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Time-like surfaces with zero mean curvature vector in four-dimensional neutral space forms

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Abstract. Let M be a Lorentz surface and $F\colon M \to N$ a time-like and conformal immersion of M into a 4-dimensional neutral space form N with zero mean curvature vector. We show that the curvature K of the induced metric on M by F is identically equal to the constant sectional curvature L_0 of N if and only if the covariant derivatives of both of the time-like twistor lifts are zero or light-like. If $K \equiv L_0$, then the normal connection $\nabla^{\perp}$ of F is flat, while the converse is not necessarily true. We also prove that a holomorphic paracomplex quartic differential Q on M defined by F is zero or null if and only if the covariant derivative of at least one of the time-like twistor lifts is zero or light-like. In addition, we get that K is identically equal to L_0 if and only if not only $\nabla^{\perp}$ is flat but also Q is zero or null.

Анотація. Нехай M – поверхня Лоренца і $F\colon M \to N$ – часоподібне та конформне занурення M у 4-вимірну нейтральну просторову форму N з нульовим вектором середньої кривини. В роботі показано, що кривина K індукованої метрики на M при відображенні F тотожна постійній секційній кривині L_0 на N тоді і тільки тоді, коли коваріантні похідні обох часоподібних твісторів підняттів нульові або світлоподібні. Якщо $K \equiv L_0$, то нормальна зв'язність $\nabla^{\perp}$ занурення F є плоскою, але зворотне твердження не обов'язково вірне. Також доведено, що голоморфний паракомплексний квартичний диференціал Q на M, визначений за допомогою F, дорівнює нулю тоді і тільки тоді, коли коваріантна похідна принаймні одного з часоподібних твісторних підняттів дорівнює нулю або є світлоподібною. Крім того, встановлено, що K тотожно дорівнює L_0 тоді і тільки тоді, коли не лише $\nabla^{\perp}$ є плоским, але також Q дорівнює нулю або має нульову норму як паракомплексне число.

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1. Introduction

Let N be a 4-dimensional neutral space form. Let also M be a Lorentz surface and $F: M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Then a holomorphic paracomplex quartic differential Q on M is defined by F. Isotropicity of F is characterized by $Q \equiv 0$. If a time-like twistor lift of F is horizontal, then F is isotropic.

However, although F is isotropic, it is possible that the covariant derivatives of the time-like twistor lifts are not zero but light-like ([1]). The covariant derivatives of both of the time-like twistor lifts are zero or light-like if and only if F has one of the following properties ([7]):

(i) the shape operator of a light-like normal vector field vanishes;

(ii) the shape operator of any normal vector field is zero or light-like.

See [1, 7] for properties (i), (ii) of F respectively. If F satisfies (i) or (ii), then the curvature K of the induced metric on M by F is identically equal to the constant sectional curvature L_0 of N.

We will see that the converse holds, that is, if K is identically equal to L_0, then the covariant derivatives of both of the time-like twistor lifts are zero or light-like (Theorem 4.3). In addition, we will study the following two properties related to F:

(I) the normal connection $\nabla^\perp$ of F is flat, that is, the curvature tensor $R^\perp$ of $\nabla^\perp$ vanishes;

(II) Q is zero or null (notice that a paracomplex number is said to be *null* if it is not zero so that its square norm vanishes).

We will show that if $K \equiv L_0$, then $\nabla^\perp$ is flat (Proposition 4.1), while the converse is not necessarily true. We will also find time-like surfaces in N with zero mean curvature vector, flat normal connection and $K \neq L_0$ (Theorem 5.1). It will also be proven that Q is zero or null if and only if the covariant derivative of at least one of the time-like twistor lifts of F is zero or light-like (Theorem 3.2).

In addition, we will find time-like surfaces in N with zero mean curvature vector such that the covariant derivative of just one time-like twistor lift is zero or light-like (Theorem 6.1). We will see that K is identically equal to L_0 if and only if F satisfies both (I) and (II), that is, not only $\nabla^\perp$ is flat but also Q is zero or null (Theorem 4.3).

Remark 1.1. Let N be a 4-dimensional Riemannian, Lorentzian or neutral space form, M be a Riemann or Lorentz surface, and $F: M \to N$ be a space-like or time-like, and conformal immersion with zero mean curvature vector.

Here F is space-like or time-like, according to whether M is Riemann or Lorentz. Let K be the curvature of the induced metric by F and L_0 the constant sectional curvature of N. If N is either Riemannian or neutral, and if M is Riemann, then by the equation of Gauss, the condition $K \equiv L_0$ means that F is totally geodesic, and then by the equation of Ricci, the normal connection is flat. Suppose that N is Lorentzian and that M is Riemann. Then the condition $K \equiv L_0$ does not necessarily imply that F is totally geodesic. By the equations of Gauss-Codazzi-Ricci, $K \equiv L_0$ is equivalent to a condition that the shape operator of a light-like normal vector field of F vanishes, that is, a holomorphic quartic differential Q on M defined by F vanishes (refer to [3]). Then by the equation of Ricci, $K \equiv L_0$ means that the normal connection is flat.

If N is neutral and if M is Lorentz, then we will obtain Proposition 4.1 and Theorem 4.3 below. If N is Lorentzian and M is Lorentz, then $K \equiv L_0$ does not necessarily imply that F is totally geodesic, while $K \equiv L_0$ means that the normal connection is flat (see the appendix below). We can refer to [12] for time-like surfaces in indefinite space forms with zero mean curvature vector and $K \equiv L_0$.

Remark 1.2. Let N be an oriented Riemannian, Lorentzian or neutral 4-manifold. Let also M, F be as in the beginning of the previous remark, and Q be a complex or paracomplex quartic differential on M defined by F. If N is Riemannian or neutral, and if M is Riemann, then isotropicity of F is equivalent to $Q \equiv 0$ and characterized by horizontality of a (space-like) twistor lift of F ([1, 11, 15]).

We can refer to [14] and [10] for twistor spaces and space-like twistor spaces respectively. Horizontality of a (space-like) twistor lift is equivalent to parallelism of the complex structure corresponding to the lift, which is equivalent to a relation between the complex structure and the second fundamental form ([5]).

Suppose that N is neutral and that M is Lorentz. Then isotropicity of F is equivalent to $Q \equiv 0$ and horizontality of a time-like twistor lift means $Q \equiv 0$, but $Q \equiv 0$ does not necessarily mean the horizontality ([1]). We can refer to [16, 17] for time-like twistor spaces. Horizontality of a time-like twistor lift is equivalent to parallelism of the paracomplex structure corresponding to the lift, which is equivalent to a relation between the paracomplex structure and the second fundamental form ([5]).

If $Q \equiv 0$ but if the time-like twistor lifts are not horizontal, then the covariant derivatives of them are zero or light-like ([1]). Suppose that N is Lorentzian. Based on the understanding of isotropicity in the above cases, the author introduced the definition of isotropicity of F in [4]. If M is Riemann, then the isotropicity just means either $Q \equiv 0$ or a relation between a mixed-type structure given by a lift of F and the second fundamental form with respect to a special local complex coordinate on a neighborhood of each point of M ([4]).

In addition, if N is a 4-dimensional Lorentzian space form and if F has no totally geodesic points, then the relation is equivalent to $Q \equiv 0$ ([4]). If M is Lorentz, then the isotropicity of F is given by a relation between a mixed-type structure given by a lift of F and the second fundamental form with respect to a special local paracomplex coordinate on a neighborhood of each point of M ([4]). In addition, if N is a 4-dimensional Lorentzian space form, then the isotropicity implies that F is totally geodesic ([4]), which is equivalent to $Q \equiv 0$.

Remark 1.3. Let N be an oriented 4-dimensional neutral manifold, and M be a Lorentz surface and $F\colon M \to N$ a time-like and conformal immersion with zero mean curvature vector. If the covariant derivative of a time-like twistor lift Θ of F is zero or light-like, and not identically equal to zero at any point, then the covariant derivative of the paracomplex structure J corresponding to Θ is locally represented as the tensor product of a nowhere zero 1-form and a nilpotent structure related to J ([8]). We can refer to [6, 8, 9] for nilpotent structures.

An oriented $4n$-dimensional neutral manifold equipped with such an almost paracomplex structure as J becomes a Walker manifold ([8]). We can find examples of such almost paracomplex structures on E^{4n}_{2n} ([8]). In addition, we can find all the pairs of sections of the two time-like twistor spaces associated with E^4_2 such that the covariant derivatives are zero or light-like, and not identically equal to zero at any point ([8]).

2. Time-like surfaces with zero mean curvature vector

Let N be a 4-dimensional neutral space form with constant sectional curvature L_0. We can suppose that

- $N = E^4_2$ if $L_0 = 0$;
- $N = \{x \in E^5_2 \mid \langle x, x\rangle = 1/L_0\}$ if $L_0 > 0$;
- $N = \{x \in E^5_3 \mid \langle x, x\rangle = 1/L_0\}$ if $L_0 < 0$;

where $\langle\ ,\ \rangle$ is the metric of E^5_2 or E^5_3.

Let h be the neutral metric of N, M a Lorentz surface, and $F\colon M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Let $\breve{w} = u + jv$ be a local paracomplex coordinate of M, where j is the paraimaginary unit of paracomplex numbers. We will denote by $\overline{\breve{w}}$ the conjugate of $\breve{w}$, so $\overline{\breve{w}} = u - jv$.

We represent the induced metric g on M by F as

$$g = e^{2\lambda} d\breve{w} d\overline{\breve{w}} = e^{2\lambda}(du^2 - dv^2)$$

for some real-valued function λ. Then

$$h(dF(\partial_{\breve{w}}), dF(\partial_{\overline{\breve{w}}})) = \frac{e^{2\lambda}}{2},$$

where

$$\partial_{\breve{w}} := \frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right), \qquad \partial_{\overline{\breve{w}}} := \frac{1}{2}\left(\frac{\partial}{\partial u} - j\frac{\partial}{\partial v}\right).$$

Let $\tilde{\nabla}$ denotes the connection of E^4_2, E^5_2 or E^5_3 in accordance with $L_0 = 0$, $L_0 > 0$ or $L_0 < 0$ respectively. Since F has zero mean curvature vector, we have

$$\tilde{\nabla}_{\partial/\partial\overline{\breve{w}}} dF(\partial/\partial\breve{w}) = -L_0 e^{2\lambda} F/2.$$

Put

$$T_1 := dF(\partial/\partial u), \qquad T_2 := dF(\partial/\partial v).$$

Let N_1, N_2 be normal vector fields of F satisfying

$$h(N_1, N_1) = -h(N_2, N_2) = e^{2\lambda}, \qquad h(N_1, N_2) = 0.$$

Then we have that

$$\begin{aligned} \tilde{\nabla}_{T_1}(T_1\ T_2\ N_1\ N_2\ F) &= (T_1\ T_2\ N_1\ N_2\ F)S, \\ \tilde{\nabla}_{T_2}(T_1\ T_2\ N_1\ N_2\ F) &= (T_1\ T_2\ N_1\ N_2\ F)T, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} S &= \begin{bmatrix} \lambda_u & \lambda_v & -\alpha_1 & \beta_1 & 1 \\ \lambda_v & \lambda_u & \alpha_2 & -\beta_2 & 0 \\ \alpha_1 & \alpha_2 & \lambda_u & \mu_1 & 0 \\ \beta_1 & \beta_2 & \mu_1 & \lambda_u & 0 \\ -L_0 e^{2\lambda} & 0 & 0 & 0 & 0 \end{bmatrix}, \\ T &= \begin{bmatrix} \lambda_v & \lambda_u & -\alpha_2 & \beta_2 & 0 \\ \lambda_u & \lambda_v & \alpha_1 & -\beta_1 & 1 \\ \alpha_2 & \alpha_1 & \lambda_v & \mu_2 & 0 \\ \beta_2 & \beta_1 & \mu_2 & \lambda_v & 0 \\ 0 & L_0 e^{2\lambda} & 0 & 0 & 0 \end{bmatrix}, \end{aligned} \tag{2.2}$$

and α_k, β_k, μ_k $(k = 1, 2)$ are real-valued functions. From (2.1), we obtain $S_v - T_u = ST - TS$. This is equivalent to the system of the equations of Gauss, Codazzi and Ricci. The equation of Gauss is given by

$$\lambda_{uu} - \lambda_{vv} + L_0 e^{2\lambda} = \alpha_1^2 - \alpha_2^2 - (\beta_1^2 - \beta_2^2), \tag{2.3}$$

the equations of Codazzi are given by

$$\begin{aligned}(\alpha_1)_v - (\alpha_2)_u &= \alpha_2\lambda_u - \alpha_1\lambda_v + \beta_2\mu_1 - \beta_1\mu_2,\\ (\alpha_2)_v - (\alpha_1)_u &= \alpha_1\lambda_u - \alpha_2\lambda_v + \beta_1\mu_1 - \beta_2\mu_2,\\ (\beta_1)_v - (\beta_2)_u &= \beta_2\lambda_u - \beta_1\lambda_v + \alpha_2\mu_1 - \alpha_1\mu_2,\\ (\beta_2)_v - (\beta_1)_u &= \beta_1\lambda_u - \beta_2\lambda_v + \alpha_1\mu_1 - \alpha_2\mu_2,\end{aligned} \tag{2.4}$$

while the equation of Ricci is given by

$$(\mu_1)_v - (\mu_2)_u = 2\alpha_1\beta_2 - 2\alpha_2\beta_1. \tag{2.5}$$

We set

$$X_\pm := \alpha_1 \pm \beta_2, \qquad Y_\pm := \alpha_2 \pm \beta_1 \tag{2.6}$$

and

$$\phi_\pm := \lambda_u \mp \mu_2, \qquad \psi_\pm := \lambda_v \mp \mu_1. \tag{2.7}$$

Then the system (2.4) is rewritten into

$$\begin{aligned}(X_\pm)_v - (Y_\pm)_u &= \phi_\pm Y_\pm - \psi_\pm X_\pm,\\ (Y_\pm)_v - (X_\pm)_u &= \phi_\pm X_\pm - \psi_\pm Y_\pm.\end{aligned} \tag{2.8}$$

We set

$$s := \frac{1}{\sqrt{2}}(u+v), \qquad t := \frac{1}{\sqrt{2}}(u-v).$$

Then we have

$$\frac{\partial}{\partial u} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial v} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right). \tag{2.9}$$

Therefore, (2.8) is rewritten into

$$\begin{aligned}(X_\pm + Y_\pm)_t &= -\frac{1}{\sqrt{2}}(\phi_\pm - \psi_\pm)(X_\pm + Y_\pm),\\ (X_\pm - Y_\pm)_s &= -\frac{1}{\sqrt{2}}(\phi_\pm + \psi_\pm)(X_\pm - Y_\pm).\end{aligned} \tag{2.10}$$

Hence, a system consisting of (2.3) and (2.5) can be written as follows:

$$\lambda_{uu} - \lambda_{vv} + L_0 e^{2\lambda} \pm ((\mu_1)_v - (\mu_2)_u) = X_\pm^2 - Y_\pm^2. \tag{2.11}$$

Applying (2.7) to (2.11), we finally obtain

$$X_\pm^2 - Y_\pm^2 = (\phi_\pm)_u - (\psi_\pm)_v + L_0 e^{2\lambda}. \tag{2.12}$$

Suppose that F satisfies (i) in Section 1, that is, the shape operator of a light-like normal vector field vanishes. Then we have $\beta_k = \varepsilon\alpha_k$ for $k = 1, 2$ and $\varepsilon \in \{+, -\}$. From (2.3), we get that

$$\lambda_{uu} - \lambda_{vv} + L_0 e^{2\lambda} = 0. \tag{2.13}$$

Then it follows from (2.5) that $\gamma_u = \mu_1$, $\gamma_v = \mu_2$ for a function γ of two variables u, v. Hence, (2.4) implies that

$$(\alpha_+)_t + \alpha_+\lambda_t + \varepsilon\alpha_+\gamma_t = 0, \qquad (\alpha_-)_s + \alpha_-\lambda_s + \varepsilon\alpha_-\gamma_s = 0,$$

where $\alpha_\pm := \alpha_1 \pm \alpha_2$. Then there exist functions $p_\pm$ of one variable satisfying $\alpha_\pm = p_\pm(u \pm v)e^{-\lambda-\varepsilon\gamma}$. Therefore,

$$\alpha_1 = \frac{1}{2}(p_+(u+v) + p_-(u-v))e^{-\lambda-\varepsilon\gamma},$$
$$\alpha_2 = \frac{1}{2}(p_+(u+v) - p_-(u-v))e^{-\lambda-\varepsilon\gamma}.$$

Suppose that F satisfies (ii) in Section 1, that is, the shape operator of any normal vector field is zero or light-like. Then, as in [7], we obtain (2.13) and

$$\alpha_2 = \varepsilon\alpha_1, \qquad \beta_2 = \varepsilon\beta_1, \qquad \mu_1 = \gamma_u, \qquad \mu_2 = \gamma_v$$

for $\varepsilon \in \{+,-\}$ and a function γ of two variables u, v, and α_1, β_1 are represented as

$$\alpha_1 = \frac{1}{2e^\lambda}(\phi(u+\varepsilon v)e^\gamma + \psi(u+\varepsilon v)e^{-\gamma}),$$
$$\beta_1 = -\frac{1}{2e^\lambda}(\phi(u+\varepsilon v)e^\gamma - \psi(u+\varepsilon v)e^{-\gamma})$$

for functions ϕ, ψ of one variable.

3. Time-like twistor lifts

Let N, h, M, F be as in the beginning of the previous section. Then the 2-fold exterior power $\bigwedge^2 F^*TN$ of the pull-back bundle F^*TN on M by F is a vector bundle on M of rank 6. Let $\hat{h}$ be the metric of $\bigwedge^2 F^*TN$ induced by h. Then $\hat{h}$ has signature $(2,4)$. Noticing the double covering

$$SO_0(2,2) \to SO_0(1,2) \times SO_0(1,2),$$

we see that $\bigwedge^2 F^*TN$ is decomposed into two subbundles $\bigwedge^2_\pm F^*TN$ of rank 3. These are orthogonal to each other with respect to $\hat{h}$ and the restriction of $\hat{h}$ on each of them has signature $(1,2)$.

The time-like twistor spaces associated with F^*TN are fiber bundles on M defined by

$$U_-\left(\textstyle\bigwedge^2_\pm F^*TN\right) := \left\{\theta \in \textstyle\bigwedge^2_\pm F^*TN \;\middle|\; \hat{h}(\theta,\theta) = -1\right\}.$$

Let (e_1, e_2, e_3, e_4) be an ordered local pseudo-orthonormal frame field of F^*TN giving the orientation of N such that e_1, e_2 (respectively, e_3, e_4) are space-like (respectively, time-like). Set

$$\Theta_{\pm,1} := \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4),$$
$$\Theta_{\pm,2} := \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2),$$
$$\Theta_{\pm,3} := \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

Then $\bigwedge^2_\pm F^*TN$ are locally generated by $\Theta_{\mp,1}$, $\Theta_{\pm,2}$, $\Theta_{\pm,3}$, respectively. The time-like twistor lifts $\Theta_{F,\pm}$ of F are sections of $U_-\big(\bigwedge^2_\pm F^*TN\big)$ and locally represented as $\Theta_{F,\pm} = \Theta_{\pm,2}$, where we suppose that e_1, e_3 satisfy $e_1, e_3 \in dF(TM)$ so that (e_1, e_3) gives the orientation of M.

Let ∇ be the Levi-Civita connection of N. Then ∇ induces a connection $\hat{\nabla}$ of $\bigwedge^2 F^*TN$. In addition, $\hat{\nabla}$ gives connections of $\bigwedge^2_\pm F^*TN$. If we set

$$e_1 := \frac{1}{e^\lambda}T_1, \qquad e_3 := \frac{1}{e^\lambda}T_2, \qquad e_2 := \frac{1}{e^\lambda}N_1, \qquad e_4 := \frac{1}{e^\lambda}N_2,$$

then the covariant derivatives of $\Theta_{F,\pm} = \Theta_{\pm,2}$ with respect to $\hat{\nabla}$ are given by

$$\hat{\nabla}_{T_1}\Theta_{F,\pm} = Y_\pm\Theta_{\mp,1} \pm X_\pm\Theta_{\pm,3},$$
$$\hat{\nabla}_{T_2}\Theta_{F,\pm} = X_\pm\Theta_{\mp,1} \pm Y_\pm\Theta_{\pm,3},$$

where $X_\pm$, $Y_\pm$ are as in (2.6). Therefore we obtain

Proposition 3.1. *The following statements hold:*

(a) *one of $\hat{\nabla}\Theta_{F,\pm}$ is zero or light-like if and only if F satisfies either $X_+^2 = Y_+^2$ or $X_-^2 = Y_-^2$;*

(b) *both of $\hat{\nabla}\Theta_{F,\pm}$ are zero or light-like if and only if F satisfies both of $X_\pm^2 = Y_\pm^2$.*

We see that F satisfies $X_+^2 = Y_+^2$ if and only if it satisfies one of the identities:

$$\alpha_1 + \alpha_2 = -(\beta_1 + \beta_2), \qquad \alpha_1 - \alpha_2 = \beta_1 - \beta_2. \tag{3.1}$$

Similarly, F satisfies $X_-^2 = Y_-^2$ if and only if it satisfies one of the identities:

$$\alpha_1 + \alpha_2 = \beta_1 + \beta_2, \qquad \alpha_1 - \alpha_2 = -(\beta_1 - \beta_2). \tag{3.2}$$

We set $\gamma_k := \alpha_k + j\beta_k$ for $k = 1, 2$. Then F satisfies one relation in (3.1) and (3.2) if and only if one of $\gamma_1 \pm \gamma_2$ is zero or null. Let σ be the second fundamental form of F. Set

$$\sigma_{\breve{w}\breve{w}} := \sigma(\partial/\partial\breve{w}, \partial/\partial\breve{w}) \qquad \text{and} \qquad Q := h(\sigma_{\breve{w}\breve{w}}, \sigma_{\breve{w}\breve{w}})d\breve{w}^4.$$

Then Q is a paracomplex quartic differential defined on M. Since N is a neutral space form, we see by the equations of Codazzi that Q is paraholomorphic. From (2.1) with (2.2), we have

$$\sigma_{\breve{w}\breve{w}} = \frac{1}{2}(\alpha_1 N_1 + \beta_1 N_2 + j(\alpha_2 N_1 + \beta_2 N_2))$$

and therefore we obtain that

$$Q = \frac{e^{2\lambda}}{4}(|\gamma_1|^2 + |\gamma_2|^2 + 2j\mathrm{Re}\,(\gamma_1\overline{\gamma}_2))d\breve{w}^4, \tag{3.3}$$

where $|\gamma_k|^2$ is the square norm of γ_k: $|\gamma_k|^2 := \alpha_k^2 - \beta_k^2$. From (3.3) and

$$|\gamma_1 \pm \gamma_2|^2 = |\gamma_1|^2 + |\gamma_2|^2 \pm 2\mathrm{Re}\,(\gamma_1\overline{\gamma}_2),$$

we see that one of $\gamma_1 \pm \gamma_2$ is zero or null if and only if Q is zero or null. Therefore from (a) of Proposition 3.1, we obtain

Theorem 3.2. *One of $\hat{\nabla}\Theta_{F,\pm}$ is zero or light-like if and only if Q is zero or null.*

4. Time-like surfaces with zero mean curvature vector and $K \equiv L_0$

Suppose that the curvature K of the induced metric g is identically equal to L_0. Then we have (2.13). By this and (2.11) we get

$$(\mu_1)_v - (\mu_2)_u = \pm(X_\pm^2 - Y_\pm^2). \tag{4.1}$$

It then follows from (4.1) that

$$X_+^2 - Y_+^2 = -(X_-^2 - Y_-^2). \tag{4.2}$$

Proposition 4.1. *If $K \equiv L_0$, then $(\mu_1)_v \equiv (\mu_2)_u$.*

Proof. Suppose $(\mu_1)_v \neq (\mu_2)_u$ on a neighborhood of a point of M. Then from (4.1) we get that $X_+^2 \neq Y_+^2$ and $X_-^2 \neq Y_-^2$. Set

$$P_\pm := \log|X_\pm + Y_\pm|, \qquad Q_\pm := \log|X_\pm - Y_\pm|. \tag{4.3}$$

Then noticing (4.2) we obtain

$$P_+ + Q_+ = P_- + Q_- = \log|X_\pm^2 - Y_\pm^2|. \tag{4.4}$$

Therefore (4.2) and (4.4) imply that

$$X_\pm^2 - Y_\pm^2 = \pm\varepsilon e^{P_\pm + Q_\pm}, \tag{4.5}$$

where $\varepsilon \in \{+,-\}$. It follows from (4.3) that (2.10) is rewritten into

$$(P_\pm)_t = -\frac{1}{\sqrt{2}}(\phi_\pm - \psi_\pm), \qquad (Q_\pm)_s = -\frac{1}{\sqrt{2}}(\phi_\pm + \psi_\pm).$$

Hence,

$$\phi_\pm = -\frac{1}{\sqrt{2}}((Q_\pm)_s + (P_\pm)_t), \qquad \psi_\pm = -\frac{1}{\sqrt{2}}((Q_\pm)_s - (P_\pm)_t). \tag{4.6}$$

Now, from (2.9) and (4.6), we get that

$$(\phi_\pm)_u - (\psi_\pm)_v = -(P_\pm + Q_\pm)_{st}. \tag{4.7}$$

Applying (4.5) and (4.7) to (2.12) we obtain

$$(P_\pm + Q_\pm)_{st} \pm \varepsilon e^{P_\pm + Q_\pm} = L_0 e^{2\lambda}. \tag{4.8}$$

However, (4.8) contradicts to (4.4). Therefore we can not suppose that $(\mu_1)_v \neq (\mu_2)_u$. Hence, $(\mu_1)_v \equiv (\mu_2)_u$. □

Remark 4.2. We see that F satisfies $(\mu_1)_v \equiv (\mu_2)_u$ if and only if the curvature tensor $R^\perp$ of the normal connection $\nabla^\perp$ of F vanishes. Although F satisfies $(\mu_1)_v \equiv (\mu_2)_u$, K is not necessarily equal to L_0 (see Section 5).

Theorem 4.3. *Let N be a 4-dimensional neutral space form with constant sectional curvature L_0. Let M be a Lorentz surface and $F\colon M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Then the following conditions are equivalent:*

(a) *the curvature K of the induced metric by F is identically equal to L_0;*
(b) *the covariant derivatives of both of the time-like twistor lifts are zero or light-like;*
(c) *not only $\nabla^\perp$ is flat but also Q is zero or null.*

Proof. Suppose $K \equiv L_0$. Then by (2.11) and Proposition 4.1, we have both $X_+^2 = Y_+^2$ and $X_-^2 = Y_-^2$. Therefore we see from (b) of Proposition 3.1 that the covariant derivatives of both of the time-like twistor lifts of F are zero or light-like. If the covariant derivatives of both of the time-like twistor lifts are zero or light-like, then from (2.11) and Proposition 3.1, we obtain $K \equiv L_0$. Hence we have proved that (a) and (b) are equivalent.

By (2.11), Proposition 3.1, Theorem 3.2 and Proposition 4.1, we obtain (c) from (a). In addition, by (2.11), Proposition 3.1 and Theorem 3.2, we obtain (a) from (c). Hence we have proved that (a) and (c) are equivalent. □

Remark 4.4. Let λ be a function of two variables u, v satisfying (2.13). If $L_0 = 0$, then λ is represented as the sum of a function of s and a function of t. If $L_0 \neq 0$, then (2.13) is easily transformed to Liouville's equation:

- if $L_0 > 0$, then λ is represented as in the form of

$$e^{2\lambda} = \frac{2p'(\sqrt{L_0}s)q'(\sqrt{L_0}t)}{(p(\sqrt{L_0}s) - q(\sqrt{L_0}t))^2},$$

where p, q are functions of one variable;

- and if $L_0 < 0$, then λ is represented as in the form of
$$e^{2\lambda} = \frac{2p'(\sqrt{|L_0|}s)q'(\sqrt{|L_0|}t)}{(p(\sqrt{|L_0|}s) + q(\sqrt{|L_0|}t))^2},$$
where p, q are as above.

5. Time-like surfaces with zero mean curvature vector and flat normal connection

Let F be as in the beginning of Section 2. Suppose that the normal connection of F is flat. Then F satisfies $(\mu_1)_v \equiv (\mu_2)_u$. From (2.11) we have
$$\lambda_{uu} - \lambda_{vv} + L_0 e^{2\lambda} = X_\pm^2 - Y_\pm^2. \tag{5.1}$$
Hence (5.1) implies that
$$X_+^2 - Y_+^2 = X_-^2 - Y_-^2. \tag{5.2}$$
Suppose $X_\pm^2 \neq Y_\pm^2$, i.e., $K \neq L_0$. Let $P_\pm$, $Q_\pm$ be as in (4.3). Then (4.4) holds. Together with (5.2) it implies that
$$X_\pm^2 - Y_\pm^2 = \varepsilon e^R, \tag{5.3}$$
where $\varepsilon \in \{+, -\}$ and $R := P_\pm + Q_\pm$. By (2.10) and (4.3), we obtain (4.6) and (4.7). Also, by (2.7) and (4.6), we get that
$$\begin{aligned} \lambda_u &= -\frac{1}{2\sqrt{2}}((Q_+ + Q_-)_s + (P_+ + P_-)_t), \\ \lambda_v &= -\frac{1}{2\sqrt{2}}((Q_+ + Q_-)_s - (P_+ + P_-)_t). \end{aligned} \tag{5.4}$$
Applying (5.4) to $\lambda_{uv} = \lambda_{vu}$ and using (2.9), we obtain
$$(P_+ + P_-)_{st} = (Q_+ + Q_-)_{st}.$$
By this and (4.4), we see that there exist functions x_1, y_1 of one variable s, t respectively satisfying
$$Q_- = P_+ + x_1(s) + y_1(t), \qquad Q_+ = P_- + x_1(s) + y_1(t). \tag{5.5}$$
Applying $R = P_\pm + Q_\pm$ to (4.7), we obtain that
$$(\phi_\pm)_u - (\psi_\pm)_v = -R_{st}. \tag{5.6}$$
Since $(\mu_1)_v \equiv (\mu_2)_u$ we get from (2.7) that
$$(\phi_\pm)_u - (\psi_\pm)_v = \lambda_{uu} - \lambda_{vv}. \tag{5.7}$$
Now (5.1), (5.3), (5.6) and (5.7) imply that
$$R_{st} = -2\lambda_{st} = L_0 e^{2\lambda} - \varepsilon e^R. \tag{5.8}$$

From the first equation in (5.8), we have

$$R = -2\lambda + x_2(s) + y_2(t) \tag{5.9}$$

for functions x_2, y_2 of one variable. Applying this to the second equation in (5.8), we obtain that

$$\lambda_{st} = -\frac{L_0}{2}e^{2\lambda} + \frac{\varepsilon}{2}e^{x_2(s)}e^{y_2(t)}e^{-2\lambda}. \tag{5.10}$$

From $R = P_\pm + Q_\pm$ and (5.5), we obtain

$$P_+ + P_- = R - x_1(s) - y_1(t), \quad Q_+ + Q_- = R + x_1(s) + y_1(t). \tag{5.11}$$

Applying (5.9) and (5.11) to (5.4) we obtain

$$x_2(s) = -x_1(s) + a, \qquad y_2(t) = y_1(t) + b$$

for constants a, b. Let $\tilde{s}$, $\tilde{t}$ be functions of s, t respectively satisfying

$$\frac{d\tilde{s}}{ds} = e^{\frac{1}{2}x_2(s)}, \qquad \frac{d\tilde{t}}{dt} = e^{\frac{1}{2}y_2(t)}.$$

Then $(\tilde{s}, \tilde{t})$ are local coordinates and from (5.10) we obtain

$$\tilde{\lambda}_{\tilde{s}\tilde{t}} = -\frac{L_0}{2}e^{2\tilde{\lambda}} + \frac{\varepsilon}{2}e^{-2\tilde{\lambda}},$$

where $\tilde{\lambda} := \lambda - (1/4)(x_2(s) + y_2(t))$. Set

$$\tilde{u} := (1/\sqrt{2})(\tilde{s} + \tilde{t}), \qquad \tilde{v} := (1/\sqrt{2})(\tilde{s} - \tilde{t}).$$

Then $(\tilde{u}, \tilde{v})$ are local coordinates satisfying $g = e^{2\tilde{\lambda}}(d\tilde{u}^2 - d\tilde{v}^2)$. Let $\tilde{X}_\pm$, $\tilde{Y}_\pm$ be defined as in (2.6) for $(\tilde{u}, \tilde{v})$. Then we have

$$\tilde{\lambda}_{\tilde{u}\tilde{u}} - \tilde{\lambda}_{\tilde{v}\tilde{v}} + L_0 e^{2\tilde{\lambda}} = \tilde{X}_\pm^2 - \tilde{Y}_\pm^2. \tag{5.12}$$

Comparing (5.1) with (5.12) we get that

$$\tilde{X}_\pm^2 - \tilde{Y}_\pm^2 = (X_\pm^2 - Y_\pm^2)e^{-(1/2)(x_2(s)+y_2(t))}.$$

Let $\tilde{P}_\pm$, $\tilde{Q}_\pm$ be defined as in (4.3) for $(\tilde{u}, \tilde{v})$. Put $\tilde{R} := \tilde{P}_\pm + \tilde{Q}_\pm$. Then

$$\tilde{R} = R - (1/2)(x_2(s) + y_2(t)).$$

Therefore by this and (5.9) we obtain that $\tilde{R} = -2\tilde{\lambda}$.

Let λ be a function of two variables s, t satisfying

$$\lambda_{st} = -\frac{L_0}{2}e^{2\lambda} + \frac{\varepsilon}{2}e^{-2\lambda} \tag{5.13}$$

for $\varepsilon \in \{+, -\}$. Denote $R := -2\lambda$. Then λ and R satisfy (5.8). Let $P_\pm$, $Q_\pm$ be functions satisfying

$$R = P_\pm + Q_\pm, \qquad Q_- = P_+ + c, \qquad Q_+ = P_- + c \tag{5.14}$$

for a constant c. Then $P_\pm$ and $Q_\pm$ satisfy (5.4). Let $\phi_\pm$, $\psi_\pm$ be functions as in (4.6). Then we get from (5.4) that

$$\lambda_u - \phi_+ = -(\lambda_u - \phi_-), \qquad \lambda_v - \psi_+ = -(\lambda_v - \psi_-). \tag{5.15}$$

Noticing (5.15), we see that

$$\begin{aligned} \mu_1 &:= (\lambda_v - \psi_+ =)\ \frac{1}{2\sqrt{2}}((Q_+ - Q_-)_s - (P_+ - P_-)_t), \\ \mu_2 &:= (\lambda_u - \phi_+ =)\ \frac{1}{2\sqrt{2}}((Q_+ - Q_-)_s + (P_+ - P_-)_t) \end{aligned} \tag{5.16}$$

satisfy (2.7). In addition, noticing

$$(\phi_+)_u - (\psi_+)_v = -R_{st} = \lambda_{uu} - \lambda_{vv},$$

we obtain that $(\mu_1)_v \equiv (\mu_2)_u$.

Set

$$X_\pm := \frac{\varepsilon'_\pm}{2}(e^{P_\pm} + \varepsilon e^{Q_\pm}), \qquad Y_\pm := \frac{\varepsilon'_\pm}{2}(e^{P_\pm} - \varepsilon e^{Q_\pm}), \tag{5.17}$$

where ε'_+, $\varepsilon'_- \in \{+,-\}$. Then (5.3) holds. Therefore we see from (5.13) that λ, $X_\pm$, $Y_\pm$ satisfy (5.1).

Put

$$\begin{aligned} \alpha_1 &:= \frac{1}{2}(X_+ + X_-), & \alpha_2 &:= \frac{1}{2}(Y_+ + Y_-), \\ \beta_1 &:= \frac{1}{2}(Y_+ - Y_-), & \beta_2 &:= \frac{1}{2}(X_+ - X_-). \end{aligned} \tag{5.18}$$

Then S, T with (2.2) satisfy $S_v - T_u = ST - TS$. Hence we obtain the following

Theorem 5.1. *Let N be a 4-dimensional neutral space form with constant sectional curvature L_0. Let also M be a Lorentz surface and $F\colon M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Then the following statements are equivalent:*

(a) *F satisfies both $K \neq L_0$ and $R^\perp = 0$;*

(b) *the induced metric g by F is locally represented as $g = e^{2\lambda} d\check{w} d\overline{\check{w}}$ for a real-valued function λ satisfying*

$$\lambda_{uu} - \lambda_{vv} = -L_0 e^{2\lambda} + \varepsilon e^{-2\lambda}, \quad (\varepsilon \in \{+,-\}) \tag{5.19}$$

and α_k, β_k, μ_k $(k = 1, 2)$ satisfy (5.16), (5.18) with (5.14), (5.17) and $R = -2\lambda$.

In addition, a function λ of two variables u, v satisfying (5.19) and functions α_k, β_k, μ_k $(k = 1, 2)$ of u, v given by (5.16), (5.18) with (5.14), (5.17) and $R = -2\lambda$ locally define a time-like surface in N with zero mean curvature vector, $K \neq L_0$ and $R^\perp = 0$, which is unique up to an isometry of N.

6. Time-like surfaces with zero mean curvature vector and $X_+^2 \neq Y_+^2$, $X_- = Y_- \neq 0$

Let F be as at the beginning of Section 2. Suppose that the covariant derivative of just one of the time-like twistor lifts of F is zero or light-like. In this section, we assume that F satisfies $X_+^2 \neq Y_+^2$ and $X_- = Y_- \neq 0$.

Let $P_\pm$, Q_+ be as in (4.3). Then ϕ_+, ψ_+ are given by (4.6), and therefore by (2.7), ϕ_-, ψ_- are given by

$$\begin{aligned} \phi_- &= 2\lambda_u + \frac{1}{\sqrt{2}}((Q_+)_s + (P_+)_t), \\ \psi_- &= 2\lambda_v + \frac{1}{\sqrt{2}}((Q_+)_s - (P_+)_t). \end{aligned} \tag{6.1}$$

Noticing $P_- = \log 2|X_-|$, from one relation in (2.10) and (6.1) we obtain

$$(P_-)_t = -2\lambda_t - (P_+)_t.$$

Hence,

$$\lambda = -\frac{1}{2}(P_+ + P_-) + a(s), \tag{6.2}$$

where a is a function of one variable. Applying (6.2) to (6.1) we obtain

$$\begin{aligned} \phi_- &= -\frac{1}{\sqrt{2}}(P_+ - Q_+)_s - (P_-)_u + \sqrt{2}a'(s), \\ \psi_- &= -\frac{1}{\sqrt{2}}(P_+ - Q_+)_s - (P_-)_v + \sqrt{2}a'(s). \end{aligned} \tag{6.3}$$

Applying (6.2), (6.3) and $X_- = Y_-$ to one relation in (2.12), we obtain

$$(P_+ - Q_+ + 2\tilde{P}_-)_{st} = L_0 e^{-P_+ - \tilde{P}_-}, \tag{6.4}$$

where $\tilde{P}_- := P_- - 2a(s)$. Applying (4.3), (4.6), (6.2) to the other relation in (2.12), we obtain

$$(P_+ + Q_+)_{st} + \varepsilon e^{P_+ + Q_+} = L_0 e^{-P_+ - \tilde{P}_-} \tag{6.5}$$

for $\varepsilon \in \{+, -\}$. If we set

$$f_1 = P_+ - Q_+ + 2\tilde{P}_-, \qquad f_2 = Q_+ - \tilde{P}_-, \tag{6.6}$$

then we get from (6.4) and (6.5) that

$$(f_1)_{st} = L_0 e^{-f_1 - f_2}, \qquad (f_2)_{st} = -\frac{\varepsilon}{2} e^{f_1 + 2f_2}. \tag{6.7}$$

These equations form a semilinear hyperbolic system.

Let f_1, f_2 be functions of two variables s, t satisfying (6.7) for some $\varepsilon \in \{+, -\}$. See [13, Chapter 5] or [2] for existence and uniqueness of solutions of semilinear hyperbolic systems. Let P_+, Q_+, $\tilde{P}_-$ be functions satisfying (6.6). Then P_+, Q_+, $\tilde{P}_-$ satisfy (6.4) and (6.5). Put

$$\begin{aligned} \lambda &:= -\frac{1}{2}(P_+ + \tilde{P}_-), \\ \phi_- &:= -\frac{1}{\sqrt{2}}(P_+ - Q_+)_s - (\tilde{P}_-)_u, \\ \psi_- &:= -\frac{1}{\sqrt{2}}(P_+ - Q_+)_s - (\tilde{P}_-)_v. \end{aligned} \tag{6.8}$$

Let ϕ_+, ψ_+ be as in (4.6). Then λ, $\phi_\pm$ and $\psi_\pm$ satisfy (5.15). Therefore we define μ_1 and μ_2 by

$$\begin{aligned} \mu_1 &:= (\lambda_v - \psi_+ =) \ -\frac{1}{2}(P_+)_u + \frac{1}{\sqrt{2}}(Q_+)_s - \frac{1}{2}(\tilde{P}_-)_v, \\ \mu_2 &:= (\lambda_u - \phi_+ =) \ -\frac{1}{2}(P_+)_v + \frac{1}{\sqrt{2}}(Q_+)_s - \frac{1}{2}(\tilde{P}_-)_u. \end{aligned} \tag{6.9}$$

Define also X_+ and Y_+ by

$$X_+ := \frac{\varepsilon'}{2}(e^{P_+} + \varepsilon e^{Q_+}), \qquad Y_+ := \frac{\varepsilon'}{2}(e^{P_+} - \varepsilon e^{Q_+}) \tag{6.10}$$

for $\varepsilon' \in \{+, -\}$. Then we have $X_+^2 \neq Y_+^2$. Set

$$X_- = Y_- := (\varepsilon''/2)e^{\tilde{P}_-}$$

for $\varepsilon'' \in \{+, -\}$. Define also α_k, β_k ($k = 1, 2$) by (5.18). Then S, T with (2.2) satisfy $S_v - T_u = ST - TS$.

Hence we obtain the following

Theorem 6.1. *Let N be a 4-dimensional neutral space form with constant sectional curvature L_0. Let also M be a Lorentz surface and $F\colon M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Then the following conditions are equivalent:*

(a) *F satisfies $X_+^2 \neq Y_+^2$ and $X_- = Y_- \neq 0$;*

(b) *the induced metric g by F is locally represented as $g = e^{2\lambda} d\check{w} d\overline{\check{w}}$ for a real-valued function λ given by (6.8) and α_k, β_k, μ_k ($k = 1, 2$) satisfy (5.18), (6.9) with (6.6), (6.7), (6.10) and $X_- = Y_- := (\varepsilon''/2)e^{\tilde{P}_-}$.*

In addition, functions λ, α_k, β_k, μ_k ($k = 1, 2$) as in (b) locally define a time-like surface in N with zero mean curvature vector, $X_+^2 \neq Y_+^2$ and $X_- = Y_- \neq 0$, which is unique up to an isometry of N.

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7. Appendix. Time-like surfaces with zero mean curvature vector and $K \equiv L_0$ in a 4-dimensional Lorentzian space form with constant sectional curvature L_0

In this appendix, let N be a 4-dimensional Lorentzian space form with constant sectional curvature L_0.

- If $L_0 = 0$, then we can suppose $N = E^4_1$;
- if $L_0 > 0$, then we can assume that

$$N = \{x \in E^5_1 : \langle x, x \rangle = 1/L_0\};$$

- finally, if $L_0 < 0$, then we can suppose

$$N = \{x \in E^5_2 : \langle x, x \rangle = 1/L_0\},$$

where $\langle \cdot, \cdot \rangle$ is the metric of E^5_1 or E^5_2.

Let h be the Lorentzian metric of N. Let also M be a Lorentz surface and $F \colon M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Denote by $\breve{w} = u + jv$ a local paracomplex coordinate of M. We represent the induced metric g on M by F as

$$g = e^{2\lambda} d\breve{w} d\overline{\breve{w}} = e^{2\lambda}(du^2 - dv^2)$$

for a real-valued function λ. Let $\tilde{\nabla}$ denotes the connection of E^4_1, E^5_1 or E^5_2 according to $L_0 = 0$, > 0 or < 0. Set

$$T_1 := dF(\partial/\partial u), \qquad T_2 := dF(\partial/\partial v).$$

Let N_1, N_2 be normal vector fields of F satisfying

$$h(N_1, N_1) = h(N_2, N_2) = e^{2\lambda}, \qquad h(N_1, N_2) = 0.$$

Then we have

$$\begin{aligned} \tilde{\nabla}_{T_1}(T_1\ T_2\ N_1\ N_2\ F) &= (T_1\ T_2\ N_1\ N_2\ F)S, \\ \tilde{\nabla}_{T_2}(T_1\ T_2\ N_1\ N_2\ F) &= (T_1\ T_2\ N_1\ N_2\ F)T, \end{aligned} \tag{7.1}$$

where

$$
\begin{aligned}
S &= \begin{bmatrix} \lambda_u & \lambda_v & -\alpha_1 & -\beta_1 & 1 \\ \lambda_v & \lambda_u & \alpha_2 & \beta_2 & 0 \\ \alpha_1 & \alpha_2 & \lambda_u & -\mu_1 & 0 \\ \beta_1 & \beta_2 & \mu_1 & \lambda_u & 0 \\ -L_0 e^{2\lambda} & 0 & 0 & 0 & 0 \end{bmatrix}, \\
T &= \begin{bmatrix} \lambda_v & \lambda_u & -\alpha_2 & -\beta_2 & 0 \\ \lambda_u & \lambda_v & \alpha_1 & \beta_1 & 1 \\ \alpha_2 & \alpha_1 & \lambda_v & -\mu_2 & 0 \\ \beta_2 & \beta_1 & \mu_2 & \lambda_v & 0 \\ 0 & L_0 e^{2\lambda} & 0 & 0 & 0 \end{bmatrix},
\end{aligned}
\tag{7.2}
$$

and α_k, β_k, μ_k $(k = 1, 2)$ are real-valued functions. From (7.1), we obtain that

$$S_v - T_u = ST - TS.$$

Suppose that the curvature K of g is identically equal to L_0. Then the equation of Gauss is given by

$$\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2, \tag{7.3}$$

while the equations of Codazzi are given by

$$
\begin{aligned}
(\alpha_1)_v - (\alpha_2)_u &= \alpha_2\lambda_u - \alpha_1\lambda_v - \beta_2\mu_1 + \beta_1\mu_2, \\
(\alpha_2)_v - (\alpha_1)_u &= \alpha_1\lambda_u - \alpha_2\lambda_v - \beta_1\mu_1 + \beta_2\mu_2, \\
(\beta_1)_v - (\beta_2)_u &= \beta_2\lambda_u - \beta_1\lambda_v + \alpha_2\mu_1 - \alpha_1\mu_2, \\
(\beta_2)_v - (\beta_1)_u &= \beta_1\lambda_u - \beta_2\lambda_v + \alpha_1\mu_1 - \alpha_2\mu_2,
\end{aligned}
\tag{7.4}
$$

and the equation of Ricci is given by

$$(\mu_1)_v - (\mu_2)_u = 2\alpha_1\beta_2 - 2\alpha_2\beta_1. \tag{7.5}$$

Notice that (7.5) coincides with (2.5). Put

$$p_\pm := e^\lambda(\alpha_1 \pm \alpha_2), \qquad q_\pm := e^\lambda(\beta_1 \pm \beta_2), \qquad \mu_\pm := \mu_1 \pm \mu_2.$$

Then the system (7.4) is rewritten into a system which consists of

$$(p_+)_t = \frac{1}{\sqrt{2}}\mu_- q_+, \qquad (q_+)_t = -\frac{1}{\sqrt{2}}\mu_- p_+ \tag{7.6}$$

and

$$(p_-)_s = \frac{1}{\sqrt{2}}\mu_+ q_-, \qquad (q_-)_s = -\frac{1}{\sqrt{2}}\mu_+ p_-. \tag{7.7}$$

By (7.6), we can find a function $\tilde{\mu}_-$ satisfying $(\tilde{\mu}_-)_t = \mu_-$ and that p_+, q_+ are represented as

$$p_+ = C_+(s)\cos\left(\frac{\tilde{\mu}_-}{\sqrt{2}}\right), \qquad q_+ = -C_+(s)\sin\left(\frac{\tilde{\mu}_-}{\sqrt{2}}\right) \tag{7.8}$$

for a function C_+ of one variable; by (7.7), we can find a function $\tilde{\mu}_+$ satisfying $(\tilde{\mu}_+)_s = \mu_+$ and that p_-, q_- are represented as

$$p_- = C_-(t)\cos\left(\frac{\tilde{\mu}_+}{\sqrt{2}}\right), \qquad q_- = -C_-(t)\sin\left(\frac{\tilde{\mu}_+}{\sqrt{2}}\right) \tag{7.9}$$

for a function C_- of one variable. Since

$$p_+p_- + q_+q_- = 0$$

due to (7.3), we see from (7.8) and (7.9) that F satisfies one of the relations:

$$C_+ = 0, \qquad C_- = 0, \qquad \cos((\tilde{\mu}_+ - \tilde{\mu}_-)/\sqrt{2}) = 0.$$

Suppose $\cos((\tilde{\mu}_+ - \tilde{\mu}_-)/\sqrt{2}) = 0$. Then $\tilde{\mu}_- = \tilde{\mu}_+ + (n + 1/2)\sqrt{2}\pi$ for some integer n, whence

$$(\tilde{\mu}_+)_s = (\tilde{\mu}_-)_s = \mu_+ \qquad \text{and} \qquad (\tilde{\mu}_+)_t = (\tilde{\mu}_-)_t = \mu_-.$$

Therefore, $(\mu_1)_v = (\mu_2)_u$. Hence

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \frac{1}{2e^{2\lambda}}(p_-q_+ - p_+q_-) = \frac{C_+(s)C_-(t)}{2e^{2\lambda}}\sin\left(\frac{1}{\sqrt{2}}(\tilde{\mu}_+ - \tilde{\mu}_-)\right).$$

Since $\tilde{\mu}_- = \tilde{\mu}_+ + (n + 1/2)\sqrt{2}\pi$, we obtain that

$$\sin((1/\sqrt{2})(\tilde{\mu}_+ - \tilde{\mu}_-)) \neq 0.$$

Therefore from (7.5) and $(\mu_1)_v = (\mu_2)_u$, we get that $C_+ = 0$ or $C_- = 0$. Thus

$$(\alpha_2, \beta_2) = \varepsilon(\alpha_1, \beta_1) \tag{7.10}$$

for $\varepsilon \in \{+, -\}$. Since $(\mu_1)_v = (\mu_2)_u$, there exists a function γ satisfying $\gamma_u = \mu_1$, $\gamma_v = \mu_2$. Therefore we obtain that $\gamma_s = \mu_+/\sqrt{2}$ and $\gamma_t = \mu_-/\sqrt{2}$.

Noticing that

$$(\tilde{\mu}_+)_s = (\tilde{\mu}_-)_s = \mu_+ \qquad \text{and} \qquad (\tilde{\mu}_+)_t = (\tilde{\mu}_-)_t = \mu_-,$$

we can assume that $\tilde{\mu}_- = \sqrt{2}\gamma$ (respectively, $\tilde{\mu}_+ = \sqrt{2}\gamma$) if $\varepsilon = +$ (respectively, $-$). Then we obtain

$$\alpha_1 = \frac{C_\varepsilon(u + \varepsilon v)}{2e^\lambda}\cos\gamma, \qquad \beta_1 = -\frac{C_\varepsilon(u + \varepsilon v)}{2e^\lambda}\sin\gamma. \tag{7.11}$$

Let λ be a function of two variables u, v satisfying (2.13), and γ be a function of two variables u, v. Let also α_1, β_1 be functions given by (7.11) for a function C_ε of one variable and $\varepsilon \in \{+, -\}$, and let α_2, β_2 be as in (7.10). Put $\mu_1 = \gamma_u$, $\mu_2 = \gamma_v$. Then λ, α_k, β_k, μ_k $(k = 1, 2)$ satisfy $S_v - T_u = ST - TS$ for S, T with (7.2).

Hence we obtain the following

Theorem 7.1. *Let N be a 4-dimensional Lorentzian space form with constant sectional curvature L_0. Let M be a Lorentz surface and $F\colon M \to N$ a time-like and conformal immersion of M into N with zero mean curvature vector. Then the following are equivalent*:

(a) *the curvature K of the induced metric g by F is identically equals L_0;*

(b) *g is locally represented as $g = e^{2\lambda} d\check{w} d\overline{\check{w}}$ for a real-valued function λ satisfying* (2.13) *and α_k, β_k, μ_k $(k = 1, 2)$ satisfy* (7.10), (7.11), *$\mu_1 = \gamma_u$, $\mu_2 = \gamma_v$ for a function γ of two variables u, v.*

In addition, functions λ, α_k, β_k, μ_k $(k = 1, 2)$ as in (b) *locally define a time-like surface in N with zero mean curvature vector and $K \equiv L_0$, which is unique up to an isometry of N.*

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