

Spheres over fields, their entire rational maps and applications

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Abstract. The paper summarizes some results on algebraic geometry presence in the homotopy theory. For the homotopy group $\pi_m(\mathbb{S}^n)$, denote by $\pi_m^{alg}(\mathbb{S}^n)$ its subset of homotopy classes represented by \mathbb{R} -entire rational maps $\mathbb{S}^m \rightarrow \mathbb{S}^n$ of spheres. The main result of this paper concerns to the study of $\pi_{n+k}^{alg}(\mathbb{S}^n)$ for $k = 0, \dots, 7$.

Анотація. В роботі підсумовано деякі результати алгебраїчної геометрії в теорії гомотопій. Позначимо через $\pi_m^{alg}(\mathbb{S}^n)$ підмножину гомотопічної групи $\pi_m(\mathbb{S}^n)$, що складається з класів гомотопії, які задаються \mathbb{R} -цілими раціональними відображеннями сфер $\mathbb{S}^m \rightarrow \mathbb{S}^n$. Основний результат роботи присвячений вивченню груп $\pi_{n+k}^{alg}(\mathbb{S}^n)$ для $k = 0, \dots, 7$.

INTRODUCTION

Algebraic sets, polynomial and K -entire rational maps are studied in algebraic geometry over any field K , but in the case of the field \mathbb{R} of reals they have rather special behaviour. The fact that there are non-constant polynomials without zeros gives rise to \mathbb{R} -entire rational functions which are everywhere defined and are not polynomials.

The main aim of this partially survey paper is representing (up to homotopy) of elements of homotopy groups $\pi_m(\mathbb{S}^n)$ by algebraic (i.e., \mathbb{R} -entire rational) maps for the n -sphere \mathbb{S}^n . Polynomial maps between real algebraic sets, mainly between unit spheres, have been studied by several authors. For the groups $O(n)$ of all $n \times n$ -orthogonal matrices with $n \geq 1$ define the homotopy group $\pi_m(O) = \lim_{n \rightarrow \infty} \pi_m(O(n))$ for $m \geq 1$. Since elements of

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$O(n)$ can be regarded as self-maps $\mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$, an element of $\pi_m(O(n))$ can be presented by a map $\mathbb{S}^m \times \mathbb{S}^{n-1} \rightarrow \mathbb{S}^{n-1}$. Applying the Hopf construction to this, we get a map $\mathbb{S}^{m+n} = \mathbb{S}^m \star \mathbb{S}^{n-1} \rightarrow \Sigma(\mathbb{S}^{n-1}) = \mathbb{S}^n$ for the join $\mathbb{S}^m \star \mathbb{S}^{n-1}$ of spheres \mathbb{S}^m and \mathbb{S}^{n-1} . This yields the Whitehead J -homomorphism

$$J: \pi_m(O(n)) \longrightarrow \pi_{m+n}(\mathbb{S}^n).$$

Taking a direct limit as n tends to infinity gives the stable J -homomorphism:

$$J: \pi_m(O) \longrightarrow \pi_m^S.$$

Baum has shown in [1] how to represent elements in the image of the stable J -homomorphism by quadratic forms. Then, Wood has studied in [17] the general problem of representing homotopy classes by polynomial maps sending sphere to sphere. Given a real algebraic set X , the authors of [4, Chapter 13] compare the set of \mathbb{R} -entire rational maps $X \rightarrow \mathbb{S}^n$ with the corresponding set of continuous or smooth mappings. Then, Bochnak and Kucharz have investigated in [2, 3, 5] an approximation concept of \mathcal{C}^∞ -maps of smooth real algebraic sets by \mathbb{R} -entire rational maps in the \mathcal{C}^∞ -topology.

Section 1 sets the stage for the developments to come. This introductory section is devoted to some necessary backgrounds on algebraic geometry needed in the sequel. In particular, polynomial and K -entire rational maps of algebraic sets are studied for any formally real closed field K .

Section 2 is based mainly on [8] and is devoted to the study of spheres

$$\mathbb{S}^n(K) = \{(x_0, \dots, x_n) \in K^{n+1} \mid x_0^2 + \dots + x_n^2 = 1\}$$

over a field K . Subsection 2.1 shows that the associated ideal

$$J(\mathbb{S}^n(K)) = \begin{cases} (X_0 + \dots + X_n + 1) & \text{if } \chi(K) = 2, \\ (X_0^2 + \dots + X_n^2 - 1) & \text{if } \chi(K) \neq 2 \end{cases}$$

is prime provided K (with the characteristic $\chi(K)$) is an infinite field. Consequently, $\mathbb{S}^n(K)$ is an irreducible algebraic set. Furthermore, Corollary 2.1.4 states that for any extension $K \subseteq L$ of an infinite field K the Zariski closure $\overline{\mathbb{S}^n(K)} = \mathbb{S}^n(L)$ of the sphere $\mathbb{S}^n(K)$ in L^{n+1} for $n \geq 0$.

In the viewpoint of algebraic geometry, the standard sphere $\mathbb{S}^n = \mathbb{S}^n(\mathbb{R})$ for $n \geq 0$ is the simplest real algebraic set and the simplest maps are \mathbb{R} -entire rational maps including polynomial maps. This paper grew out of our desire to review known and present new techniques in representing homotopy classes of maps $\mathbb{S}^m \rightarrow \mathbb{S}^n$ by \mathbb{R} -entire rational maps.

Subsection 2.2 is concentrated on polynomial maps of spheres and some results presented mainly in [17] are summarized. In particular, let $K = \mathbb{C}$, \mathbb{H} , \mathbb{O} be the field of complex numbers, the skew \mathbb{R} -algebra of quaternions

and the alternative \mathbb{R} -algebra of octonions, respectively. Then, the multiplications " \cdot ": $K \times K \rightarrow K$ lead to 2-forms: $\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2$, $\nu: \mathbb{S}^7 \rightarrow \mathbb{S}^4$ and $\sigma: \mathbb{S}^{15} \rightarrow \mathbb{S}^8$ called the *Hopf maps (fibrations)*.

Subsection 2.3, based mainly on [2–4, 10], recalls and then generalizes some results on \mathbb{R} -entire rational maps of spheres. In particular, it contains a simple but nice result, simplifying its proof stated in [10] and saying that the iterates suspension of a homogeneous polynomial map between spheres is homotopic to an \mathbb{R} -entire rational map.

Given real algebraic sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, we write $[X, Y]$ for the set of homotopy classes of continuous maps from X to Y and by $[X, Y]^{alg}$ the subset of $[X, Y]$ of homotopy classes represented by \mathbb{R} -entire rational maps from X to Y .

Section 3 estimates the size of $\pi_m^{alg}(\mathbb{S}^n) = [\mathbb{S}^m, \mathbb{S}^n]^{alg}$ in $\pi_m(\mathbb{S}^n)$ and the size of $[\mathbb{S}^m(\mathbb{C}), \mathbb{S}^n(\mathbb{C})]^{alg}$ in $[\mathbb{S}^m(\mathbb{C}), \mathbb{S}^n(\mathbb{C})]$ for the n -sphere

$$\mathbb{S}^n(\mathbb{C}) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \dots + z_n^2 = 1\}$$

over the field \mathbb{C} of complex numbers.

The aim of Subsection 3.1 is to present an extended version of [10, Corollaries 1.1–1.4]:

Theorem 3.1.5. *Elements of the homotopy groups $\pi_{n+k}(\mathbb{S}^n)$ for $n \geq 1$, $k = 0, 1, \dots, 6$ and $k = 7$ with $n \neq 5, 6$ admit (up to homotopy) \mathbb{R} -entire rational representations.*

Then, Subsection 3.2 generalizes [18, Theorem 3] as follows:

Theorem 3.2.5. *If an element in $\pi_m(\mathbb{S}^n)$ is representable by a real homogeneous polynomial map of spheres $\mathbb{S}^m \rightarrow \mathbb{S}^n$ then its $2k$ -fold suspension is representable by a complex polynomial map $\mathbb{S}^{m+2k}(\mathbb{C}) \rightarrow \mathbb{S}^{n+2k}(\mathbb{C})$.*

1. PREREQUISITES

Throughout the rest of this paper, all spaces and maps are assumed to be connected and based with the homotopy type of CW -complexes unless we assume otherwise. We also do not distinguish notationally between a continuous map and its homotopy class. We write $\Omega(X)$ (resp. $\Sigma(X)$) for the loop (resp. suspension) space on a space X and $[X, Y]$ for the set of homotopy classes of maps $X \rightarrow Y$. Given a based space X and the n -th sphere \mathbb{S}^n , we write $\pi_n(X) = [\mathbb{S}^n, X]$ for its n -th homotopy group of X and ι_X (resp. ι_n) for the identity map on X (resp. on \mathbb{S}^n with $n \geq 0$). Furthermore, we freely use notations for elements of homotopy groups of spheres from book [14] by Toda.

Now, we present some necessary backgrounds on algebraic geometry over a field K needed in the sequel.

Let K be a field and $X \subseteq K^n$ an algebraic set, i.e.,

$$X = V(I) = \{x \in K^n \mid p(x) = 0 \text{ for } p \in I\}$$

for an ideal $I \subseteq K[X_1, \dots, X_n]$. Notice that by the Hilbert's Basis Theorem, the ideal $I = (p_1, \dots, p_k)$ for its generators $p_1, \dots, p_k \in I$. Therefore,

$$X = V(I) = \{x \in K^n \mid p_i(x) = 0 \text{ for } i = 1, \dots, k\}.$$

Proposition 1.1.

(1) If $I_\lambda \subseteq K[X_1, \dots, X_n]$ are ideals for $\lambda \in \Lambda$ then

$$V\left(\sum_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} V(I_\lambda);$$

(2) if $I_1, I_2 \subseteq K[X_1, \dots, X_n]$ are ideals then $V(I_1 I_2) = V(I_1) \cup V(I_2)$, so the finite union of algebraic sets is an algebraic set;

(3) $V(0) = K^n$ and $V(K[X_1, \dots, X_n]) = \emptyset$.

Proposition 1.1 allows to define a topology on K^n (called the *Zariski topology*) whose closed sets are precisely algebraic sets.

Now, let $X \subseteq K^m$ and $Y \subseteq K^n$ be algebraic sets. A map $f: X \rightarrow Y$ is called *polynomial* if there are polynomials $p_i \in K[X_1, \dots, X_m]$ with $i = \overline{1, n}$ such that $f(x) = (p_1(x), \dots, p_n(x))$ for $x \in X$. We write $\mathcal{P}(X, Y)$ for the set of all polynomial maps from X to Y .

A polynomial map $f = (p_1, \dots, p_n): X \rightarrow Y$ is called a *d-form* if each polynomial p_i is homogeneous of degree d for $i = 1, \dots, n$.

Given an algebraic set $X \subseteq K^n$, define the K -algebra (called the *coordinate ring of X*) of polynomial maps $K[X] = \{f: X \rightarrow K\}$ and the associated ideal $J(X) = \{p \in K[X_1, \dots, X_n] : p(x) = 0 \text{ for } x \in X\}$. Then, there is an isomorphism

$$K[X] \approx K[X_1, \dots, X_n]/J(X)$$

of K -algebras.

Proposition 1.2.

(1) If $X, Y \subseteq K^n$ are algebraic sets then $J(Y) \subseteq J(X)$;

(2) $J(\emptyset) = K[X_1, \dots, X_n]$;

(3) $S \subseteq J(V(S))$ for any subset $S \subseteq K[X_1, \dots, X_n]$ and $X \subseteq V(J(X))$ for any algebraic set $X \subseteq K^n$;

(4) $V(J(X)) = \overline{X}$, the Zariski closure of any subset $X \subseteq K^n$;

(5) $V(J(V(S))) = V(S)$ for any subset $S \subseteq K[X_1, \dots, X_n]$ and

$$J(V(J(X))) = J(X)$$

for any subset of $X \subseteq K^n$.

- (6) *There is a one-to-one correspondence between polynomial maps from X to Y of algebraic sets and K -homomorphisms of $K[Y] \rightarrow K[X]$.*
- (7) *If $f = (p_1, \dots, p_n)$, $g = (q_1, \dots, q_n): X \rightarrow Y$ are polynomial maps and $f = g$ then $p_i - q_i \in J(X)$.*

An algebraic set $X \subseteq K^n$ is called *irreducible* if $X \neq \emptyset$ and X cannot be expressed as $X = X_1 \cup X_2$, where X_1, X_2 are algebraic sets not equal to X .

Proposition 1.3. *An algebraic set $X \subseteq K^n$ is irreducible if and only if its ideal $J(X)$ is prime.*

Let $X \subseteq K^m$ and $Y \subseteq K^m$ be algebraic sets. Recall that a map $f: X \rightarrow Y$ is said to be *K -entire rational* if there exist polynomials $p_i, q_i \in K[X_0, \dots, X_n]$ such that $q_i^{-1}(0) \cap X = \emptyset$ for $i = 0, \dots, n$ and

$$f(x) = \left(\frac{p_0(x)}{q_0(x)}, \dots, \frac{p_n(x)}{q_n(x)} \right)$$

with $x \in X$. Certainly, any polynomial map $X \rightarrow Y$ is entire rational as well. We write $\mathcal{R}(X, Y)$ for the set of all \mathbb{R} -entire rational maps from X to Y . For other details on algebraic geometry over an algebraically closed field K , [12, Chapter 1] is recommended.

One can consider the category of algebraic varieties over formally real closed fields K and regular maps in the sense of Serre. The usual definition of regular functions is of a local nature. Since the affine space K^n is quasi-compact (in the Zariski topology), any algebraic subset $V \subseteq K^n$ is also quasi-compact. Thus, by [4, Chapter 3, Proposition 3.2.3], the local nature (in the Zariski topology) of the notion of regular function on an algebraic set is compatible with the existence of a global denominator. Consequently, a map $X \rightarrow Y$ of algebraic sets over a field K is regular if and only if it is K -entire rational.

2. SPHERES OVER FIELDS AND THEIR ENTIRE RATIONAL MAPS

This section is based mainly on [8] and is devoted to the study of spheres

$$S^n(K) = \{(x_0, \dots, x_n) \in K^{n+1} \mid x_0^2 + \dots + x_n^2 = 1\}$$

for $n \geq 0$ over a field K .

2.1. Spheres over fields. Given a field K , write $\chi(K)$ for its characteristic. Let $(X_0 + \dots + X_n - 1)$ and $(X_0^2 + \dots + X_n^2 - 1)$ be principal ideals in $K[X_0, \dots, X_n]$ generated by the polynomials $X_0 + \dots + X_n - 1$ and $X_0^2 + \dots + X_n^2 - 1$, respectively. Notice that the polynomial

$$X_0^2 + \dots + X_n^2 - 1 \in K[X_0, \dots, X_n]$$

is irreducible for $\chi(K) \neq 2$.

Recall that a field K in which every sum of two squares is a square, is called *Pythagorean*. Equivalently, a finite number of its squares is a square. Certainly \mathbb{R} , the field of reals and any algebraically closed field K are Pythagorean.

Then, we state:

Lemma 2.1.1. *If K is a Pythagorean field then*

$$J(\mathbb{S}^n(K)) = \begin{cases} (X_0 + \cdots + X_n + 1), & \chi(K) = 2, \\ (X_0^2 + \cdots + X_n^2 - 1), & \chi(K) \neq 2. \end{cases}$$

Proof. If K is a field with $\chi(K) = 2$ then

$$\mathbb{S}^n(K) = \{(x_0, \dots, x_n) \in K^{n+1} \mid x_0 + \cdots + x_n + 1 = 0\}.$$

Since,

$$K[X_0, \dots, X_n] = K[X_0 + \cdots + X_n + 1, X_1, \dots, X_n],$$

the results follows.

Next, note that any finite field with $\chi(K) \neq 2$ is not Pythagorean. Hence, any Pythagorean field K with $\chi(K) \neq 2$ is infinite.

Let $p \in K[X_0, \dots, X_n]$ be a polynomial which vanishes on $\mathbb{S}^n(K)$, write $p = p_0 + p_1$, where p_0 has all monomials with even degree and p_1 with odd degree. If x is in $\mathbb{S}^n(K)$ then also $-x$ is in $\mathbb{S}^n(K)$. Thus, $p_0(x) + p_1(x) = 0$ and $p_0(x) - p_1(x) = 0$ imply $p_0(x) = p_1(x) = 0$ for all points x in $\mathbb{S}^n(K)$. Therefore, we may assume that all monomials of the given polynomial P have either even or odd degree. Multiplying all monomials of p by an appropriate power of $X_0^2 + \cdots + X_n^2$, we get a polynomial

$$q \equiv p \pmod{(X_0^2 + \cdots + X_n^2 - 1)}.$$

Hence, we may also assume that p is homogeneous of degree d and vanishes on both algebraic sets

$$\begin{aligned} &\{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\}, \\ &\{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 = 0\}. \end{aligned}$$

Let $x = (x_0, \dots, x_n) \in K^n$ and y be a solution of the equation

$$X^2 = x_0^2 + \cdots + x_n^2$$

in the field K provided that $x_0^2 + \cdots + x_n^2 \neq 0$. Then x/y is in the sphere $\mathbb{S}^n(K)$ and

$$P(x) = p(yx/y) = y^d p(x/y) = 0.$$

But, the field K is infinite, so the polynomial p is trivial and the proof is complete. \square

Now, for a field K with $\chi(K) \neq 2$, let

$$V_K = V_K\left(X_0 - 1, \sum_{k=1}^n X_k^2\right)$$

and

$$\mathbb{S}_i^{n-1}(K) = V_K\left(\sum_{k=1}^n X_k^2 + 1\right)$$

be affine sets in K^{n+1} and K^n , respectively. Then, the rational function

$$\Phi_n(K) = \left(\frac{X_1}{1 - X_0}, \dots, \frac{X_n}{1 - X_0}\right)$$

(determined by the familiar stereographic projection) yields the bijection

$$\phi_n(K): \mathbb{S}^n(K) \setminus V_K \rightarrow K^n \setminus \mathbb{S}_i^{n-1}(K)$$

with the inverse

$$\psi_n(K): K^n \setminus \mathbb{S}_i^{n-1}(K) \rightarrow \mathbb{S}^n(K) \setminus V_K$$

determined by the inverse

$$\Psi_n(K) = \frac{1}{1 + \sum_{k=1}^n X_k^2} \left(\sum_{k=1}^n X_k^2 - 1, 2X_1, \dots, 2X_n\right)$$

to the stereographic projection. Whence, $\mathbb{S}^n(K)$ and K^n are birationally equivalent for any $n \geq 0$ and we are in a position to state:

Proposition 2.1.2. *Let K be an infinite field with $\chi(K) \neq 2$. Then*

$$\overline{\mathbb{S}^n(K)} = \mathbb{S}^n(\overline{K}) \quad \text{for } n \geq 0$$

where \overline{K} denotes the algebraic closure of K and $\overline{\mathbb{S}^n(K)}$ the Zariski closure of $\mathbb{S}^n(K)$ in \overline{K}^{n+1} .

Proof. Obviously, $\overline{\mathbb{S}^0(K)} = \mathbb{S}^0(\overline{K})$. Next, recall that $\overline{\mathbb{S}^n(K)} = V(J(\mathbb{S}^n(K)))$ and notice that $J(\mathbb{S}^n(\overline{K})) \subseteq J(\mathbb{S}^n(K))$ for $n \geq 1$.

Now, if a polynomial $p \in \overline{K}[X_0, \dots, X_n]$ vanishes on $\mathbb{S}^n(K)$ then $p\Psi_n(K)$ vanishes on $K^{n+1} \setminus \mathbb{S}_i^n(K)$ for $n \geq 1$. Because the field K is infinite, K^n is dense in \overline{K}^n (with respect to the Zariski topology on \overline{K}^n) and consequently $K^n \setminus \mathbb{S}_i^{n-1}(K)$ is dense in $\overline{K}^n \setminus \mathbb{S}_i^{n-1}(\overline{K})$ as well.

The regular function $p\Psi_n(K)$ vanishes on $K^n \setminus \mathbb{S}_i^{n-1}(K)$ whence $p\Psi_n(\overline{K})$ vanishes on $\overline{K}^n \setminus \mathbb{S}_i^{n-1}(\overline{K})$. Therefore, the polynomial p vanishes on the

non-empty open subset $\mathbb{S}^n(\overline{K}) \setminus V_{\overline{K}}$. Because the field \overline{K} is Pythagorean, in view of Lemma 2.1.1, the principal ideal

$$J(\mathbb{S}^n(\overline{K})) = \left(\sum_{k=0}^n X_k^2 - 1 \right)$$

is prime. So, the sphere $\mathbb{S}^n(\overline{K})$ is irreducible and hence the open set $\mathbb{S}^n(\overline{K}) \setminus V_{\overline{K}}$ is dense in $\mathbb{S}^n(\overline{K})$. Thus, p vanishes on the whole sphere $\mathbb{S}^n(\overline{K})$ and so

$$p \in J(\mathbb{S}^n(\overline{K})) = \left(\sum_{k=0}^n X_k^2 - 1 \right).$$

Consequently, $\overline{\mathbb{S}^n(K)} = \mathbb{S}^n(\overline{K})$ and this completes the proof. \square

Corollary 2.1.3. *Let K be an infinite field with $\chi(K) \neq 2$. Then*

$$J(\mathbb{S}^n(K)) = (X_0^2 + \cdots + X_n^2 - 1).$$

Proof. Let $p \in K[X_0, \dots, X_n]$ be a polynomial vanishing on the sphere $\mathbb{S}^n(K)$. Since, by Proposition 2.1.2, the Zariski closure $\overline{\mathbb{S}^n(K)} = \mathbb{S}^n(\overline{K})$, the continuity of p as a polynomial map yields

$$p(\mathbb{S}^n(\overline{K})) = p(\overline{\mathbb{S}^n(K)}) \subseteq \overline{p(\mathbb{S}^n(K))} = \{0\}.$$

Next, mimicking the proof of Lemma 2.1.1, we may also assume that p is homogeneous of degree d and vanishes on both varieties

$$\begin{aligned} \mathbb{S}^n(K) &= \{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\}, \\ &\{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 = 0\}. \end{aligned}$$

Let $(x_0, \dots, x_n) \in K^n$ and y be a solution of the equation $X^2 = x_0^2 + \cdots + x_n^2$ in the field \overline{K} provided that $x_0^2 + \cdots + x_n^2 \neq 0$.

Then x/y is in the sphere $\mathbb{S}^n(\overline{K})$ and

$$p(x) = p(yx/y) = y^d p(x/y) = 0.$$

But, the field K is infinite, so the polynomial p is trivial and the proof is complete. \square

Since $K[X_0, \dots, X_n]$ is a *UFD*-ring and the polynomials

$$X_0 + \cdots + X_n + 1, \quad X_0^2 + \cdots + X_n^2 - 1 \in K[X_0, \dots, X_n]$$

for $\chi(K) \neq 2$ are irreducible, Corollary 2.1.3 yields that the n -sphere $\mathbb{S}^n(K)$ is an irreducible algebraic set provided the field K is infinite.

Next, let K be an infinite field and $K \subseteq L$ be its extension. Since, the ideal

$$J(\mathbb{S}^n(L)) = \begin{cases} (X_0 + \cdots + X_n + 1), & \chi(L) = 2, \\ (X_0^2 + \cdots + X_n^2 - 1), & \chi(K) \neq 2. \end{cases}$$

is prime, the proof of Proposition 2.1.2 leads to:

Corollary 2.1.4. *Let K be an infinite field. Then the following statements hold.*

- (1) $\overline{\mathbb{S}^n(K)} = \mathbb{S}^n(L)$ for $n \geq 0$ for any extension $K \subseteq L$ of the field K , where $\overline{\mathbb{S}^n(K)}$ denotes the Zariski closure of $\mathbb{S}^n(K)$ in L^{n+1} . In particular,

$$\overline{\mathbb{S}^n(\mathbb{Q})} = \mathbb{S}^n(K)$$

for $n \geq 0$ and any field K with $\chi(K) = 0$, where \mathbb{Q} denotes the field of rationals and $\overline{\mathbb{S}^n(\mathbb{Q})}$ the Zariski closure of $\mathbb{S}^n(\mathbb{Q})$ in K^{n+1} .

- (2) There exists a bijection between the algebraic sets of polynomial maps $\mathbb{S}^m(K) \rightarrow \mathbb{S}^n(K)$ and homomorphisms of K -algebras

$$\frac{K[X_0, X_1, \dots, X_m]}{(X_0^2 + \cdots + X_m^2 - 1)} \rightarrow \frac{K[X_0, X_1, \dots, X_n]}{(X_0^2 + \cdots + X_n^2 - 1)}$$

with $\chi(K) \neq 2$ and

$$\frac{K[X_0, X_1, \dots, X_m]}{(X_0 + \cdots + X_m - 1)} \rightarrow \frac{K[X_0, X_1, \dots, X_n]}{(X_0 + \cdots + X_n - 1)}$$

with $\chi(K) = 2$.

2.2. Spheres over reals. We summarize some results presented mainly in [17].

In what follows, we write $\mathbb{S}^n = \mathbb{S}^n(\mathbb{R})$ with $n \geq 0$ for \mathbb{R} , the field of reals. From the point of view of algebraic geometry, the standard sphere \mathbb{S}^n is the simplest real algebraic set and the simplest maps are \mathbb{R} -entire rational maps including polynomial maps.

First, the following appears:

Question 2.2.1. *Which maps of spheres $\mathbb{S}^m \rightarrow \mathbb{S}^n$ can be represented (up to homotopy) by polynomial maps?*

From the differential geometric point of view, all of them. Namely, given a compact subset $X \in \mathbb{R}^m$ and a continuous map $f: X \rightarrow \mathbb{S}^n$, consider the composition map $\tilde{f}: X \rightarrow \mathbb{S}^n \hookrightarrow \mathbb{R} \setminus \{0\}$.

Then, by the Stone-Weierstrass Approximation Theorem there exists a sequence $\{p_k\}_{k \geq 1}$ uniformly converging to \tilde{f} , where $p_k = (p_k^{(0)}, \dots, p_k^{(n)})$ and $p_k^{(i)} \in \mathbb{Q}[X_1, \dots, X_m]$ for $i = 1, \dots, n$ and $k \geq 1$.

Since $\{p_k\}_{k \geq 1}$ uniformly converges to \tilde{f} , we have that $p_k(x) \neq 0$ and $\tilde{f}(x) \neq -p_k(x)$ for $x \in X$ with $k \gg 0$. Consequently, we get a homotopy $\tilde{f} \simeq p_k$ for $k \gg 0$. But, $p_k(x) \notin \mathbb{S}^n$, while $\frac{p_k(x)}{\|p_k(x)\|} \in \mathbb{S}^n$ for $x \in X$ with $k \gg 0$.

A first result by Baum [1] suggested a wide affirmative answer to Question 2.2.1 but, later on, Wood (1968) showed:

Theorem 2.2.2. *If n is a power of 2 then all polynomial maps from \mathbb{S}^n to \mathbb{S}^{n-1} are constant.*

Consequently the real polynomial representation is impossible in many cases among them, for example, the non-trivial element of $\pi_4(\mathbb{S}^3)$.

Corollary 2.2.3.

- (1) *If $n \geq 2^d > k$ for some integer $d \geq 1$, then any polynomial map $\varphi: \mathbb{S}^n \rightarrow \mathbb{S}^k$ is constant;*
- (2) *if $n \geq 2k$ then any polynomial map $\mathbb{S}^n \rightarrow \mathbb{S}^k$ is constant.*

Proof. Let $d_0 = \min\{d; k < 2^d\}$. For $n < 2^{d_0}$ we have $2k \leq n < 2^{d_0}$ and so $k < 2^{d_0-1}$. Therefore, $2^{d_0} \leq n$. Hence, $n \geq 2^{d_0} > k$ and the proof follows from (1). \square

Example 2.2.4.

- (1) If $m < n$ then for any map $f: \mathbb{S}^m \rightarrow \mathbb{S}^n$, we have $f \simeq *$
- (2) If $f: \mathbb{S}^m \rightarrow \mathbb{S}^n$ and $f(\mathbb{S}^m) \not\subseteq \mathbb{S}^n$ then $f \simeq *$.
- (3) If $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ then $f \simeq (p, q)$ with $p, q \in \mathbb{Z}[X_0, X_1]$ for the ring \mathbb{Z} of integers.

The same for self-maps of the spheres \mathbb{S}^3 and \mathbb{S}^7 .

Proof. (3) Let $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $f: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a map with Brouwer degree n . Then, this map is represented (up to homotopy) by the self-map $f_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by $f_n(z) = z^n$ for $z \in \mathbb{S}^1$. Then, for $z = x_0 + ix_1$ with $x_0, x_1 \in \mathbb{R}$ we get

$$z^n = (x_0 + ix_1)^d = \sum_{k=0}^d \binom{d}{k} x_0^k (ix_1)^{d-k} = p_n(x_0, x_1) + iq_n(x_0, x_1).$$

Finally, $f \simeq (p_n, q_n)$ with $p_n, q_n \in \mathbb{Z}[X_0, X_1]$ and the proof follows.

For spheres \mathbb{S}^3 and \mathbb{S}^7 we follow the procedure applied above. \square

Theorem 2.2.5. *If n is odd then elements of the infinite cyclic group $\pi_n(\mathbb{S}^n) = \mathbb{Z}\{\iota_n\}$ corresponding to the integer k can be represented by a $|k|$ -form over \mathbb{Z} from \mathbb{S}^n to \mathbb{S}^n .*

Hopf forms. Let $\Phi: \mathbb{R}^{k+1} \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a normed bilinear form that is $\|\Phi(x, y)\| = \|x\| \cdot \|y\|$ for $(x, y) \in \mathbb{R}^{k+1} \times \mathbb{R}^m$. Corresponding to Φ there is a quadratic form $\varphi: \mathbb{S}^{k+m} \rightarrow \mathbb{S}^n$ given by

$$\varphi(x, y) = (\|x\|^2 - \|y\|^2, 2\Phi(x, y))$$

for (x, y) .

In particular, let $K = \mathbb{C}, \mathbb{H}$ or \mathbb{O} be the field of complex numbers, the skew \mathbb{R} -algebra of quaternions and the alternative \mathbb{R} -algebra of octonions, respectively. Then, for the multiplication $\circ: K \times K \rightarrow K$ being a Hopf form, we get quadratic forms:

$$\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2, \quad \nu: \mathbb{S}^7 \rightarrow \mathbb{S}^4, \quad \sigma: \mathbb{S}^{15} \rightarrow \mathbb{S}^8$$

called the *Hopf maps (fibrations)*.

Furthermore, the Hopf fibration $\mathbb{S}^1 \rightarrow \mathbb{S}^3 \xrightarrow{h} \mathbb{S}^2$ can be expressed as $h(q) = qi\bar{q}$, where $q \in \mathbb{S}^3$ is regarded as a unit quaternion, which suggests that the Hopf fibration $\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is non-trivial because the quaternions do not commute.

The Hopf fibration $\mathbb{S}^3 \rightarrow \mathbb{S}^7 \xrightarrow{h} \mathbb{S}^4$ can be written as $h(c) = (e_1c)(\bar{c}e_2)$, where $c \in \mathbb{S}^7$ is regarded as a unit octonion. Analogously, this suggests that its non-triviality is due to the non-associativity of the octonions.

Next, write $\eta_n = \Sigma^{n-2}\eta$ for $n \geq 2$, $\nu_n = \Sigma^{n-4}\nu$ for $n \geq 4$ and $\sigma_n = \Sigma^{n-8}\sigma$ for $n \geq 8$.

Then, in view of [11], the polynomial map $f: \mathbb{S}^7 \rightarrow \mathbb{S}^6$ which maps c to $ce_1\bar{c}$ for $c \in \mathbb{S}^7 \subseteq \mathbb{O}$ represents the generator of the group $\pi_7(\mathbb{S}^6) = \mathbb{Z}_2\{\eta_6\}$.

But, by means of Corollary 2.2.3, the Blakers-Massey elements representing the generators of $\pi_6(\mathbb{S}^3) = \mathbb{Z}_{12}\{\nu'\}$ and $\pi_{14}(\mathbb{S}^7) = \mathbb{Z}_{120}\{\sigma'\}$ cannot be represented by polynomial maps.

Nevertheless, write $U_2(\mathbb{H})$ for the unitary group of 2×2 -matrices over \mathbb{H} . Then, in view of [13, Part II], the Blakers-Massey element which represents the generators of $\pi_6(\mathbb{S}^3) = \mathbb{Z}_{12}\{\nu'\}$ is the characteristic map of the principal fibration $\mathbb{S}^3 \rightarrow U_2(\mathbb{H}) \xrightarrow{p} \mathbb{S}^7$.

On the other hand, let

$$\mathbb{S}^6 = \{(v, w) \in \mathbb{H}_0 \times \mathbb{H} \mid |v|^2 + |w|^2 = 1\},$$

where $\mathbb{H}_0 = \{v \in \mathbb{H} \mid \text{Re}(v) = 0\}$. Then, in view of [6, Proposition 19.1], the generator of the group $\pi_6(\mathbb{S}^3) = \mathbb{Z}_{12}\{\nu'\}$ is also represented by the map $f: \mathbb{S}^6 \rightarrow \mathbb{S}^3$ (being an \mathbb{R} -entire rational map) given by

$$f(v, w) = 1 - 2w(1 + v)^{-2}\bar{w} \tag{2.1}$$

for $(v, w) \in \mathbb{S}^6$ which is the characteristic map of the principal fibration $\mathbb{S}^3 \rightarrow U_2(\mathbb{H}) \xrightarrow{p} \mathbb{S}^7$. Consequently, we have the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^3 & \xlongequal{\quad} & \mathbb{S}^3 \\ \downarrow & & \downarrow \\ U_2(\mathbb{H}) & \longrightarrow & E(\mathbb{S}^3) \\ p \downarrow & & \downarrow \\ \mathbb{S}^7 & \xrightarrow{\bar{f}} & B(\mathbb{S}^3) \end{array}$$

where $\bar{f}: \mathbb{S}^7 \rightarrow B(\mathbb{S}^3)$ is the left adjunct of $\mathbb{S}^6 \xrightarrow{f} \mathbb{S}^3 \simeq \Omega(B(\mathbb{S}^3))$.

Unfortunately, the sphere \mathbb{S}^7 is not a topological group and we cannot mimic the above for the Blakers-Massey element, which represents the generator of the group $\pi_{14}(\mathbb{S}^7) = \mathbb{Z}_{120}\{\sigma'\}$. But, we make use of the following procedure.

Recall that a topological monoid \mathbf{M} has a classifying space $B(\mathbf{M})$ and there is a homotopy equivalence $\mathbf{M} \xrightarrow{\simeq} \Omega B(\mathbf{M})$ preserving the multiplications up to homotopy. Many authors have worked in this area but in this context, the result is usually attributed to Dold and Lashof in [7].

Next, given a finite CW-complex X , write $\mathbf{Aut}(X)$ for the topological monoid of self-homotopy equivalences of X . Then, the pair $(\mathbf{Aut}(X), \circ)$ with multiplication $\circ: \mathbf{Aut}(X) \times \mathbf{Aut}(X) \rightarrow \mathbf{Aut}(X)$ given by composition is a topological monoid, where $\mathbf{Aut}(X)$ is topologized as a subspace of the space $\mathbf{Map}(X, X)$ of all continuous functions with the compact-open topology.

Since, the monoid $\mathbf{Aut}(X)$ is not path-connected in general, let $\mathbf{Aut}_0(X)$ be the submonoid of $\mathbf{Aut}(X)$ given by self-equivalences of X homotopic to ι_X , the identity map on X , and let $\mathbf{Aut}_0^\bullet(X)$ be the submonoid of $\mathbf{Aut}_0(X)$ fixing a basepoint. Then, there is a natural fibration given by the evaluation map

$$\mathbf{Aut}_0^\bullet(X) \rightarrow \mathbf{Aut}_0(X) \xrightarrow{\omega} X,$$

where $\omega: \mathbf{Aut}_0(X) \rightarrow X$ is the evaluation map at the basepoint $* \in X$. This gives rise to the fibration

$$X \longrightarrow B(\mathbf{Aut}_0^\bullet(X)) \xrightarrow{p_0} B(\mathbf{Aut}_0(X))$$

which is the universal fibration with fiber X . In particular, there is the universal fibration

$$\mathbb{S}^7 \longrightarrow B(\mathbf{Aut}_0^\bullet(\mathbb{S}^7)) \xrightarrow{p_0} B(\mathbf{Aut}_0(\mathbb{S}^7))$$

with fiber \mathbb{S}^7 . Consequently, for the fibration $\mathbb{S}^7 \rightarrow U_2(\mathbb{O}) \xrightarrow{p} \mathbb{S}^{15}$ with the set $U_2(\mathbb{O})$ of unitary 2×2 -matrices over \mathbb{O} , a map $g: \mathbb{S}^{15} \rightarrow B(\mathbf{Aut}_0(\mathbb{S}^7))$

make the following diagram commute

$$\begin{array}{ccc}
 \mathbb{S}^7 & \xlongequal{\quad} & \mathbb{S}^7 \\
 \downarrow & & \downarrow \\
 U_2(\mathbb{O}) & \longrightarrow & B(\mathbf{Aut}_0^\bullet(\mathbb{S}^7)) \\
 p \downarrow & & \downarrow p_0 \\
 \mathbb{S}^{15} & \xrightarrow{g} & B(\mathbf{Aut}_0(\mathbb{S}^7))
 \end{array}$$

Then, the right adjoint $g': \mathbb{S}^{14} \rightarrow \mathbf{Aut}_0(\mathbb{S}^7)$ of $g: \mathbb{S}^{15} \rightarrow B(\mathbf{Aut}_0(\mathbb{S}^7))$ composed with the evaluation map $\omega: \mathbf{Aut}_0(\mathbb{S}^7) \rightarrow \mathbb{S}^7$ at the base point $* \in \mathbb{S}^7$ leads to the map $\bar{g}: \mathbb{S}^{14} \rightarrow \mathbb{S}^7$.

On the other hand, following the above, one can define the \mathbb{R} -entire rational map $h: \mathbb{S}^{14} \rightarrow \mathbb{S}^7$ given by

$$h(p, w) = 1 - 2w(1 + p)^{-2}\bar{w},$$

for

$$(p, w) \in \mathbb{S}^{14} = \{(p, w) \in \mathbb{O}_0 \times \mathbb{O} \mid |p|^2 + |w|^2 = 1\},$$

where $\mathbb{O}_0 = \{p \in \mathbb{O} \mid \operatorname{Re}(p) = 0\}$.

Conjecture 2.2.6. *The maps $\bar{g}, h: \mathbb{S}^{14} \rightarrow \mathbb{S}^7$ defined above are homotopic and represent a generator of the group $\pi_{14}(\mathbb{S}^7) = \mathbb{Z}_{120}\{\sigma'\}$.*

To state a result for even spheres, define

$$\mathbb{S}_r^n = \{(x_0, \dots, x_n) \mid (x_0^2 + \dots + x_{r-1})^2 + x_r^2 + \dots + x_n^2 = 1\}$$

for $0 \leq r \leq n - 1$. Then, by [15, Theorem 1], we have:

Theorem 2.2.7. *Let k be an integer and suppose that $n \geq 2$ is even. Then*

- (1) *if k is odd then there exists a homogeneous polynomial self-map of \mathbb{S}^n with Brouwer degree k and algebraic degree $2|k| - 1$;*
- (2) *if k is even then, for each $2 \leq 2r \leq n$, there exists a polynomial map from \mathbb{S}_{2r}^n to \mathbb{S}^n of Brouwer degree k .*

2.3. Entire rational maps of spheres. Basing mainly on [2–4, 10], we recall and then generalize some results on \mathbb{R} -entire rational maps of spheres.

First, Bochnak and Kucharz [2, Proposition 4.1] obtained a simple but very nice result.

Proposition 2.3.1. *Let $h: \mathbb{S}^n \rightarrow \mathbb{S}^k$ be a d -form being the restriction of a homogenous polynomial map $\varphi: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{k+1}$. Then the suspension map*

$\Sigma h: \mathbb{S}^{n+1} \rightarrow \mathbb{S}^{k+1}$ is homotopic to \mathbb{R} -entire rational map $\tilde{h}: \mathbb{S}^{n+1} \rightarrow \mathbb{S}^{k+1}$ given by

$$\tilde{h}(x, y) = \frac{(2\varphi(x), (1+y)^d - (1-y)^d)}{(1+y)^d + (1-y)^d}$$

for $(x, y) \in \mathbb{S}^{n+1}$.

Proof. By a direct and elementary computation one checks that the map \tilde{h} is well defined.

Further, notice that the restriction

$$\tilde{h}|_{\mathbb{S}^n} = q, \quad \tilde{h}(\mathbb{S}_+^{n+1}) \subseteq \mathbb{S}_+^{k+1}, \quad \tilde{h}(\mathbb{S}_-^{n+1}) \subseteq \mathbb{S}_-^{k+1},$$

for the upper (resp. lower) hemisphere \mathbb{S}_+^{n+1} (resp. \mathbb{S}_-^{n+1}) of \mathbb{S}^{n+1} . Since the points $(\Sigma h)(x)$ and \tilde{q} cannot be antipodal for $x \in \mathbb{S}^n$, the map

$$H: \mathbb{S}^{n+1} \times I \rightarrow \mathbb{S}^{k+1}$$

given by

$$H(x, t) = \frac{(1-t)(\Sigma h)(x) + t\tilde{h}(x)}{\|(1-t)(\Sigma h)(x) + t\tilde{h}(x)\|}$$

for $(x, t) \in \mathbb{S}^{n+1} \times I$, is a well defined desired homotopy and the proof follows. \square

Following the proof of Proposition 2.3.1 and applying the inductive procedure, we derive its extended version:

Corollary 2.3.2. *If $h: \mathbb{S}^n \rightarrow \mathbb{S}^k$ is a d -form, then its r -fold suspension $\Sigma^r h: \Sigma^r(\mathbb{S}^n) \rightarrow \Sigma^r(\mathbb{S}^k)$ is homotopic to the map*

$$(\Sigma^r h)(x_0, \dots, x_{n+r}) = \left(\frac{2h(x_0, \dots, x_n)}{(1 + \sqrt{1 - \sum_{i=0}^n x_i^2})^d + (1 - \sqrt{1 - \sum_{i=0}^n x_i^2})^d}, \frac{(1 + \sqrt{1 - \sum_{i=0}^n x_i^2})^d - (1 - \sqrt{1 - \sum_{i=0}^n x_i^2})^d}{\sqrt{1 - \sum_{i=0}^n x_i^2}} (x_{n+1}, \dots, x_{n+r}) \right).$$

Let $(x_0, \dots, x_{n+r}) = (x, y) \in \Sigma^r(\mathbb{S}^n)$, where

$$x = (x_0, \dots, x_n) \quad \text{and} \quad y = (x_{n+1}, \dots, x_{n+r}).$$

Then, the above formula can be written as follows:

$$\Sigma^r(h)(x, y) = \frac{1}{(1 + \|y\|)^d + (1 - \|y\|)^d} \left(2h(x), \frac{((1 + \|y\|)^d - (1 - \|y\|)^d)y}{\|y\|} \right)$$

$$= \left(\frac{h(x)}{\sum \binom{d}{2j} \|y\|^{2j}}, \frac{\sum \binom{d}{2j+1} \|y\|^{2j}}{\sum \binom{d}{2j} \|y\|^{2j}} y \right).$$

We point out that the result stated in Corollary 2.3.2 has been already shown in [10, Theorem 1.1], but by much more advanced methods.

Given two real algebraic sets X and Y , we regard $\mathcal{R}(X, Y)$ as a subset of the space $\mathcal{C}^0(X, Y)$ of all continuous maps endowed with the \mathcal{C}^0 - (that is, compact-open) topology. Say that a continuous map can be *approximated by \mathbb{R} -entire rational maps* in the \mathcal{C}^0 -topology if, for every neighborhood \mathcal{U} of $f \in \mathcal{C}^0(X, Y)$, there is an \mathbb{R} -entire rational map which belongs to \mathcal{U} .

Then, [5, Theorem 1.1] states:

Theorem 2.3.3. *Let X be a real algebraic set (resp. nonsingular real algebraic set) and let Y be a homogeneous space for some linear real algebraic group. Then, for a continuous (resp. smooth) map $f: X \rightarrow Y$, the following conditions are equivalent:*

- (1) *f can be approximated by regular maps in the \mathcal{C}^0 - (resp. \mathcal{C}^∞ -) topology;*
- (2) *f is homotopic to an \mathbb{R} -entire rational map.*

Taking $Y = \mathbb{S}^n = SO(n+1)/SO(n)$ for the special orthogonal group $SO(k)$ with $k \geq 1$, in view of the above, we have solutions of several problems that have been open since the 1980's and which concern approximation of maps with values in the unit spheres. This has also several consequences for approximation of maps between spheres.

If both algebraic sets X and Y are nonsingular, then $\mathcal{R}(X, Y)$ is a subset of the space $\mathcal{C}^\infty(X, Y)$ of all \mathcal{C}^∞ -maps endowed with the \mathcal{C}^∞ -topology; therefore the concept of an approximation of a \mathcal{C}^∞ -map by \mathbb{R} -entire rational maps in the \mathcal{C}^∞ topology is well-defined. Let $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ be real algebraic sets and $\mathcal{R}(X, Y)$ be the set of all \mathbb{R} -entire rational maps from X to Y . If X and Y are compact and non-singular, we write $\mathcal{C}^\infty(X, Y)$ for the set of all \mathcal{C}^∞ -maps from X to Y .

Then, [5, Conjecture I] states:

Conjecture 2.3.4. *Let Y be a homogeneous space for some linear real algebraic group. Then, for every positive integer n , the set of \mathbb{R} -entire rational maps $\mathcal{R}(\mathbb{S}^n, Y)$ is dense in the space $\mathcal{C}^\infty(\mathbb{S}^n, Y)$.*

In particular, $\mathcal{R}(\mathbb{S}^m, \mathbb{S}^n)$ is dense in $\mathcal{C}^\infty(\mathbb{S}^m, \mathbb{S}^n)$ for every pair (m, n) of positive integers.

Now, recall the useful fact stated in [9, Proposition 17.8, p. 213]:

Proposition 2.3.5. *Let M, N be smooth manifolds and $f: M \rightarrow N$ a continuous map. Then f is continuously homotopic to a \mathcal{C}^∞ -map.*

In virtue of [3, Theorem 1.1], we have:

Theorem 2.3.6. *For each positive n the set $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^m)$ is C^∞ -dense in $C^\infty(\mathbb{S}^n, \mathbb{S}^m)$ provided $m = 1, 2$ or 4 .*

It would be exciting to extend the result above for $m = 8$ and unify its proof with that for $m = 1, 2$ or 4 . Recall that the proof of Theorem 2.3.6 for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $k \leq n$ (as the Referee has pointed out) is based on the following:

- (a) if $G_{n,k}(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid A^2 = A = \bar{A}^t, \text{tr}(A) = k\}$ for $k \leq n$ is the Grassmannian and X is a compact space then a continuous map $f: X \rightarrow G_{n,k}(\mathbb{F})$ can be approximated by regular maps if and only if the pullback under f of the universal \mathbb{F} -vector bundle is topologically isomorphic to an algebraic \mathbb{F} -vector bundle on X ;
- (b) every topological \mathbb{F} -vector bundle on the sphere \mathbb{S}^n is topologically isomorphic to an algebraic \mathbb{F} -vector bundle.

Now, the question is, can we repeat the arguments above for the octonions \mathbb{O} ? One can consider something that resembles “ \mathbb{O} -vector bundles”, so the issue is to justify (a) and (b) for $\mathbb{F} = \mathbb{O}$. Since only maps with values in the Grassmannian

$$G_{2,1}(\mathbb{O}) = \{A \in M_2(\mathbb{O}) \mid A^2 = A = \bar{A}^t, \text{tr}(A) = 1\} = \mathbb{S}^8$$

are of interest, the task may be simpler.

We close this section with the following:

Conjecture 2.3.7. *For each positive n the set $\mathcal{R}(\mathbb{S}^n, \mathbb{S}^m)$ is C^∞ -dense in $C^\infty(\mathbb{S}^n, \mathbb{S}^m)$ provided $m = 1, 2, 4$ or 8 .*

3. MAIN RESULTS

Given real algebraic sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, consider the set $[X, Y]$ of homotopy classes of all continuous maps from X to Y and its subset $[X, Y]^{alg}$ determined by homotopy classes of all maps represented by \mathbb{R} -entire rational maps from X to Y . Further, let

$$\mathbb{S}^n(\mathbb{C}) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \dots + z_n^2 = 1\}$$

be the n -sphere over the field \mathbb{C} of complex numbers.

Although it is not known whether $\pi_m^{alg}(\mathbb{S}^n)$ (resp. $[\mathbb{S}^m(\mathbb{C}), \mathbb{S}^n(\mathbb{C})]^{alg}$) is always a subgroup of $\pi_m(\mathbb{S}^n)$ (resp. $[\mathbb{S}^m(\mathbb{C}), \mathbb{S}^n(\mathbb{C})]$), we aim to estimate the size of

$$\pi_m^{alg}(\mathbb{S}^n) = [\mathbb{S}^m, \mathbb{S}^n]^{alg} \quad \text{in} \quad \pi_m(\mathbb{S}^n)$$

and

$$[\mathbb{S}^m(\mathbb{C}), \mathbb{S}^n(\mathbb{C})]^{alg} \quad \text{in} \quad [\mathbb{S}^m(\mathbb{C}), \mathbb{S}^n(\mathbb{C})].$$

3.1. $\pi_m^{alg}(\mathbb{S}^n)$ in $\pi_m(\mathbb{S}^n)$. First, recall that by [2, Proposition 2.6], we have:

Proposition 3.1.1. *Let n be any positive integer and let $m = 3$ or 7 . Then:*

- (1) $2\pi_n(\mathbb{S}^m) \subseteq \pi_n^{alg}(\mathbb{S}^m)$;
- (2) if $\pi_n(\mathbb{S}^m)$ is cyclic, then $\pi_n(\mathbb{S}^m) = \pi_n^{alg}(\mathbb{S}^m)$ or $2\pi_n(\mathbb{S}^m) = \pi_n^{alg}(\mathbb{S}^m)$;
- (3) if $\pi_n(\mathbb{S}^m)$ is cyclic of odd order then $\pi_n(\mathbb{S}^m) = \pi_n^{alg}(\mathbb{S}^m)$.

Next, [16, Theorems (8.2) and (8.3)] yields:

Proposition 3.1.2. *Let $\alpha_1, \alpha_2 \in \pi_m(X)$.*

- (1) *If $\beta \in \pi_{n-1}(\mathbb{S}^{m-1})$ then $(\alpha_1 + \alpha_2) \circ \Sigma\beta = \alpha_1 \circ \Sigma\beta + \alpha_2 \circ \Sigma\beta$.*
- (2) *If X is an H -space and $\gamma \in \pi_n(\mathbb{S}^m)$ then $(\alpha_1 + \alpha_2) \circ \gamma = \alpha_1 \circ \gamma + \alpha_2 \circ \gamma$.*

Given $\alpha \in \pi_m(\mathbb{S}^m)$ with $m \geq 1$, write $\deg(\alpha)$ for its Brouwer degree. Then, Proposition 3.1.2 leads to:

Corollary 3.1.3. *Let $\alpha \in \pi_m(\mathbb{S}^m)$ with $m \geq 2$.*

- (1) *If $\beta \in \pi_{n-1}(\mathbb{S}^{m-1})$ with $n \geq 2$ then the homotopy class of $\alpha \circ \Sigma\beta$ is represented by $\deg(\alpha)\Sigma(\beta)$.*
- (2) *If $\gamma \in \pi_n(\mathbb{S}^m)$ with $m = 3, 7$ and $n \geq 3$ then the homotopy class of $\alpha \circ \gamma$ is represented by $\deg(\alpha)\gamma$.*

Now, let $\eta_n = \Sigma^{n-2}\eta$ for $n \geq 2$, $\nu_n = \Sigma^{n-4}\nu$ for $n \geq 4$ and $\sigma_n = \Sigma^{n-8}\sigma$ for $n \geq 8$. Then, in view of [10, Theorem 2.1, Corollaries 2.1-2.2], we can state:

Proposition 3.1.4. *The Whitehead product $[\iota_n, \iota_n]$ for $n \geq 1$ admits (up to homotopy) an \mathbb{R} -entire rational representation.*

The Whitehead products $[\iota_n, \eta_n]$ for $n \geq 3$, $[\iota_n, \nu_n]$ for $n \geq 5$ and $[\iota_n, \sigma_n]$ for $n \geq 9$ admit (up to homotopy) \mathbb{R} -entire rational representations.

Now, we are in a position to present a generalization of results stated in [10, Corollaries 1.1–1.4]:

Theorem 3.1.5. *Elements of the homotopy groups $\pi_{n+k}(\mathbb{S}^n)$ for $n \geq 1$, $k = 0, 1, \dots, 6$ and $k = 7$ with $n \neq 5, 6$ admit (up to homotopy) \mathbb{R} -entire rational representations.*

Proof. We freely use notations for elements of homotopy groups of spheres from the book [14] by Toda.

If $k = 0$ then Theorem 2.2.5 and Proposition 3.1.1 yield

$$\pi_n(\mathbb{S}^n) = \pi_n^{alg}(\mathbb{S}^n), \quad n \geq 1.$$

If $k = 1, 2$ then by [14, Propositions 5.1–5.2 and Table of $\pi_{n+k}(\mathbb{S}^n)$, I], we have:

$$\pi_{n+1}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}\{\eta\}, & n = 2, \\ \mathbb{Z}_2\{\eta_n\}, & n \geq 3, \end{cases}$$

and $\pi_{n+2}(\mathbb{S}^n) = \mathbb{Z}_2\{\eta_n^2\}$ for $n \geq 2$.

Since the Hopf map $\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is a 2-form, and by Theorem 2.2.5, there exists a k -form $\mathbb{S}^3 \rightarrow \mathbb{S}^3$ of the Brouwer degree k , we get that

$$\pi_3(\mathbb{S}^2) = \mathbb{Z}\{\eta\} = \pi_3^{alg}(\mathbb{S}^2).$$

Next, Corollary 2.3.2 implies that the homotopy class of $\eta_n = \Sigma^{n-2}\eta$ is represented by an \mathbb{R} -entire rational map for $n \geq 3$. Therefore,

$$\pi_{n+1}(\mathbb{S}^n) = \pi_{n+1}^{alg}(\mathbb{S}^n) \quad \text{and} \quad \pi_{n+2}(\mathbb{S}^n) = \pi_{n+2}^{alg}(\mathbb{S}^n) \quad \text{for} \quad n \geq 3.$$

If $k = 3$ then, by [14, Proposition 5.6 and Table of $\pi_{n+k}(\mathbb{S}^n)$, I], we have:

$$\pi_{n+3}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}_2\{\eta_2^3\}, & n = 2, \\ \mathbb{Z}_{12}\{\nu'\}, & n = 3, \\ \mathbb{Z}\{\nu\} \oplus \mathbb{Z}_{12}\{\Sigma\nu'\}, & n = 4, \\ \mathbb{Z}_{24}\{\nu_n\}, & n \geq 5. \end{cases}$$

Since the Hopf map $\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2$ is a 2-form, Corollary 2.3.2 implies that $\pi_5(\mathbb{S}^2) = \pi_5^{alg}(\mathbb{S}^2)$. Next, in view of (2.1), the Blakers-Massey element ν' is represented by an \mathbb{R} -entire rational map and by Theorem 2.2.5, there exists a $|k|$ -form $\mathbb{S}^3 \rightarrow \mathbb{S}^3$ of the Brouwer degree k , we get that $\pi_6(\mathbb{S}^3) = \pi_6^{alg}(\mathbb{S}^3)$. Furthermore, by Theorems 2.3.6 and 2.3.3, we have $\pi_7(\mathbb{S}^4) = \pi_7^{alg}(\mathbb{S}^4)$.

Now, since the Hopf map $\nu: \mathbb{S}^7 \rightarrow \mathbb{S}^4$ is a 2-form, Corollary 2.3.2 implies that the homotopy class of the map

$$\nu_n = \Sigma^{n-4}\nu: \mathbb{S}^{n+3} \rightarrow \mathbb{S}^n$$

is represented by an \mathbb{R} -entire rational map for $n \geq 4$. Therefore, in view of Corollary 3.1.3 and case $k = 0$, we have $\pi_{n+3}(\mathbb{S}^n) = \pi_{n+3}^{alg}(\mathbb{S}^n)$ for $n \geq 5$.

Consequently,

$$\pi_{n+3}(\mathbb{S}^n) = \pi_{n+3}^{alg}(\mathbb{S}^n) \quad \text{for} \quad n \geq 2.$$

If $k = 4$ then, by [14, Proposition 5.8 and Table of $\pi_{n+k}(\mathbb{S}^n)$, I], we have:

$$\pi_{n+4}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}_{12}\{\eta\nu'\}, & n = 2, \\ \mathbb{Z}_2\{\nu'\eta_6\}, & n = 3, \\ \mathbb{Z}_2\{\nu\eta_7\} \oplus \mathbb{Z}_2\{\Sigma\nu'\eta_7\}, & n = 4, \\ \mathbb{Z}_2\{\nu_5\eta_8\}, & n = 5, \\ 0, & n \geq 6. \end{cases}$$

First, notice that Theorems 2.3.6 and 2.3.3 imply $\pi_8(\mathbb{S}^4) = \pi_8^{alg}(\mathbb{S}^4)$. Since \mathbb{R} -entire rational maps are preserved by compositions, arguments applied in (3) yield,

$$\pi_{n+4}(\mathbb{S}^n) = \pi_{n+4}^{alg}(\mathbb{S}^n) \text{ for } n \geq 2.$$

If $k = 5$ then, by [14, Proposition 5.6 and Table of $\pi_{n+k}(\mathbb{S}^n)$, I], we have:

$$\pi_{n+5}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}_2\{\eta\nu'\eta_6\}, & n = 2, \\ \mathbb{Z}_2\{\nu'\eta_6^2\}, & n = 3, \\ \mathbb{Z}_2\{\nu\eta_7^2\} \oplus \mathbb{Z}_2\{\Sigma\nu'\eta_7^2\}, & n = 4, \\ \mathbb{Z}_2\{\nu_5\eta_8^2\}, & n = 5, \\ \mathbb{Z}\{\iota_6, \iota_6\}, & n = 6, \\ 0, & n \geq 7. \end{cases}$$

First, notice that Theorems 2.3.6 and 2.3.3 imply $\pi_9(\mathbb{S}^4) = \pi_9^{alg}(\mathbb{S}^4)$. Then, arguments applied in case $k = 4$, Corollary 3.1.3 and Proposition 3.1.4 yield,

$$\pi_{n+5}(\mathbb{S}^n) = \pi_{n+5}^{alg}(\mathbb{S}^n) \text{ for } n \geq 2.$$

If $k = 6$ then, by [14, Proposition 5.11 and Table of $\pi_{n+k}(\mathbb{S}^n)$, I] we have:

$$\pi_{n+6}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}_2\{\eta\nu'\eta_6^2\}, & n = 2, \\ \mathbb{Z}_3, & n = 3, \\ \mathbb{Z}_{24}\{\nu_4^2\} \oplus \mathbb{Z}_3, & n = 4, \\ \mathbb{Z}_2\{\nu_n^2\}, & n \geq 5. \end{cases}$$

Since \mathbb{R} -entire rational maps are preserved by compositions, we have $\pi_8(\mathbb{S}^2) = \pi_8^{alg}(\mathbb{S}^2)$.

Furthermore, by:

- Theorems 2.3.6 and 2.3.3, $\pi_8(\mathbb{S}^2) = \pi_8^{alg}(\mathbb{S}^2)$;
- Proposition 3.1.1(3), $\pi_9(\mathbb{S}^3) = \pi_9^{alg}(\mathbb{S}^3)$;
- Theorems 2.3.6 and 2.3.3, $\pi_{10}(\mathbb{S}^4) = \pi_{10}^{alg}(\mathbb{S}^4)$;
- Corollary 2.3.2, the homotopy class of the map $\nu_n^2: \mathbb{S}^{n+6} \rightarrow \mathbb{S}^n$ is represented by an \mathbb{R} -entire rational map for $n \geq 5$.

Consequently,

$$\pi_{n+6}(\mathbb{S}^n) = \pi_{n+6}^{alg}(\mathbb{S}^n), \quad n \geq 2.$$

If $k = 7$ then, by [14, Proposition 5.15 and Table of $\pi_{n+k}(\mathbb{S}^n)$, I], we have:

$$\pi_{n+7}(\mathbb{S}^n) = \begin{cases} \mathbb{Z}_3, & n = 2, \\ \mathbb{Z}_{15}, & n = 3, \\ \mathbb{Z}_{15}, & n = 4, \\ \mathbb{Z}_2\{\sigma'''\} \oplus \mathbb{Z}_{15}, & n = 5, \\ \mathbb{Z}_4\{\sigma''\} \oplus \mathbb{Z}_{15}, & n = 6, \\ \mathbb{Z}_{120}\{\sigma'\}, & n = 7, \\ \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}_{120}\{\Sigma\sigma'\}, & n = 8, \\ \mathbb{Z}_{240}\{\sigma_n\}, & n \geq 9. \end{cases}$$

Then, by:

- Theorems 2.3.6 and 2.3.3, $\pi_{n+7}(\mathbb{S}^n) = \pi_{n+7}^{alg}(\mathbb{S}^n)$ if $n = 2, 4$;
- Proposition 3.1.1(3), Corollary 3.1.3 and case $k = 0$, $\pi_{10}(\mathbb{S}^3) = \pi_{10}^{alg}(\mathbb{S}^3)$;
- Proposition 3.1.1(2), $\pi_{14}(\mathbb{S}^7) = \pi_{14}^{alg}(\mathbb{S}^7)$ or $2\pi_{14}(\mathbb{S}^7) = \pi_{14}^{alg}(\mathbb{S}^7)$. Furthermore, Conjecture 2.2.6 (provided its affirmative claim), Corollary 3.1.3 and case $k = 0$ yield $\pi_{14}(\mathbb{S}^7) = \pi_{14}^{alg}(\mathbb{S}^7)$;
- Conjecture 2.3.7, $\pi_{15}(\mathbb{S}^8) = \pi_{15}^{alg}(\mathbb{S}^8)$;
- Corollary 2.3.2, the homotopy class of the map $\sigma_n = \Sigma^{n-8}\sigma: \mathbb{S}^{n-8} \rightarrow \mathbb{S}^8$ is represented by an \mathbb{R} -entire rational maps for $n \geq 8$.

Consequently, in view of Corollary 3.1.3 and case $k = 0$, we have

$$\pi_{n+7}(\mathbb{S}^n) = \pi_{n+7}^{alg}(\mathbb{S}^n)$$

for $n \geq 9$ and the proof is complete. \square

3.2. Spheres over complex numbers. First, we show:

Proposition 3.2.1. *If K is a algebraically closed field then the only entire rational maps of algebraic sets are polynomial maps.*

Proof. Let $X = V(I) \subseteq K^m$ and $Y \subseteq K^n$ be algebraic sets. Consider a rational map $f = (f_1, \dots, f_n): X \rightarrow Y$, where $p_i, q_i \in K[X_1, \dots, X_m]$, $f_i(x) = \frac{p_i(x)}{q_i(x)}$ and $q_i(x) \neq 0$ for $x \in X$ with $i = 1, \dots, n$. Then,

$$X \cap V(q_i) = V(I) \cap V(q_i) = V(I + (q_i)) = \emptyset.$$

Hence, by the Hilbert's Nullstellensatz,

$$J(V(I + (q_i))) = \sqrt{I + (q_i)} = K[X_1, \dots, X_m]$$

with the radical $\sqrt{I + (q_i)}$ of the ideal $I + (q_i)$ for $i = 1, \dots, n$. This implies that $1 = \alpha_i q_i + r_i$ with some $\alpha_i \in K[X_1, \dots, X_m]$, $r_i \in I$ for $i = 1, \dots, m$. Consequently, $\frac{p_i}{q_i}|_X = p_i \alpha_i|_X$ for $i = 1, \dots, m$ and the proof follows. \square

Since the field \mathbb{C} of complex number is algebraically closed, we get:

Corollary 3.2.2. *Any \mathbb{C} -entire rational map $\mathbb{S}^m(\mathbb{C}) \rightarrow \mathbb{S}^n(\mathbb{C})$ is represented by a polynomial map.*

Now, consider the n -sphere

$$\mathbb{S}^n(\mathbb{C}) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \dots + z_n^2 = 1\}$$

over the field \mathbb{C} of complex numbers. Then, for $(z_0, \dots, z_n) \in \mathbb{S}^n(\mathbb{C})$ and $z_k = x_k + iy_k$ with $k = 0, \dots, n$ we have

$$x_0^2 + \dots + x_n^2 = y_0^2 + \dots + y_n^2 + 1$$

and

$$x_0 y_0 + \dots + x_n y_n = 0.$$

This yields a diffeomorphism $\mathbb{S}^n(\mathbb{C}) \approx T\mathbb{S}^n$ for the tangent bundle $T\mathbb{S}^n$ of \mathbb{S}^n yields a homotopy equivalence $\mathbb{S}^n(\mathbb{C}) \simeq \mathbb{S}^n$.

Since a real polynomial representation of maps $\mathbb{S}^m \rightarrow \mathbb{S}^n$ is impossible in many cases, it seems reasonable to consider a complex polynomial representation $\mathbb{S}^m(\mathbb{C}) \rightarrow \mathbb{S}^n(\mathbb{C})$, where many obstacles disappear and nothing changes up to homotopy.

Recall that by [18, Theorem 1], we have:

Theorem 3.2.3. *If an element in $\pi_{n-1}(\mathbb{S}^{2k-1})$ is representable by a real homogeneous polynomial map of spheres $\mathbb{S}^{n-1} \rightarrow \mathbb{S}^{2k-1}$, then its suspension is representable by a complex polynomial map of quadrics $\mathbb{S}^n(\mathbb{C}) \rightarrow \mathbb{S}^{2k}(\mathbb{C})$.*

But, by Theorem 2.2.5, every Brouwer degree on an odd-dimensional sphere can be realised by a homogeneous polynomial map. Therefore, we derive:

Corollary 3.2.4. *Every element in $\pi_n(\mathbb{S}^n)$ can be represented by a complex polynomial map of $\mathbb{S}^n(\mathbb{C})$ for all positive integers n .*

Next, [18, Theorem 3] can be generalized as follows:

Theorem 3.2.5. *If an element in $\pi_m(\mathbb{S}^n)$ is representable by a real homogeneous polynomial map of spheres $\mathbb{S}^m \rightarrow \mathbb{S}^n$, then its $2k$ -fold suspension is representable by a complex polynomial map $\mathbb{S}^{m+2k}(\mathbb{C}) \rightarrow \mathbb{S}^{n+2k}(\mathbb{C})$.*

Proof. Given a d -form $h = (h_0, \dots, h_n): \mathbb{S}^m \rightarrow \mathbb{S}^n$, we consider the associated d -form $H = (H_0, \dots, H_n): \mathbb{S}^m(\mathbb{C}) \rightarrow \mathbb{S}^n(\mathbb{C})$ with

$$H_0^2 + \dots + H_m^2 = (z_0^2 + \dots + z_m^2)^d$$

and introduce new variables

$$u = z_{m+1} + iz_{m+2} \quad \text{and} \quad v = z_{m+1} - iz_{m+2}.$$

Then, $uv = z_{m+1}^2 + z_{m+2}^2$ and we have the expansion

$$(z_1^2 + \dots + z_n^2 + z_{m+1}^2 + z_{m+2}^2)^d = (z_1^2 + \dots + z_n^2)^d + uw,$$

where w is a polynomial in the variables z_1, \dots, z_{m+2} .

Now, writing $H_{n+1} = \frac{u+w}{2}$ and $H_{n+2} = i\frac{u-w}{2}$, we get the polynomial map

$$(H_0, \dots, H_n, H_{n+1}, H_{n+2}): \mathbb{S}^{m+2}(\mathbb{C}) \rightarrow \mathbb{S}^{n+2}(\mathbb{C}),$$

which represents (according to the construction from [18]) the double suspension of the original map H .

Proceedings by induction, we get the polynomial map

$$(H_0, \dots, H_{n+2k}): \mathbb{S}^{m+2k}(\mathbb{C}) \rightarrow \mathbb{S}^{n+2k}(\mathbb{C}),$$

which represents the $2k$ -suspension of the original map H and the proof follows. \square

Since the Hopf maps

$$\eta: \mathbb{S}^3 \rightarrow \mathbb{S}^2, \quad \nu: \mathbb{S}^7 \rightarrow \mathbb{S}^4, \quad \sigma: \mathbb{S}^{15} \rightarrow \mathbb{S}^8$$

are homogeneous polynomial map, an immediate conclusion is the following:

Corollary 3.2.6. *Every element in $\pi_{2n+1}(\mathbb{S}^{2n})$, $\pi_{2n+3}(\mathbb{S}^{2n})$ and $\pi_{2n+7}(\mathbb{S}^{2n})$ can be represented by a complex polynomial map $\mathbb{S}^{2n+1}(\mathbb{C}) \rightarrow \mathbb{S}^{2n}(\mathbb{C})$, $\mathbb{S}^{2n+3}(\mathbb{C}) \rightarrow \mathbb{S}^{2n}(\mathbb{C})$ and $\mathbb{S}^{2n+7}(\mathbb{C}) \rightarrow \mathbb{S}^{2n}(\mathbb{C})$, respectively.*

REFERENCES

- [1] Paul F. Baum. Quadratic maps and stable homotopy groups of spheres. *Illinois J. Math.*, 11:586–595, 1967. doi:10.1215/ijm/1256054449.
- [2] J. Bochnak and W. Kucharz. Realization of homotopy classes by algebraic mappings. *J. Reine Angew. Math.*, 377:159–169, 1987. doi:10.1515/crll.1987.377.159.
- [3] J. Bochnak and Wojciech Kucharz. Algebraic approximation of mappings into spheres. *Michigan Math. J.*, 34(1):119–125, 1987. doi:10.1307/mmj/1029003489.
- [4] Jacek Bochnak, Michel Coste, and Marie-Françoise Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors. doi:10.1007/978-3-662-03718-8.
- [5] Jacek Bochnak and Wojciech Kucharz. On approximation of maps into real algebraic homogeneous spaces. *J. Math. Pures Appl. (9)*, 161:111–134, 2022. With an appendix by János Kollár. doi:10.1016/j.matpur.2022.03.002.

- [6] A. Borel and J.-P. Serre. Groupes de Lie et puissances réduites de Steenrod. *Amer. J. Math.*, 75:409–448, 1953. doi:10.2307/2372495.
- [7] Albrecht Dold and Richard Lashof. Principal quasi-fibrations and fibre homotopy equivalence of bundles. *Illinois J. Math.*, 3:285–305, 1959. doi:10.1215/ijm/1255455128.
- [8] Marek Golasinski and Francisco Gómez Ruiz. Polynomial and regular maps into Grassmannians. *K-Theory*, 26(1):51–68, 2002. doi:10.1023/A:1016305323458.
- [9] Morris W. Hirsch. *Differential topology*, volume No. 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1976. doi:10.1007/978-1-4684-9449-5.
- [10] Jiagui Peng and Zizhou Tang. Algebraic maps from spheres to spheres. *Sci. China Ser. A*, 42(11):1147–1154, 1999. doi:10.1007/BF02875982.
- [11] A. Rigas and Lucas M. Chaves. Hopf maps and triality. *Math. J. Okayama Univ.*, 38:197–208, 1996. URL: <https://www.math.okayama-u.ac.jp/mjou/mjou-38.html>.
- [12] Igor R. Shafarevich. *Basic algebraic geometry, 1*. Springer, Heidelberg, 2013. Varieties in projective space. doi:10.1007/978-3-642-37956-7.
- [13] Norman Steenrod. *The topology of fibre bundles*, volume vol. 14 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1951. doi:10.1515/9781400883875.
- [14] Hiroshi Toda. *Composition methods in homotopy groups of spheres*, volume No. 49 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1962. doi:10.1515/9781400882625.
- [15] Francisco-Javier Turiel. Polynomial maps and even dimensional spheres. *Proc. Amer. Math. Soc.*, 135(8):2665–2667, 2007. doi:10.1090/S0002-9939-07-08812-0.
- [16] George W. Whitehead. *Elements of homotopy theory*, volume 61 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. doi:10.1007/978-1-4612-6318-0.
- [17] R. Wood. Polynomial maps from spheres to spheres. *Invent. Math.*, 5:163–168, 1968. doi:10.1007/BF01425547.
- [18] R. M. W. Wood. Polynomial maps of affine quadrics. *Bull. London Math. Soc.*, 25(5):491–497, 1993. doi:10.1112/blms/25.5.491.

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