's Hamiltonian of a free particle, defined on the entire tempered distribution space, automatically implies the conservation of Born probability flux (which doesn't mean the conservation of particles number, rather it implies the conservation of the total relativistic energy of the solution wave). We, then, deduce the continuity equation for the Born probability density and study some its different (but equivalent) expressions. We determine some possible forms of probability currents and flux velocity fields associated with the particle evolution. We provide the relativistic invariant expression for both Schrödinger equation and probability flux continuity equations.

Анотація. В даній роботі обговорюється проблема квантування релятивістського гамільтоніана вільної масивної частинки (у якої маса спокою відмінна від нуля), що є надзвичайно складною проблемою при стандартному підході до квантової механіки. У просторі розподілів повільного росту знайдено природний спосіб визначення релятивістського оператора Гамільтона та пов'язаного з ним рівняння Шредінгера. Існування лінійного неперервного ермітового оператора, пов'язаного з гамільтоніаном Ейнштейна вільної частинки і визначеного на всьому просторі розподілів повільного росту, автоматично передбачає збереження борнівського потоку ймовірностей: при цьому число частинок не обов'язково повинно з зберігатись, зберігається повна релятивістська енергія хвилі.

Далі в статті виводяться рівняння неперервності для щільності ймовірності Борна та вивчаються деякі його різні (але еквівалентні) вирази, описано деякі можливі форми ймовірнісних струмів і полів швидкості

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потоку, пов'язаних з еволюцією частинок, а також наведено релятивістський інваріантний вираз як для рівняння Шредінгера, так і для рівнянь неперервності потоку ймовірностей.

1. Introduction

In literature, the problem of finding the relativistic counterpart of the original classic Schrödinger equation was still not solved, neither it seems fully understood and simplified the problem of a relativistic equation for spin 1 particles (and subsequent ones). In this paper, we show an extremely natural way to formulate (essentially) one (Lorentz invariant) relativistic equation for all the above cases, starting from the original idea of Schrödinger himself and from some fruitful developments of Laurent Schwartz Distribution Theory.

Indeed, the classic Dirac equation deals with half spin particles, on the contrary, the basic relativistic Schrödinger equation we propose here deals with zero spin relativistic particles. When applying the same equation to bi-waves (bi-tempered distributions) or tri-waves (tri-tempered distributions) we simply obtain the free evolution equation for "positive" fermions (Dirac particles with positive energy) and "positive" spin 1 particles.

1.1. **Our main equation in Lorentz invariant form.** The equation we propose (and solve in [19]), in its Lorentz invariant form, is:

$$\left(\sqrt{\boldsymbol{\mu}^2(P)} - (m_0 c)\mathbb{I}\right)\psi = 0,$$

where

- for the moment, the domain of the equation is intended to be the space of all tempered distributions ψ obtainable by a continuous Schwartz superposition of all de Broglie waves $(\beta_p)_{p\in\mathbb{P}_4}$, characterized by strictly positive Minkowsky square norms $\boldsymbol{\mu}(p,p) > 0$;
- the square root operation is intended in its positive definition;
- P is the operator 4-momentum;
- $\boldsymbol{\mu}$ is the Minkowsky metric tensor (or bilinear form, if you prefer) and $\boldsymbol{\mu}^2$ represents its associated quadratic form;
- the square root of a strictly positive operator takes its proper place here, since the quadratic form $\boldsymbol{\mu}^2$ calculated upon the 4-momentum operator P is a strictly positive operator upon our chosen domain (its eigenvalues upon the chosen basis vectors β_p are all strictly positive and indeed they are equal to the Minkowsky square norm of the four vector p);
- $\mathbb{I}$ stands for the identity operator upon the chosen domain.

The equation can be immediately generalized to a curved spacetime (M, g) by the following equation

$$\left(\sqrt{g^2(P)} - (m_0 c)\mathbb{I}\right)\psi = 0,$$

which gives the de Broglie waves satisfying the Einstein relation in the curved space time under exam, as possible Schwartz basis solution.

1.2. **A brief history of the relativistic Schrödinger equation.** As a matter of fact, the problem of a relativistic description of elementary quantum particles has not been completely solved, in history, by Dirac, not in the 20s nor later. The story is much more complicated, interesting and rich.

In the beginning, Schrödinger studied the problem of finding an equation that admitted the relativistic de Broglie waves as possible solutions (that was the natural path, finding an equation whose solution space was generated by the de Broglie family of relativistic harmonic plane waves by continuous linear superpositions). He tried to use the Einstein relativistic dispersion relation and then he tried to build up a differential equation by substituting (in the standard way) the four-momentum by the partial derivative operators.

Unfortunately, such a simple substitution leads to an "operator-form" which is not a differential operator, because of the problematic square root, it was a case practically intractable at that time for many reasons (the theory of fractional differential operators and pseudo-differential operators are recent and not even imaginable in those days, based on distribution theory which was fully developed and accepted in the fifties of last century).

This failure induced Schrödinger to abandon the idea of obtaining a relativistic equation for quantum particles (those not endowed with spin, he never introduced the spin in the question) and to propose its celebrated (but non-relativistic) equation (which, by the way, is not at all – in the applications or in its meaning – surpassed or canceled by the Dirac equation, it remains a fundamental tool and useful piece of quantum mechanics nowadays).

1.3. **The Pauli equation.** We underline here that the $1/2$ spin particles can be introduced also in the non-relativistic setting by using the square S^2 of the Schrödinger state space S and the Pauli matrices, which act naturally upon the state space S^2, Cartesian square of the space S (whatever this state space S might be). Indeed, when the electromagnetic field is absent, the non-relativistic analogous of the non-relativistic Schrödinger equation for spin $1/2$ particles is the famous Pauli equation (which, in its standard form is nothing more than the non-relativistic Schrödinger equation acting on a "Pauli bi-wave" function: indeed, an element of the space S^2).

1.4. **Problems with the original Schrödinger equation.** On the other hand, the relativistic de Broglie waves are not belong to the set of solutions of the non-relativistic Schrödinger equation and Schrödinger recognized immediately that its equation should be a first approximation of some other irrational equation definable by the sum of a power series of differential operators (regrettably, nothing about that series was understandable at that time, not even the Hilbert space approach was in place at that time and, however, the problem of studying that operator series goes far beyond the Hilbert space theory).

At this purpose, we observe that the differential operators used by Schrödinger (in its classic equation) are not even specified in their domain: Schrödinger considered those operators simply defined on "differentiable functions", without considering any well structured function space or good topology on it, so that, the problem of studying that series of operators became intractable and utterly meaningless.

1.5. **Towards various solutions of the relativistic issue.** Clearly, to many theoretical physicists of that time, the basic problem of finding a relativistic Schrödinger equation for spin-less particles was of capital importance, in the development of quantum mechanics. Many fathers of quantum mechanics faced (failing) such a problem, between them we remember (in fact) Pauli, Dirac, Klein, Gordon, Heisenberg and many others.

As we told before, Dirac proposed a very smart algebraic way to circumnavigate the square root problem, but it worked in S^4 (I desire to emphasize again that the method proposed by Dirac solves the fermion problem in the fourth power of the Schrödinger state-space S of zero spin particle, not the original problem itself in S and not in the Pauli space S^2).

Moreover, the ingenious Dirac method introduces some negative energy states to be dangerously interpreted (we need to observe here that - from a strictly mathematical point to view - the presence of negative energy is highly questionable because the Einstein dispersion relation uses the "positive square root" operation, not the complex one, so, when we calculate this "positive square root", we must discard any negative outcome, by the very definition of the operation involved, but this is another story).

In this moment, we do not want to underline the good and bad sides of Dirac equation, we desire to underline that it is not the much desired and desirable relativistic Schrödinger equation for zero spin particles and it addresses explicitly the half spin particle evolution problem in an unexpected space S^4 instead of S^2 (the natural space of action for the Pauli matrices).

1.6. **The Kleine Gordon approach.** Then, we desire to recall (also) the approach of Klein and Gordon, they considered directly the argument under the problematic square root and they found an evolution-type equation for zero spin particles (0, yes indeed), however, as we know, that equation offers other kinds of issues and many interpretative questions from a probabilistic point of view.

We do not want to emphasize the famous issues that such equation provides, we desire only to underline that the Kleine Gordon equation is not the desired relativistic Schrödinger equation. In some sense, it is the square of the dreamed relativistic Schrödinger equation.

1.7. **Our contribution.** Again, Dirac, using a very smart algebraic trick, solves the relativistic energy momentum relation issue in the power fourth of the state space S of zero spin particles, S^4, which shall reveal itself as the state space of fermions and not anymore the space of the zero spin particles, but he didn't prove that the square root issue can only be fulfilled using $(4, 4)$ matrices.

To prove that the problem can be conveniently transformed, changed, stated and solved (without square roots of operators) in S^4 did not prove that the problem cannot be somehow stated and solved just in S, as the classic Schrödinger equation does.

Remark 1.7.1. We do not consider, explicitly, the fermion problem here, because it was solved by Dirac. On the other hand, it is a fact that we found a new simpler and more natural relativistic Schrödinger-form of the Dirac equation in the spin $1/2$ space S^2 (the Pauli matrix state-space). Moreover, we officially adopt the tempered distribution spaces, on $\mathbb{M}_2$ and $\mathbb{M}_4$, as the right and most natural domain S of the Schrödinger equation (relativistic and non-relativistic), assuming those vast spaces as the sufficient and necessary state spaces for quantum systems without spin. For describing n different such systems the natural space will be the tensor power $\otimes^n S$, which is nothing more than the distribution space $\mathcal{S}'(\mathbb{M}_4^n)$ on the Cartesian power n of the chronotope $\mathbb{M}_4$.

Actually, here we propose the original problem proposed by Schrödinger (exactly the original one, not transformed or changed) in the space of tempered distributions (which therefore becomes the zero spin particles state-space).

We think that the majority of the scientific community has simply discarded the problem because it was apparently mathematically intractable, and it preferred to consider the problem not closable.

Our paper clarifies many misunderstood aspect of relativistic quantum mechanics, not only because it solves the fundamental problem proposed more than 100 years ago, also because it answers to the question:

Is there a relativistic counterpart of the original (without spin) non-relativistic Schrödinger equation?

Why the Klein Gordon equation is not the relativistic Schrödinger equation and we can't prove the probability currents conservation in relativistic quantum mechanics?

1.8. **Conservation of Born probabilities.** In fact, by means of the relativistic Schrödinger equation for 0 spin particle, we can now prove the conservation of probability currents. That conservation does not refer to the number of particles, that may change, but to the conservation of energy of the quantum field associated with the particle.

The conservation of probability currents is again the conservation of the Born probability density $|\psi|^2$, during the evolution of the field ψ, while satisfying our relativistic Schrödinger equation.

It indeed follows immediately by the Hermite symmetry of our relativistic Hamiltonian $\hat{H}$, since

$$\imath\hbar\frac{\partial}{\partial \mathrm{t}}\int_{\mathbb{R}}\rho_{\psi}\partial\mathbf{x} = \imath\hbar\frac{\partial}{\partial \mathrm{t}}\langle\psi|\psi\rangle = \langle\psi|\hat{H}\psi\rangle - \langle\hat{H}\psi|\psi\rangle = 0,$$

for every regular tempered distribution ψ, whose section $\psi(ct,\cdot)$ is a Schwartz function.

So that, as soon as the Hamiltonian exists and reveals Hermitian then the probability flux is conserved.

We here show exactly the constructive existence of a specific Hermitian Hamiltonian associated to the Einstein's energy.

1.8.1. *Some certainty states which are not Lebesgue normalizable.* The above relation can be duplicated for another vast and important class of regular distributions, this time non-normalizable wavefunctions, in the form:

$$\imath\hbar\frac{\partial}{\partial \mathrm{t}}\|\psi\|_{\beta}^2 = \langle\psi|\hat{H}\psi\rangle_{\beta} - \langle\hat{H}\psi|\psi\rangle_{\beta} = 0.$$

where we have consider the pre-Hilbert time-dependent quasi-norm

$$\|\psi\|_{\beta}^2 = 2\pi\hbar\lim_{r\to\infty}(2r)^{-1}\int_{[-r,r]}|\psi|^2\partial\mathbf{x},$$

for every regular distribution ψ induced by a complex continuous function defined on the Minkowski chronotope for which that function is finite valued, let us denote the space of those tempered distribution by $\mathcal{B}_{\beta}^2(\mathbb{M}_2)$ and where the inducing time-dependent pre-Hilbert quasi inner product is defined by

$$\langle\psi|\phi\rangle_\beta = 2\pi\hbar \lim_{r\to\infty}(2r)^{-1}\int_{[-r,r]}\psi^*\phi\ \partial\mathbf{x},$$

for every $\psi, \phi \in \mathcal{B}^2_\beta(\mathbb{M}_2)$.

We could show that the above space contains every de Broglie wave and the linear span of the unitary de Broglie family β. Moreover, β is a unitary orthogonal system of that quasi (time-dependent) pre-Hilbert space. Furthermore, if we complete such space, with respect to the chosen inner product, we obtain the Besikovich space of almost periodic functions with respect to the space argument, in which the family β is an orthonormal basis.

We do not want to go into technicalities too much, but we desire to observe that by following Laurent Schwartz distribution theory and our approach, we can reconsider some fundamental hole in quantum theory, for example the natural normalizability of de Broglie waves (with respect to the right inner product) that (after all) should be all normalizable because they simply represent certainty, that is certainty to find the momentum (or energy) of a particle with a certain value.

1.9. **Final remarks.** By the way, the new equation, in any case (for any spin) is accompanied by a conjugate equation, which admits as solutions negative energy solutions (or time reverse solution).

The relativistic Schrödinger equation allows to avoid wrong or vague explanations about the developments of Schrödinger equation or Dirac equation, or Pauli equation, or Klein-Gordon equation.

For the moment, because of mysterious and vague reasons, we should believe that the non-relativistic Schrödinger equation can be defined correctly, while its simple relativistic twin cannot be defined, without considering other problems or features such as non-0 spin particles and so on.

2. Background and literature review

The intent of our analysis is to start from Laurent Schwartz distribution spaces, based also on the Minkowski space-time, following the spirit of Linear Algebra and Geometry, and offer precise meanings and rigorous support to many calculus methods of Quantum Mechanics.

2.1. **The role of Schwartz Distribution Theory.** The mathematical pre-context of our analysis is, so, the Laurent Schwartz distribution theory, in particular, that based on the Minkowski space-time. For distribution theory, we refer to Schwartz [29–34], Barros-Neto [1], Dieudonné [23, 24], Bourbaki [3–5], Horváth [26], Kesavan [27], Boccara [2], Lang [28], Trèves [36], Yosida [37] and Zeidler [38].

Our approach not only provides a rigorous and efficient justification for the use of many quantum mechanics mathematical tools substantially as they usually show up in the physical practice, but – by a "correct" formulation of the calculus methods in terms of contemporary mathematics – it helps to reach a deeper comprehension of the physical structures studied in Quantum Mechanics.

2.2. **Schwartz Linear Algebra and physical Hilbert space.** Distribution Theory allows us to provide an efficient and deep calculus method for Quantum Mechanics, inspired by the renowned Dirac Calculus [25]. Specifically, we move inside the realm of Schwartz Linear Algebra, which helps us to reach a deeper comprehension of the physical structures studied in Quantum Mechanics.

In particular, in Schwartz Linear Algebra, we consider a new definition of state spaces for quantum systems. These structures are often identified with separable Hilbert spaces, leading immediately to the so called "non-normalizable" states – which revealed fundamental in the development of the quantum mechanics. Indeed, those particular states should be normalizable, but with respect other convenient scalar products, orthogonal to the initial one.

Some researchers in theoretical quantum mechanics already understand that the state space of a Quantum system should be a larger structure than Hilbert spaces, sometimes they call it "physical Hilbert space" ([35]), without introducing a clear definition.

We have already shown in the past [6–18, 20–22] that "physical Hilbert spaces" can smoothly be identified with distribution spaces on suitable Euclidean spaces or Minkowski spaces, depending on the nature of the quantum system considered. Such distribution spaces should be endowed with some algebraic-topological structures such as: continuous operations of superposition; extended Dirac products; partial inner products; etc.

2.3. **Extended Dirac products and associated non-separable Hilbert structures.** In particular, the extended Dirac product allows us to introduce new usual scalar products upon some distinguished subspaces of distribution spaces, endowing those subspaces with non-separable Hilbert structures, which clarify definitely the role of the so-called non-normalizable states.

For example, the singular Dirac distributions and the celebrated de Broglie waves become elements of those new non-separable Hilbert spaces and, consequently, they acquire the status of normalizable states, as it seems completely natural because of the physical usual probability interpretation of such states.

2.4. **Continuous Superpositions.** The new operation of continuous – superposition revealed the right tool which allows us to build – in a mathematically rigorous way – the extended Linear Algebra of Dirac, in distribution spaces, using the Schwartz natural topological-linear structures of those spaces.

More precisely, we saw that the natural algebraic-topological structure of those spaces allows to define an extension of the finite linear combination, when the sets indexing the families of vectors are continuous sets, even in the case in which the systems of coefficients show a continuous-infinity of terms different from zero.

Now, it appears utterly clear how our approach to distribution theory and QM induces a renewed geometrical vision of the functional analysis developments of the two theories themselves: we realize the existence of continuous bases of distributions and corresponding coordinate systems, continuous matrices of continuous operators, tangent and cotangent spaces of infinite dimensional manifolds, modeled upon distribution spaces, endowed with suitable and comfortable continuous bases, leading to a infinite dimensional differential geometry theory much closer to the finite dimensional one.

As a possible application, we solve the problem of quantizing the relativistic Hamiltonian of a free massive particle massive particle (rest mass different from 0). In distribution state spaces, we find a natural way to define the relativistic Hamiltonian operator and its associated Schrödinger equation. We, then, deduce the equivalent continuity equation for the Born probability density and study some its different (but equivalent) expressions. We determine the possible probability currents and flux velocity fields associated with the particle-field evolution.

3. Preliminaries

3.1. **Relativistic Hamiltonian operator in $\mathcal{S}'_2$.** In this work, we face the problem of quantizing the relativistic Hamiltonian,

$$H\colon \mathbb{R} \to \mathbb{R}, \qquad \boldsymbol{p} \mapsto c\sqrt{\boldsymbol{p}^2 + (m_0 c)^2},$$

of a free particle evolving upon a straight line and the relativistic Hamiltonian,

$$H\colon \mathbb{R}^3 \to \mathbb{R}, \qquad \boldsymbol{p} \mapsto c\sqrt{|\boldsymbol{p}|^2 + (m_0 c)^2},$$

of a free particle evolving upon a 3-dimensional space.

Differently from past approaches, we shall move in the convenient space of tempered distributions $\mathcal{S}'(\mathbb{M}_2)$ ($\mathcal{S}'(\mathbb{M}_4)$, for the general four-dimensional case, respectively).

By $\mathcal{S}'(\mathbb{M}_2)$, we mean the space of Schwartz complex tempered distributions defined upon a time-space Minkowski plane $\mathbb{M}_2$ (upon the 4-dimensional Minkowski vector space $\mathbb{M}_4$, for the general case, respectively).

Remark 3.1.1. In such Schwartz distribution spaces, we find all the quantum-states we need for an efficient formulation of quantum single particle models, we find efficient superposition operations, continuous quantum basis of eigenstates, Dirac generalized scalar products, Dirac orthonormality, a convenient and well behaved wide class of linear continuous observables and (of extreme importance in this paper) a suitable handy manageable functional calculus for the observables.

3.1.2. *Strictly positive operators and massive free particles.* We show here that, in the Schwartz distribution context, we can associate with the relativistic Hamiltonian function H a unique linear continuous positive[1] regular Hermitian[2] operator

$$\hat{H} : \mathcal{S}'(\mathbb{M}_2) \to \mathcal{S}'(\mathbb{M}_2), \qquad \hat{H} = c\sqrt{P^2 + (m_0 c)^2 \mathbb{I}}$$

after naturally defining a **unique principal root of Schwartz diagonalizable strictly-positive operators**.

In the above definition, P represents the momentum operator upon the tempered distribution space $\mathcal{S}'(\mathbb{M}_2)$ and $\mathbb{I}$ the identity operator of the same space.

The eigenvalues of the above Hamiltonian operator $\hat{H}$ will be strictly positive: "negative-energy" eigenstates will always show positive Hamiltonian eigenvalues. Moreover, we shall not see associated negative probabilities at all: the conserved probability density remains the Born one, as in the non-relativistic case. In other terms, there exist eigenstates $\chi_{\boldsymbol{p}}$, of definite momentum $\boldsymbol{p}$ and positive hamiltonian eigenvalue $H_{\boldsymbol{p}}$, whose conjugate $\bar{\chi}_{\boldsymbol{p}}$ will determine negative eigenvalues of the energy operator $\hat{E}$ and again positive values for the Hamiltonian operator $\hat{H}$. Clearly, those conjugate eigenstates will not solve the Schrödinger equation, but they will solve its so called conjugate equation.

[1]In the sense that any eigenvalue is strictly positive.

[2]In the sense that it is bilaterally restrictable to a continuous operator from $\mathcal{S}(\mathbb{M}_2)$ to $\mathcal{S}(\mathbb{M}_2)$ and its bilateral restriction to the Schwartz space $\mathcal{S}(\mathbb{M}_2)$ is Hermitian with respect to the standard $L^2(\mathbb{M}_2)$ inner product.

4. The relativistic Schrödinger equation

The relativistic Schrödinger equation

$$\mathcal{E}_H : \hat{E} = \hat{H}$$

corresponding to the Hamiltonian operator $\hat{H}$ and defined by

$$\imath\hbar \frac{\partial}{\partial \mathrm{t}} \psi = c\sqrt{P^2 + (m_0 c)^2 \mathbb{I}}\ \psi,$$

for any $\psi \in \mathcal{S}'(\mathbb{M}_2)$, will be not a differential equation but a linear (continuous) operator equation defined upon the entire space $\mathcal{S}'_2$.

By conjugating, we obtain the conjugate Schrödinger equation. The relativistic conjugate Schrödinger equation

$$\bar{\mathcal{E}}_H : -\hat{E} = \hat{H},$$

corresponding to the Hamiltonian operator $\hat{H}$, will be defined by

$$\imath\hbar \frac{\partial}{\partial \bar{\mathrm{t}}} \psi = c\sqrt{P^2 + (m_0 c)^2 \mathbb{I}}\ \psi,$$

for any $\psi \in \mathcal{S}'(\mathbb{M}_2)$, where $\bar{\mathrm{t}}$ is the reverse time coordinate of t. Also the above equation will be not a differential equation but a linear (continuous) operator equation defined upon the entire space $\mathcal{S}'(\mathbb{M}_2)$. A complex probability wave ψ solves $\mathcal{E}_H$ iff $\bar{\psi}$ solves $\bar{\mathcal{E}}_H$. Any solution of the above two equations will solve the Klein-Gordon equation.

4.1. **Solutions of the free relativistic Schrödinger equation.** Our relativistic Schrödinger equation $\mathcal{E}_H$ admits every normalized de Broglie wave

$$\chi_{\boldsymbol{p}} = \frac{1}{2\pi\hbar} e^{(\imath/\hbar)(\boldsymbol{p}\mathrm{x} - H_{\boldsymbol{p}}\mathrm{t})},$$

with momentum $\boldsymbol{p}$ and energy

$$H_{\boldsymbol{p}} := H(\boldsymbol{p}),$$

as solution, (for every real E). Moreover, any Schwartz superposition of the family χ will solve the relativistic Schrödinger equation.

4.2. **Lorentz invariant forms of relativistic equation.** Moreover, the operator equation $\mathcal{E}_H$, in its four dimensional version, possesses Lorentz invariant natural forms:

$$\sqrt{\sum_{\mu \in I_4} \hat{p}_\mu \hat{p}^\mu} = m_0 c\ \mathbb{I}, \qquad \imath\hbar\sqrt{\sum_{\mu \in I_4} \partial_\mu \partial^\mu} = m_0 c\ \mathbb{I},$$

where

$$\hat{p} = (\hat{p}^\mu)_{\mu \in I_4} = (m_0 c\,\Gamma, P_1, P_2, P_3)\Gamma = \frac{\hat{E}}{m_0 c^2}$$

is the (contravariant) four-momentum operator of the particle and

$$\partial = (\partial_\mu)_{\mu \in I_4}, \qquad I_4 = \{0, 1, 2, 3\},$$

represents the four-gradient of Minkowski vector space.

In the second invariant form of $\mathcal{E}_H$, the Minkowski product

$$\boldsymbol{\mu}(\partial, \partial) = \sum_{\mu \in I_4} \partial_\mu \partial^\mu = \partial_0^2 - \nabla^2$$

is a negative operator and its principal root is defined by

$$\sqrt{\boldsymbol{\mu}(\partial, \partial)} := \imath \sqrt{-\boldsymbol{\mu}(\partial, \partial)},$$

taking into account that the Minkowski product

$$-\boldsymbol{\mu}(\partial, \partial)$$

is a positive operator and we can define (see appendices) the square root for positive operators.

Both Lorentz invariant equations come from the energy operator balance relation

$$\sqrt{\hat{E}^2 - \sum_{\mu \in I_4^*} (cP_\mu)^2} = m_0 c^2 \ \mathbb{I}, \qquad I_4^* = \{1, 2, 3\},$$

written in operator form, and defined at every tempered distribution ψ belonging to the domain

$$\operatorname{dom} \sqrt{\boldsymbol{\mu}(\partial, \partial)} = \{\psi \in \mathcal{S}_4' \mid (\psi)_\beta|_{\ker(\mathbf{p})} = 0\},$$

that is the subspace of the tempered distributions whose momentum representation vanishes in a neighborhood of the kernel of the momentum standard projection.

The Lorentz invariant forms, in 2 dimensions, are straightforwardly analogous.

For regular smooth states ψ "slowly increasing at infinity" – that is, for wavefunctions

$$\psi \colon \mathbb{M}_2 \to \mathbb{C}$$

belonging to the function space $\mathcal{O}_M(\mathbb{M}^2, \mathbb{C})$ – we prove, in the above context, some equivalent probability conservation theorems in the form of classic continuity equations of the type

$$\frac{\partial \rho_\psi}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathbf{x}} \mathbf{j}_\psi = \sigma_\psi, \qquad \frac{\partial \rho_\psi}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathbf{x}} \mathbf{j}_\psi = 0, \qquad \frac{\partial \rho_\psi}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathbf{x}} (\rho \mathbf{v}_\psi) = 0,$$

concerning just the classic Born probability density

$$\rho_\psi = \bar{\psi}\psi,$$

similarly to the non-relativistic Schrödinger equation.

This is the main advance in the matter, thanks to the new relativistic Schrödinger equation, with respect to the Klein-Gordon equation, which allows indeed to recover a continuity equation, but not for the Born probability equation, and indeed, not for a probability field at all.

4.3. **First continuity equation with source term.** Explicitly, we obtain our first conservation relation $\mathbf{C}_1$ in the form

$$\mathbf{C}_1\colon \frac{\partial}{\partial \mathrm{t}}(\bar{\psi}\psi) + \frac{\partial}{\partial \mathbf{x}}(\bar{\psi}\ \mathbf{V}\psi - \psi\ \mathbf{V}\bar{\psi}) = \frac{\partial \bar{\psi}}{\partial \mathbf{x}}\ \mathbf{V}\psi - \frac{\partial \psi}{\partial \mathbf{x}}\ \mathbf{V}\bar{\psi},$$

defined for every smooth "(at most) slowly increasing" complex field ψ, belonging to the domain $\mathcal{V}$ of the "phase velocity operator" $\mathbf{V}$ and true for every such wavefunction ψ solving the relativistic Schrödinger equation.

Above, the "relativistic phase velocity operator" $\mathbf{V}$ is defined by

$$\mathbf{V} = c\frac{\sqrt{P^2 + P_0^2}}{P} = \frac{\hat{H}}{P^2}P,$$

upon a convenient subspace $\mathcal{V}$ of the tempered distribution space $\mathcal{S}'(\mathbb{M}_2)$, roughly speaking, the largest subspace in which the momentum operator possesses an inverse operator (a sufficiently large subspace to contain every de Broglie wave $\beta_{(E/c,\boldsymbol{p})}$, with the exception of those with $\boldsymbol{p} = 0$).

Specifically, the subspace

$$\mathcal{V} = \{\psi \in \mathcal{S}'(\mathbb{M}_2) \mid (\psi)_\beta|_{\ker(\mathbf{p})} = 0\},$$

will be our choice for the relativistic phase velocity operator's domain, where $\mathbf{p}$ represents the usual second projection and the expression

$$a|_{\ker(\mathbf{p})} = 0$$

means that there exists an open neighborhood of the kernel $\ker \mathbf{p}$ in which the distribution $a \in \mathcal{S}'(\mathbb{P}_2)$ vanishes.

Observe that the kernel of the projection $\mathbf{p}$ is nothing more than the "energy" axis of the Minkowsky momentum space $\mathbb{P}_2$.

4.4. **Non-relativistic case.** Our continuity equation is valid in relativistic conditions and reduces to the non-relativistic Schrödinger continuity equation

$$\mathbf{C}\colon \frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}\frac{\bar{\psi}\ P\psi - \psi\ P\bar{\psi}}{2m} = 0,$$

when adopting the non-relativistic Schrödinger Hamiltonian

$$\hat{H}' = \frac{1}{2m}P^2,$$

instead of our relativistic (principal root) Hamiltonian operator $\hat{H}$ above. The non-relativistic operator $\hat{H}'$ determines an associated non-relativistic phase-velocity operator

$$\mathbf{V}' = \frac{\hat{H}'}{P} = \frac{1}{2m}P.$$

Note that, only by a fortuitous chance - a singularly lucky accident - the source term

$$\frac{\partial\bar{\psi}}{\partial\mathbf{x}}\,\mathbf{V}'\psi - \frac{\partial\psi}{\partial\mathbf{x}}\,\mathbf{V}'\bar{\psi} = -\frac{\imath\hbar}{2m}\left[\frac{\partial\bar{\psi}}{\partial\mathbf{x}}\frac{\partial}{\partial\mathbf{x}}\psi - \frac{\partial\psi}{\partial\mathbf{x}}\frac{\partial}{\partial\mathbf{x}}\bar{\psi}\right]$$

vanishes, in the non-relativistic case, since only in the non-relativistic approximation the phase velocity operator is luckily and simply proportional to the momentum operator P. In the relativistic case, we need now to manage and interpret the non-zero, endogenous source terms.

5. Partial inversion of the momentum operator

In order to better understand the definition of the relativistic phase velocity operator (which requires the division by the momentum operator of the Hamiltonian operator), we firstly desire to examine a possible meaning of the inverse momentum operator.

The momentum operator P, in the space $\mathcal{S}'(\mathbb{M}_2)$ of tempered distributions, reveals linear, continuous, surjective but not injective. We need to restrict the domain and co-domain of P in order to obtain a bijective momentum operator. Let

$$\mathcal{V} = \{\psi \in \mathcal{S}'(\mathbb{M}_2) \mid (\psi)_\beta|_{\ker(\mathbf{p})} = 0\}$$

be our choice for the new domain, where $\mathbf{p}$ represents the usual second projection and the expression

$$a|_{\ker(\mathbf{p})} = 0$$

means that there exists an open neighborhood of the kernel $\ker\mathbf{p}$ in which the distribution $a \in \mathcal{S}'(\mathbb{P}_4)$ vanishes.

Theorem 5.1. *The restriction of the Momentum operator to $\mathcal{V}$ is injective.*

Proof. In order to prove that the restriction of momentum operator to $\mathcal{V}$ is injective, we need to show that the only wave in $\mathcal{V}$ sent to 0 by P is the null wave. Indeed, if

$$P(\psi) = 0,$$

then

$$\mathbf{p}(\psi)_\beta = 0.$$

Consequently, by the distribution zero product property, we see that

$$(\psi)_\beta = 0$$

upon the co-level 0 of $\mathbf{p}$; but, the representation ψ_β is zero, by definition of $\mathcal{V}$, also on an open neighborhood Ω of $\ker \mathbf{p}$ and hence – by the locality of distributions – the component distribution ψ_β should equal zero upon the union

$$\Omega \ \cup \ {}^c \ker \mathbf{p},$$

that means just everywhere. □

We can sum up that the operator

$$P_\mathcal{V} \colon \mathcal{V} \to P\mathcal{V}, \qquad \psi \mapsto P\psi,$$

is invertible. Henceforth

$$P_\mathcal{V}^{-1} \colon P\mathcal{V} \to \mathcal{V}, \qquad P\psi \mapsto \psi,$$

moreover

$$P(P_\mathcal{V}^{-1}\varphi) = \varphi,$$

as soon as $\varphi \in P\mathcal{V}$.

We can unfold the definition of the inverse operator $P_\mathcal{V}^{-1}(\psi)$, for every ψ in the domain $\mathcal{V}$ of $P_\mathcal{V}^{-1}$, as follows

$$P_\mathcal{V}^{-1}(\psi) = \int_{\mathbb{P}^2} (f/\mathbf{p})\,\tilde{}\,\, (\psi)_\beta \ \beta,$$

where:

- the component system $(\psi)_\beta$ vanishes upon an open neighborhood Ω of the kernel of the projection $\mathbf{p}$;
- f is any smooth "cutoff" function which is
  - (a) equal 0 upon the neighborhood Ω_3 (of $\ker \mathbf{p}$) obtained by Ω by contracting the second coordinate according to the factor $1/3$ (ψ_β vanishes in Ω_3)
  - (b) equal 1 outside the open neighborhood Ω_2 obtained by Ω by contracting the second coordinate according to the factor $1/2$ (it contains Ω_3 and the coordinate distribution ψ_β still vanishes there);
- the product function

  $$f/\mathbf{p} = (1/\mathbf{p})f$$

  reveals
  - (1) everywhere defined upon $\mathbb{P}_2$, except on $\ker \mathbf{p}$;
  - (2) equal 0 upon Ω_3, except on the kernel of $\mathbf{p}$;
  - (3) equal $1/\mathbf{p}$ outside Ω_2.

  Therefore, we can consider the unique smooth prolongation $(f/\mathbf{p})\tilde{}$ of $f/\mathbf{p}$, which reveals:

(1) everywhere defined upon $\mathbb{P}_2$;
(2) equal 0 upon Ω_3;
(3) equal to $1/\mathbf{p}$ outside Ω_2;
(4) smooth and slowly increasing at infinity.

Hence, the tempered distribution

$$(f/\mathbf{p})\tilde{}\ (\psi)_\beta$$

exists as the standardly defined product of an $\mathcal{O}_M$ function times a tempered distribution. We observe that the above distribution vanishes on Ω and its restriction to the complement of $\bar{\Omega}$ equals the distribution

$$(1/\mathbf{p})|_{(\mathbb{P}_2\setminus\bar{\Omega})}\ \psi_\beta|_{(\mathbb{P}_2\setminus\bar{\Omega})}.$$

We can now formalize the result.

Theorem 5.2. *In the above conditions, we have*

$$P_{\mathcal{V}}^{-1}(\psi) = \int_{\mathbb{P}^2} (f/\mathbf{p})\tilde{}\ (\psi)_\beta\ \beta,$$

for every ψ in $\mathcal{V}$.

Proof. To prove that the above equality holds true, we apply the Momentum operator to the superposition

$$\int_{\mathbb{P}^2} (f/\mathbf{p})\tilde{}\ (\psi)_\beta\ \beta,$$

we get

$$P\int_{\mathbb{P}^2} (f/\mathbf{p})\tilde{}\ (\psi)_\beta\ \beta = \int_{\mathbb{P}^2} \mathbf{p}\ (f/\mathbf{p})\tilde{}\ (\psi)_\beta\ \beta = \int_{\mathbb{P}^2} f\ (\psi)_\beta\ \beta = \int_{\mathbb{P}^2} (\psi)_\beta\ \beta = \psi,$$

for every $\psi \in \mathcal{V}$, after observing that $f\psi_\beta = \psi_\beta$.

Analogously, we see that

$$\int_{\mathbb{P}^2} (f/\mathbf{p})\tilde{}\ (P\psi)_\beta\ \beta = \int_{\mathbb{P}^2} (f/\mathbf{p})\tilde{}\ \mathbf{p}\,(\psi)_\beta\ \beta = \int_{\mathbb{P}^2} f\ (\psi)_\beta\ \beta = \int_{\mathbb{P}^2} (\psi)_\beta\ \beta = \psi,$$

for every $\psi \in \mathcal{V}$, after observing that

$$(P\psi)_\beta = \mathbf{p}\,(\psi)_\beta.$$

This concludes the proof. □

In general, we can define the following integral (superposition) upon the complement of a smooth graph.

Definition 5.3. Suppose $a \in \mathcal{S}'_m$, $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$, and h is a complex smooth function slowly increasing at infinity, defined upon the (open) complement

$$^cG = \mathbb{R}^m \setminus G,$$

of a smooth graph G. We can consider the integral (superposition)

$$\int_{^cG} (ha)v,$$

as soon as a vanishes upon G, that is, upon a neighborhood of G. We use a cutoff function f analogous to that used before and define

$$\int_{^cG} (ha)\, v := \int_{\mathbb{R}^m} (fh)\tilde{}\; a\, v,$$

as we have chosen before.

6. Definition of the phase velocity operator

Here we state and prove the relativistic continuity equation for the probability density ρ_ψ of an $\mathcal{O}_M$ state ψ.

Here, we desire to underline the key points of the statement.

- The statement is compatible with the non-relativistic Schrödinger equation, in the sense that, if we substitute the non-relativistic Hamiltonian instead of our relativistic Hamiltonian $\hat{H}$, we obtain exactly the well known continuity equation of the non-relativistic Schrödinger equation, with the source term sigma equal to zero, as in the classic non-relativistic case.
- In our statement we need to introduce the (relativistic) phase-velocity operator

  $$\mathbf{V} = \frac{\hat{H}}{P} = c\frac{\sqrt{P^2 + P_0^2}}{P}.$$

  We can define the phase velocity operator thanks to the *Schwartz simultaneous diagonalizability of the Hamiltonian operator $\hat{H}$ and momentum operator P.* The only serious difficult to overcome – from a mathematical point of view – is the exotic division by the momentum operator P, whose system of eigenvalues vanishes at every de Broglie wave of type $\beta_{(E/c,0)}$. In other terms, on the way to define the new operator

  $$\mathbf{V} = \hat{H}/P$$

  on every member of the energy-momentum basis β, we must taking into account that for every wave $\beta_{E/c,\boldsymbol{p}}$, with $(E/c, p)$ in the kernel of the projection $\mathbf{p}$, we would obtain

$$P\beta_{E,\boldsymbol{p}} = 0.$$

The previous difficulty will be eliminated by containing the domain $\mathcal{V}$ of the phase-velocity operator $\mathbf{V}$ to those state vectors ψ whose component systems $(\psi)_\eta$ vanish in an open neighborhood of the "energy axis" $\ker \mathbf{p}$, which is a closed subset of the energy-momentum plane. Then, we simply write down the definition for every psi in the domain $\mathcal{V}$ of $\mathbf{V}$, we can define:

$$\mathbf{V}(\psi) = c \int_{\mathbb{R}^2 \setminus \ker \mathbf{p}} \frac{\sqrt{\mathbf{p}^2 + (m_0 c)^2}}{\mathbf{p}} \ (\psi)_\beta \ \beta,$$

for every ψ such that the component system $(\psi)_\beta$ vanishes upon the kernel of the projection $\mathbf{p}$ - where E and $\mathbf{p}$ are the energy and momentum standard plane projections (coordinates).

7. Proof of the first continuity equation

Now, we can state and prove our theorem.

Theorem 7.1. *Let $\psi \in \mathcal{V}$ be a smooth $\mathcal{O}_M$ solution of the relativistic Schrödinger equation,*

$$\hat{E} = \hat{H},$$

in distribution space $\mathcal{S}'_2$. Then, the solution ψ satisfies the continuity equation

$$\frac{\partial}{\partial \mathrm{t}} \rho_\psi + \frac{\partial}{\partial \mathbf{x}} \ \boldsymbol{J}_\psi = (\boldsymbol{\sigma}_\psi - \boldsymbol{\sigma}_{\bar{\psi}}),$$

where:

- *the density ρ_ψ is the "smooth slowly-increasing" function defined, following Born, by*

$$\rho_\psi = \psi \bar{\psi} = |\psi|^2;$$

- *the probability current $\boldsymbol{J}_\psi$ is defined by*

$$\boldsymbol{J}_\psi = \bar{\psi} \ \mathbf{V}\psi - \psi \ \mathbf{V}\bar{\psi} = 2 \ \mathcal{R}\left(\bar{\psi} \ \mathbf{V}\psi\right),$$

via the phase-velocity operator

$$\mathbf{V} \colon \mathcal{V} \to \mathcal{S}'_2, \qquad \mathbf{V} = c\frac{\sqrt{P^2 + P_0^2}}{P},$$

with

$$\mathcal{V} = \{\psi \in \mathcal{S}'(\mathbb{M}_2) \mid (\psi)_\eta|_{\ker(\mathbf{p})} = 0\}, \qquad P_0 = m_0 c^2 \mathbb{I};$$

- *the source term* $\boldsymbol{\sigma}_\psi - \boldsymbol{\sigma}_{\bar{\psi}}$ *is defined by*

$$\boldsymbol{\sigma}_\psi - \boldsymbol{\sigma}_{\bar{\psi}} = \frac{\partial \bar{\psi}}{\partial \mathbf{x}} \ \mathbf{V}\psi - \frac{\partial \psi}{\partial \mathbf{x}} \ \mathbf{V}\bar{\psi} = 2 \ \mathcal{R} \left(\frac{\partial \bar{\psi}}{\partial \mathbf{x}} \ \mathbf{V}\psi \right).$$

If the smooth function ψ *reveal invertible with respect to the point-wise product, then the equation assume the more familiar form*

$$\frac{\partial}{\partial \mathrm{t}} \rho_\psi + \frac{\partial}{\partial \mathbf{x}} \ (\rho_\psi \mathbf{v}_\psi) = 2 \ \mathcal{R} \left(\boldsymbol{\sigma}_\psi \right),$$

where:

$$\mathbf{v}_\psi = \frac{\mathbf{V}\psi}{\psi} - \frac{\mathbf{V}\bar{\psi}}{\bar{\psi}} = 2 \ \mathcal{R} \left(\frac{\mathbf{V}\psi}{\psi} \right).$$

Proof. We know that

$$\imath\hbar c \ \partial_0 \psi = \hat{H}\psi = c\sqrt{P^2 + P_0^2} \ (\psi) = cP \frac{\sqrt{P^2 + P_0^2}}{P} \ (\psi),$$

for every solution ψ of our evolution equation, belonging to the subspace $\mathcal{V}$. We can rewrite the above identity as

$$c\partial_0 \psi = -\frac{\partial}{\partial \mathbf{x}} \mathbf{V}\psi,$$

the complex conjugate of the above is

$$c\partial_0 \bar{\psi} = -\frac{\partial}{\partial \mathbf{x}} \overline{\mathbf{V}\psi} = \frac{\partial}{\partial \mathbf{x}} \mathbf{V}\bar{\psi}.$$

From the above two relations we get

$$c\bar{\psi}\partial_0 \psi = -\bar{\psi} \frac{\partial}{\partial \mathbf{x}} \mathbf{V}\psi, \qquad\qquad c\psi\partial_0 \bar{\psi} = \psi \frac{\partial}{\partial \mathbf{x}} \mathbf{V}\bar{\psi}.$$

Adding the two above relations, we obtain

$$c\partial_0 (\bar{\psi}\psi) = -\bar{\psi} \frac{\partial}{\partial \mathbf{x}} \mathbf{V}\psi + \psi \frac{\partial}{\partial \mathbf{x}} \mathbf{V}\bar{\psi}.$$

Completing the derivatives, we obtain immediately the result. For instance,

$$\bar{\psi} \ \frac{\partial}{\partial \mathbf{x}} \mathbf{V}\psi = \frac{\partial}{\partial \mathbf{x}} (\bar{\psi} \ \mathbf{V}\psi) - \frac{\partial}{\partial \mathbf{x}} \bar{\psi} \ \mathbf{V}\psi,$$

as we desired. □

7.2. **Real forms of continuity equation $\mathbf{C}_1$.** The continuity equation $\mathbf{C}_1$ can be written, in an explicitly real-valued form, as:

$$\mathbf{C}_1 \colon \frac{\partial}{\partial \mathrm{t}}(\bar{\psi}\psi) + \frac{\partial}{\partial \mathbf{x}} 2\, \mathcal{R}\left(\bar{\psi}\ \mathbf{V}\psi\right) = 2\, \mathcal{R}\left(\frac{\partial \bar{\psi}}{\partial \mathbf{x}}\ \mathbf{V}\psi\right),$$

defined for every smooth "at most slowly increasing" complex field ψ, belonging to the domain $\mathcal{V}$ of the "phase velocity operator" $\mathbf{V}$ and true for every such wavefunction ψ solving the relativistic Schrödinger equation $\mathcal{E}_H$.

Remark 7.3. We need the $\mathcal{O}_M$ assumption, of "contained" increasing behavior, for our smooth complex wave field ψ, just to guarantee that the image $\mathbf{V}\psi$, of the wave ψ by the phase-velocity operator still remain a smooth $\mathcal{O}_M$ function. We could think to adopt only smooth waves rapidly decreasing at infinity (which most of the time reveal enough for Quantum Mechanics) nevertheless we desire to take into our arena the de Broglie type waves, which are bounded but not vanishing at infinity.

8. The second form of continuity equation

The second continuity equation form $\mathbf{C}_2$ is defined by:

$$\mathbf{C}_2 \colon \frac{\partial}{\partial \mathrm{t}}(\bar{\psi}\psi) + \frac{\partial}{\partial \mathbf{x}} \mathcal{R}\left(\bar{\psi}\ \mathbf{V}\psi\right) = \mathcal{R}\left[\frac{\partial}{\partial \mathrm{x}}\bar{\psi} \cdot (c\psi, \mathbf{V}\psi)\right],$$

defined for every smooth "at most slowly increasing" complex field ψ, belonging to the domain $\mathcal{V}$ of the "phase velocity operator" $\mathbf{V}$ and true for every such wavefunction ψ solving the relativistic Schrödinger equation. The symbol $\partial/\partial\mathrm{x}$ represents the bi-gradient operator with respect to the canonical coordinate chart

$$\mathrm{x} = (c\mathrm{t}, \mathbf{x}) \colon \mathbb{M}_2 \to \mathbb{C}^2, \qquad (ct, \boldsymbol{x}) \mapsto (ct + 0\imath, \boldsymbol{x} + 0\imath).$$

We indicate by $\cdot$ the Euclidean scalar product in $\mathbb{M}_2$.

The above equation $\mathbf{C}_2$ reveals of the form

$$\frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}\mathcal{R}\left(\boldsymbol{\jmath}_\psi\right) = \mathcal{R}\sigma_\psi,$$

where

$$\boldsymbol{\jmath}_\psi := \bar{\psi}\mathbf{V}\psi, \qquad \sigma_\psi := (\partial/\partial\mathrm{x})\bar{\psi} \cdot \mathscr{V}\psi = d\bar{\psi}\,(\mathscr{V}\psi), \qquad \mathscr{V}\psi := (c\mathbb{I}\,\psi, \mathbf{V}\psi).$$

Both continuity equations $\mathbf{C}_1$ and $\mathbf{C}_2$ are equivalent to the relativistic Schrödinger equation $\mathcal{E}_H$ restricted to the subspace

$$\mathcal{V}\ \cap\ \mathcal{O}_M(\mathbb{M}_2, \mathbb{C}),$$

of the wide tempered distribution (relativistic) state space $\mathcal{S}'(\mathbb{M}_2, \mathbb{C})$.

8.1. **The non-conservative phase velocity gauge.** Here, we have fixed a **gauge choice**, for what concerns the non-conservative probability current $\boldsymbol{\jmath}_\psi$.

Indeed, that current can be defined as an equivalence class C of gauge-choices in the space $\mathcal{O}_M(\mathbb{M}_2,\mathbb{C})$: two choices of C differing from each other by a complex smooth $\mathcal{O}_M$ complex field $f\colon \mathbb{M}_2 \to \mathbb{C}$ such that

$$\frac{\partial}{\partial \mathbf{x}} f = 0.$$

We call this gauge choice

$$\boldsymbol{\jmath}_\psi := \bar{\psi}\ \mathbf{V}\psi$$

"*the non-conservative phase velocity gauge*".

8.2. **Lorentz invariant form of the source term.** The source term σ_ψ can be viewed, in a relativistic more convenient fashion, as a Minkowski inner product

$$\sigma_\psi := \frac{\partial}{\partial \mathrm{x}}\ \bar{\psi}\cdot \mathscr{V}\psi = d\bar{\psi}\,(\mathscr{V}\psi) = \frac{1}{\imath\hbar}\ \boldsymbol{\mu}(\hat{p}\bar{\psi},\mathscr{V}\psi) = \frac{1}{\imath\hbar}\ \langle(\hat{p}\bar{\psi})^\flat,\mathscr{V}\psi\rangle,$$

where $\hat{p}$ represent the 2-momentum operator and $(\hat{p}\bar{\psi})^\flat$ is the covariant vector field canonically associated with the contravariant vector field $\hat{p}\bar{\psi}$ – with respect to the Minkowski inner product $\boldsymbol{\mu}$ – which naturally acts upon the contravariant vector fields as a linear differential form (linear form field).

Remark 8.3. We invite the reader to appreciate the perfect duality created by the pair of inner products $(\cdot,\boldsymbol{\mu})$, when moving from a purely differential operator view to a quantum mechanics observable one.

8.4. **The conjugate equation $\bar{\mathbf{C}}_2$.** For what concerns the conjugate wave $\bar{\psi}$, we observe that

$$\boldsymbol{\jmath}_{\bar{\psi}} := \psi\ \mathbf{V}\bar{\psi} = -\psi\ \overline{\mathbf{V}\psi}\ = -\overline{\boldsymbol{\jmath}_\psi}.$$

On the contrary, we apparently don't note any possible natural relation between the two sources σ_ψ and $\sigma_{\bar{\psi}}$. But, if we consider the "conjugate source"

$$\bar{\sigma}_\psi := (\partial/\partial\bar{\mathrm{x}})\ \bar{\psi}\cdot\mathscr{V}\psi, \qquad \mathscr{V}\psi := (c\mathbb{I}\ \psi,\mathbf{V}\psi), \qquad \bar{\mathrm{x}} = (-\mathrm{t},\mathbf{x}),$$

obtained by inverting the time coordinate t, we immediately obtain the continuity equation for the conjugate wavefunction $\bar{\psi}$:

$$\bar{\mathbf{C}}_2\colon \frac{\partial}{\partial\bar{\mathrm{t}}}(\rho_{\bar{\psi}}) + \frac{\partial}{\partial\mathbf{x}}\mathcal{R}\left(\psi\ \mathbf{V}\bar{\psi}\right) = \mathcal{R}[(\partial/\partial\bar{\mathrm{x}})\psi\cdot(c\bar{\psi},\mathbf{V}\bar{\psi})],$$

defined for every smooth wave in $\mathcal{O}_M \cap \mathcal{V}$ and true for every solution ψ of the conjugate relativistic evolution equation $\bar{\mathcal{E}}_H$ which belongs to that intersection.

That kind of time conjugation, somehow, does not surprise, since the conjugate Schrödinger equation $\bar{\mathcal{E}}$ comes from the Schrödinger equation $\mathcal{E}$ also by simply inverting the time coordinate.

9. Third form of the continuity equation

We can see the first continuity equation $\mathbf{C}_1$ as the difference of two complex "conjugated" continuity equations, relative to the propagation of ψ and its conjugate $\bar{\psi}$, respectively:

$$\mathbf{C}_3\colon \frac{\partial}{\partial \mathrm{t}}(\bar{\psi}\psi) + \frac{\partial}{\partial \mathbf{x}}(\bar{\psi}\ \mathbf{V}\psi) = \frac{\partial \bar{\psi}}{\partial \mathrm{t}}\ \psi + \frac{\partial \bar{\psi}}{\partial \mathbf{x}}\ \mathbf{V}\psi,$$

for the propagation of ψ, future directed and

$$\bar{\mathbf{C}}_3\colon \frac{\partial}{\partial \bar{\mathrm{t}}}(\psi\bar{\psi}) + \frac{\partial}{\partial \mathbf{x}}(\psi\ \mathbf{V}\bar{\psi}) = \frac{\partial \psi}{\partial \bar{\mathrm{t}}}\ \bar{\psi} + \frac{\partial \psi}{\partial \mathbf{x}}\ \mathbf{V}\bar{\psi},$$

for the propagation of $\bar{\psi}$, past directed, where $\bar{\mathrm{t}} = -\mathrm{t}$.

Remark 9.1. In this paper, we shall interpret any of the above twin equations in two different ways: a non-conservative one and a conservative one. Here (in section 3) we are discussing about the non-conservative interpretation. In the next sections we shall deal with the conservative ones.

In the above non-conservative forms, focusing on the first equation $\mathbf{C}_3$, we therefore note an endogenous probability source

$$\sigma_\psi = (\partial/\partial \mathrm{x})\bar{\psi} \cdot (c\psi, \mathbf{V}\psi),$$

with a non-conservative probability current (within the phase velocity gauge)

$$\boldsymbol{\jmath}_\psi = \bar{\psi}\ \mathbf{V}\psi,$$

for the future directed wave ψ and analogously for the past directed $\bar{\psi}$.

We observe that equations $\mathbf{C}_2$ and its time conjugate derive immediately from equations $\mathbf{C}_3$ and $\bar{\mathbf{C}}_3$, just applying the real part operator $\mathcal{R}$. Moreover, applying the imaginary part operator $\Im$, we get, respectively:

$$\Im\mathbf{C}_3\colon \Im\ \frac{\partial}{\partial \mathbf{x}}(\bar{\psi}\ \mathbf{V}\psi) = \Im\sigma_\psi$$

for the propagation of ψ, future directed and

$$\Im\bar{\mathbf{C}}_3\colon \Im\ \frac{\partial}{\partial \mathbf{x}}(\psi\ \mathbf{V}\bar{\psi}) = \Im\ \bar{\sigma}_{\bar{\psi}}$$

for the propagation of $\bar{\psi}$, past directed. Therefore, in equations $\mathbf{C}_3, \bar{\mathbf{C}}_3$, all the imaginary parts cancel each other. Therefore, equations $\mathbf{C}_3, \bar{\mathbf{C}}_3$ are indeed equal (not only equivalent) to equations $\mathbf{C}_2, \bar{\mathbf{C}}_2$ and they are, so, actually two real-valued equations, although the fields $\boldsymbol{\jmath}_\psi$ and σ_ψ remains two veritable complex fields.

9.2. **Lorentz invariant form of the equation $\mathbf{C}_3$.** As we already proved, the source term σ_ψ can be viewed, in a relativistic more convenient fashion, as the Minkowski inner product

$$\sigma_\psi = \frac{1}{\imath\hbar}\, \boldsymbol{\mu}(\hat{p}\bar{\psi}, \mathscr{V}\psi) = \frac{1}{\imath\hbar}\, \langle (\hat{p}\bar{\psi})^\flat, \mathscr{V}\psi \rangle.$$

We can immediately write down two elegant Lorentz invariant forms of $\mathbf{C}_3$:

$$\boldsymbol{\mu}(\hat{p}, \bar{\psi}\mathscr{V}\psi) = \boldsymbol{\mu}(\hat{p}\bar{\psi}, \mathscr{V}\psi),$$

where the 2-momentum operator $\hat{p}$ acts on the bi-current (in the phase velocity gauge)

$$\jmath_\psi = \bar{\psi}\mathscr{V}\psi = (\bar{\psi}\mathscr{V}^0\psi,\ \bar{\psi}\mathscr{V}^1\psi),$$

as the covariant vector $\hat{p}^\flat$ operator $\boldsymbol{\mu}$-associated with the contravariant vector operator $\hat{p}$, and

$$\langle \partial, \bar{\psi}\mathscr{V}\psi \rangle = \langle \partial\bar{\psi}, \mathscr{V}\psi \rangle,$$

where the gradient of $\bar{\psi}$ acts as a covariant vector fields upon the contravariant vector field $\mathscr{V}\psi$.

We can also write:

$$\partial \cdot \jmath_\psi = d\bar{\psi}\,(\mathscr{V}\psi),$$

where, at the left hand side, we see the canonical divergence of the bi-current $\jmath_\psi$ and, at the right hand side, we see the standard differential of the wave $\bar{\psi}$ calculated at the phase velocity vector field $\mathscr{V}\psi$. While, treating $\partial\bar{\psi}$, as a vector field of $\mathbb{M}_2$, we can write:

$$\partial \cdot \jmath_\psi = \partial\bar{\psi} \cdot \mathscr{V}\psi.$$

10. Non-conservative velocity fields

From our continuity equations, when the wave ψ is multiplicatively invertible, we deduce the respective continuity equations with explicit flux velocity fields.

We obtain the following equations:

- in the continuity equation $\mathbf{C}_1$, we read:

$$\frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}\left[\rho_\psi \frac{\mathbf{V}\psi}{\psi} - \rho_{\bar{\psi}}\frac{\mathbf{V}\bar{\psi}}{\bar{\psi}}\right] = \frac{\partial\bar{\psi}}{\partial \mathbf{x}}\, \mathbf{V}\psi - \frac{\partial\psi}{\partial \mathbf{x}}\, \mathbf{V}\bar{\psi}.$$

It can be also viewed in the classic form

$$\frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}(\rho_\psi \mathbf{v}_\psi - \rho_{\bar{\psi}}\mathbf{v}_{\bar{\psi}}) = \boldsymbol{\sigma}_\psi - \boldsymbol{\sigma}_{\bar{\psi}},$$

which is real-valued by its nature and characterized by a non-conservative current

$$\mathbf{j}_\psi = \bar{\psi}\mathbf{V}\psi = \rho_\psi \mathbf{v}_y,$$

our "non-conservative phase velocity gauge current". We can also write

$$\frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}(\rho_\psi \mathbf{u}_\psi) = 2\mathcal{R}\boldsymbol{\sigma}_\psi,$$

where

$$\mathbf{u}_\psi := \mathbf{v}_\psi - \mathbf{v}_{\bar{\psi}} = 2\mathcal{R}\mathbf{v}_\psi$$

is the real general non-conservative velocity field of the probability flux $\mathbf{j}_\psi$.

- explicitly real-valued continuity equations $\mathbf{C}_2$:

$$\frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}\mathcal{R}(\rho_\psi \mathbf{v}_\psi) = \mathcal{R}(\sigma_\psi),$$

and

$$\frac{\partial}{\partial \bar{\mathrm{t}}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}\mathcal{R}(\rho_{\bar{\psi}} \mathbf{v}_{\bar{\psi}}) = \mathcal{R}(\bar{\sigma}_{\bar{\psi}}),$$

which are a rewriting of the above equation $\mathbf{C}_1$ by two endogenous probability sources

$$\sigma_\psi = (\partial/\partial \mathrm{x})\bar{\psi}\cdot(c\psi, \mathbf{V}\psi), \qquad \bar{\sigma}_{\bar{\psi}} = (\partial/\partial \bar{\mathrm{x}})\psi \cdot (c\bar{\psi}, \mathbf{V}\bar{\psi}),$$

for every $\psi \in \mathcal{O}_M$;

- a pair of complex-valued continuity equations concerning a wavefunction ψ and its conjugate:

$$\begin{aligned} \mathbf{C}_3\colon\ & \frac{\partial}{\partial \mathrm{t}}\rho_\psi + \frac{\partial}{\partial \mathbf{x}}\ (\rho_\psi \mathbf{v}_\psi) = \sigma_\psi, \\ \bar{\mathbf{C}}_3(\bar{\psi})\colon\ & \frac{\partial}{\partial \bar{\mathrm{t}}}\rho_{\bar{\psi}} + \frac{\partial}{\partial \mathbf{x}}\ (\rho_{\bar{\psi}} \mathbf{v}_{\bar{\psi}}) = \bar{\sigma}_{\bar{\psi}}; \end{aligned}$$

- For what concerns the wave ψ, a 2-vector expression of the equations $\mathbf{C}_3$ reads:

$$\boldsymbol{\mu}\left(\hat{p}, \rho_\psi \frac{\mathscr{V}\psi}{\psi}\right) = \boldsymbol{\mu}\left(\hat{p}\bar{\psi}, \psi \frac{\mathscr{V}\psi}{\psi}\right), \qquad \langle \partial\,, \rho_\psi \frac{\mathscr{V}\psi}{\psi}\rangle \ = \ \langle \partial\bar{\psi}, \mathscr{V}\psi\rangle,$$

within a non-conservative gauge choice

$$\mathbf{j}_\psi = \bar{\psi}\mathscr{V}\psi = (\bar{\psi}\mathscr{V}^0\psi, \bar{\psi}\mathscr{V}^1\psi).$$

11. Discussion

11.1. **Dirac equation and its solution space issues.** The problem of a relativistic description of fermions (half spin particles) has been substantially cracked by Dirac in the twenties of the last century, by the formulation of the celebrated equation after his name.

We say "substantially" because Dirac couldn't specify properly the functional spaces in which his equation was defined, and that isn't a secondary issue for many reasons:

- firstly, as we well know, in any equation, even in algebraic equations upon finite dimensional spaces, the solution space reveals strongly dependent on the domain of the equation itself, so that, in Dirac equation, we don't even know officially where to find our solutions;
- secondarily, the very meaning and properties of the operators in any functional equation are strongly linked with the spaces in which we desire to consider and define the operators themselves;
- thirdly, for sure, the absence of a clear definition of the functional spaces in which the Dirac equation has been originally written, has determined a lot of comprehensible confusion and embarrassment among researchers, teachers, students and scholars , at the point that is almost universally accepted that the natural spaces of such equation should be the Hilbert spaces of square integrable functions, defined upon the four dimensional Minkowsky space, while the very basic solutions of this equation are (universally as well) recognized in the de Broglie waves, which are not square integrable at all! moreover, all of that mess in the very same page of any quantum physics book dealing with the celebrated equation! We, clearly, do not want to be over-precise and pedant as some mathematicians use to be, but, on the other hand, we neither want brutally contradict ourselves in the space of two adjacent lines of the same page, when dealing with such delicate theories of physics.

Therefore, we can conclude that, although the form of Dirac equation should be accepted as it is – in its naked beautiful algebraic relations – involving some (not better defined) Differential operators in Minkowski Space, we must admit that even the classic Dirac equation for fermions, after 100 years from its formulation, needs now a more careful treatment and understanding in its very fundamental formulation.

11.2. **Dirac equation and Pauli equation.** The Dirac equation can be rightfully considered the relativistic counterpart of the Pauli equation, although it strangely acts on the state space S^4, instead of the Pauli state space S^2, where S is the Schrödinger equation state space (here we pretend to know what that state space is ... but, useless to say, it cannot be a presumed magical Hilbert space of infinitely differentiable functions that are also square-integrable and containing the de Broglie plane waves).

11.3. **Dirac's solution to the square root problem for $1/2$ spin particles.** In 1928, Dirac came across an equation, which addressed the relativistic particles with 1/2 spin, only by an algebraic fortuitous coincidence: in order to eliminate the Einstein-Schrödinger problematic square root of the dispersion relation, it revealed enough the fourth power S^4 of the Schrödinger state space S. The state space S^4 – in Dirac equation – can be viewed as decomposable into $S^2 \times S^2$, the first space S^2 for the half spin particles with positive energies and the second space S^2 for the half spin particles with negative energies (whatever this meant).

Useless to say that the relativistic de Broglie waves (as components of state vectors in S^4) determine the very standard basis of Dirac equation solution space, although we cannot understand how to generate all possible solutions of Dirac equation, since we don't know the space S^4 and its algebraic topological structures: we do not know how vast the space S^4 could be and we do not know how the possible solutions can be approximated by finite superpositions of the basic de Broglie tetra-waves.

At this purpose, we want only to observe that, if the space S were a functional separable Hilbert space (the standard setting of quantum mechanics), then the de Broglie waves couldn't even belong to S (because they are non-normalizable with respect to the standard L^2 product; moreover the concept of continuous superposition of states couldn't find a rightful citizenship in S, because the only natural linear superpositions in Hilbert spaces are the Hilbert sums (denumerable sums) of (possibly orthogonal) sequences of states. Hilbert spaces can face only problems with a discrete basis of eigenstates (and with considerable fatigue, due to the circumstance that the differential operators are not everywhere defined and even not continuous there) and they cannot reasonably face problems with a continuous basis of eigenstates, which is exactly the case in the Dirac equation, it is a typical example in the very informal quantum theory of observables.

12. Conclusions

The article deals with a Schwartz distribution approach to some fundamental mathematical methods of quantum mechanics, helping to achieve a deeper understanding of the quantum physical structure.

Indeed, the adoption of distribution spaces rehabilitates, in a natural way, a lot of essential elements in the usual mathematical formulation (and construction) of quantum mechanics - bra-ket calculus with continuous indexing, generalized "delta"-orthogonality, generalized "continuous" linear combinations, generalized "continuous" bases, generalized Hermiticity, eigenvectors of "unbounded" quantum observable and practical quantum functional calculus - which cannot find any justification in separable Hilbert spaces (and, in fact, not even in non-separable Hilbert spaces, because in Hilbert spaces we don't have any reasonable or viable approach to the use of continuous bases, or continuous superpositions, or continuous matrices, or generalized inner products between generalized bases and vectors, not to talk about non-normalizable vectors in Hilbert spaces, where each vector is clearly normalizable).

Our approach aims at the recognition that the spaces traditionally used in quantum theory (to describe state spaces) are in fact not Hilbert spaces but the spaces of generalized functions (on convenient Euclidean spaces or Minkowski spaces, depending on the quantum system.)

The invaluable advantage of distribution spaces is not only that we can rightfully include, in this expanded context, some fundamental quantum states and procedures usually adopted in the context of quantum mechanics that, clearly, cannot live all together in any conceivable Hilbert space (the position basis, the momentum basis and the creation operator basis cannot live in the same Hilbert space, not even pair-wise!), but, more importantly, we can manipulate the intuitive and effective standard quantum approaches into a very efficient and widely used mathematical environment: Schwartz distribution theory.

Such distribution approaches were previously studied by the author in works [6, 22].

12.1. **A geometric approach.** The "new" geometric approach has been developed in the last 25 years and, at the end of the day, can be considered as a natural and straightforward prosecution of the Laurent Schwartz analytical study of generalized functions, with an eye on the mathematical methods of theoretical physics and to the standard theoretical architecture of Linear Algebra and Geometry.

Even the basic definition of continuous superposition (generalized linear combination, which is at the very foundations of our geometric structure) can be found in the Laurent Schwartz treatise, although just rapidly sketched and with a different aim (action of a usual distribution upon a vector valued function). The definition of continuous superposition is indeed a generalization of the Bourbaki's integral of a vector valued function with respect to a Radon measure.

In the article, the problem of "quantization of the relativistic Hamiltonian of a free massive particle" and the related Schrödinger equation is considered as a possible application of the specified Schwarz spaces.

Here, we only desire to observe that, in usual approaches to quantum mechanics, already the non-relativistic Schrödinger's equation is not even well specified in its domain: we don't even know if the differential operators written there are intended in the classic point-wise Leibniz-Newton-Cauchy sense or in a generalized Sobolev sense. For sure, in the usual practice of quantum mechanics, those operators are adopted in the classic calculus elementary meaning: that practice brutally excludes Hilbert spaces from the widely adopted quantum context and it brings to incompleteness, embarrassments and doubts, when the derivatives of the wavefunctions present some discontinuity.

On the other hand, the Hilbert space approach reveals totally incapable of providing differential operators with most desirable continuity or spectral properties.

Furthermore, the Sobolev Calculus in Hilbert spaces is incapable even to handle the necessity of actual eigenstates for the momentum operator (which essentially is a differential operator) or for the non-relativistic Hamiltonian operator of a free particle (they are the most elementary quantum operators, and they, from the Leibniz point of view in differentiable function spaces, possess the very same wonderful basis of eigenstates: the de Broglie family of harmonic waves!).

On the other hand, from both the classic Leibniz approach and Sobolev approach in Hilbert spaces, which essentially work in the "small" spaces of differentiable functions or in the small space of square integrable functions, we cannot provide a reasonable justification for the renowned and celebrated "position basis" of quantum mechanics, because its elements are not square integrable functions and neither smooth functions in the classical sense.

Summarizing: in usual contexts, we don't know even where to define and solve the elementary non-relativistic Schrödinger's equation, in first place, just think about the insurmountable difficulties in the relativistic case, where we didn't even know how to reasonable define formally the problematical square-root Einstein-Hamilton operator!

Schwartz distribution approach solves all the above problems at once.

12.2. **The main specific goal of our work.** For free 0-spin relativistic Particle we study possible representations of the continuity equation induced by the relativistic Schrödinger equation.

Since the mere existence of the relativistic Hamiltonian operator and its Hermiticity, on temperate distribution spaces, immediately imply the conservation of probability, in this paper, we study various different forms of continuity equation which presents a fictitious "source term", this we do in order to create a more stringent parallel-comparison between the classic non-relativistic treatment, here we desire to underline that the source term doesn't mean that the probability are not conserved, we have chosen a form of probability current, which is not conservative, in order to create a more comprehensible parallel between the classic form of the probability current continuity equation and our relativistic case.

The conservative probability current is determined immediately by the use of the Hamiltonian operator in a classic Hermitian expression, but this almost obvious aspect is not the center of our paper.

12.3. **From classic to relativistic.** In temperate distribution space, the non-relativistic Schrödinger equation is everywhere (and continuously) defined, the "right" subfamily of the de Broglie wave family generates its solution space, in the sense of Schwartz superpositions.

The de Broglie family is indeed a basis of the whole of distribution space, in the generalized Schwartz-Bourbaki sense, and it is an eigenbasis of every linear differential operator (including the momentum, energy and many Hamiltonian operators). Therefore, indeed, we finally define and solve satisfactorily the Schrödinger equation also in the non-relativistic case, by distribution approach.

In the relativistic case, we follow the same path, we need only to solve the problem of the correct definition for the principal square root of a linear continuous strictly positive operator, and, indeed, we did it by observing that, in tempered distribution space, there exists one and only one linear continuous operator having the relativistic de Broglie family as eigenbasis with respect to a family of eigenvalues given by the Einstein energy level of a particle with a fixed rest mass. This completes the landscape of the possible free Schrödinger equations in distribution spaces (which, at this point, should be considered the first natural environment of such equations.)

A collateral advantage of this approach is that the new relativistic Hamiltonian operator is Hermitian (in a straightforward generalized sense), and this immediately implies the conservation of Born probability flux, in its very classic definition.

12.4. **Comparison with the Klein-Gordon equation.** In particular, for what concerns a comparison with other equations concerning spin zero particles, we should underline that the problem has remained unsolved until now, for almost a hundred year: already immediately after its formulation, quantum physicists have understood the bad issues of Klein Gordon equation (in comparison with the Schrödinger equation): it is not a quantum evolution equation in the first place; it involves the second derivatives with respect to time, instead of the first derivative, which implies that the Klein Gordon equation does not provide or suggest any relativistic Hamiltonian operator (indeed, Klein Gordon equation is not the counterpart of the non-relativistic Schrödinger equation, but, rather, a generalization of the wave equation for the electromagnetic waves, to the case of a massive wave field). If we consider the second derivative with respect to time on the left-hand side, and the generalized "massive" Laplacian operator on the right-hand side, we can state that our relativistic Schrödinger equation is nothing more than the principal square root of the Klein Gordon equation in that form. Klein Gordon equation doesn't allow to define a probability flux continuity equation, it was abandoned, just after its formulation, and it is now widely recognized that it cannot represent a spin zero particle, but, rather, the evolution of a quantum field.

On the other hand, our relativistic Schrödinger equation is in the form of a quantum evolution

$$\imath\hbar\ (\partial/\partial t)u = \hat{H}u,$$

and this form offers us immediately:

- the right relativistic Hamiltonian operator (directly coming from the Einstein-Hamilton energy);
- the conservation of energy and probability;
- the possibility to reformulate (in a relativistic setting) the current density and its corresponding continuity equation in several different forms.

All the above fundamental features are not given by the Klein Gordon equation: it cannot be shaped in the form of a Schrödinger type equation. Not mentioning that the "current density" given from the Klein Gordon equation cannot be a probability current density, as it has been observed at the time of its early formulation.

12.5. **Distributions to describe systems of many quantum particles.** For what concerns the evolution of systems containing many quantum particles, we should observe that the Laurent Schwartz distribution theory, as well recognized by mathematicians and experts of functional analysis, provides the most complete and efficient theory of tensor products: not only the basis states of many particle systems can be easily represented by the tensor products of single particle states, but such a fortunate happening enjoys the advantages of the powerful and unifying Schwartz kernel theorem.

Any n-particle state can be viewed as an element of an n-tensor power of distribution spaces, each based on Minkowski space-time, or, equivalently, as an element of a distribution space based on the Cartesian n-power of Minkowski space-time. Moreover, there exists a complete identification between distribution kernels (distributions "with two n-dimensional arguments") and "Schwartz operator kernels", which can be considered the most general form of observables (and Heisenberg continuous matrices) in quantum mechanics.

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