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Fundamental theorems of quasi-geodesic mappings of generalized-recurrent-parabolic spaces

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Abstract. In previous papers we studied mappings of pseudo-Riemannian spaces being mutually quasi-geodesic and almost geodesic of the 2nd type. As a result, we arrived at the quasi-geodesic mapping

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$$

of spaces with an affine structure, which was called generalized-recurrent. Quasi-geodesic mappings are divided into two types: general and canonical. In this article, the fundamental issues of the theory of quasi-geodesic mappings of generalized-recurrent-parabolic spaces are considered. First, the fundamental equations of quasi-geodesic mappings are reduced to a form that allows effective investigation. Then, using a new form of the fundamental equations, we prove theorems that allow for any generalized-recurrent-parabolic space (V_n, g_{ij}, F_i^h) or to find all spaces $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ onto which V_n admits a quasi-geodesic mapping of the general form, or prove that there are no such spaces.

Анотація. В попередніх статтях досліджено відображення псевдоріманових просторів, які є одночасно квазі-геодезичними та майже геодезичними другого типу, а також побудовано квазі-геодезичне відображення $f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$ просторів з певною афінорною структурою, яку назвали узагальнено-рекурентною. Квазі-геодезичні відображення можуть бути двох типів: загального виду і канонічні. В даній статті досліджуються фундаментальні питання теорії квазі-геодезичних відображень узагальнено-рекурентно-параболічних просторів. Основні рівняння квазі-геодезичних відображень приводяться до вигляду, який допускає ефективне дослідження. Використовуючи нову форму основних рівнянь, ми доводимо теореми, які дозволяють для будь-якого узагальнено-рекурентно-параболічного простору (V_n, g_{ij}, F_i^h) або знайти всі простори $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$, на які V_n допускає квазі-геодезичне відображення загального виду, або довести, що таких просторів немає.

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1. Introduction

1.1. We continue to study diffeomorphisms of pseudo-Riemannian spaces which are also a quasi-geodesic mappings (QGM) [5, 12, 13, 15, 16, 20, 26] with the reciprocity condition and almost-geodesic mappings of the second type [2, 3, 11, 19, 27–29].

Say that QGM

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$$

satisfies the *reciprocity condition* if the reverse mapping f^{-1} is also QGM. In [15] we obtained the fundamental equations of such a mapping

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$$

in the common coordinate system (x^i) with respect to f:

$$\overline{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_{(i}(x)\delta_{j)}^h + \phi_{(i}(x)F_{j)}^h(x), \tag{1.1}$$

$$F_i^h(x) = \overline{F}_i^h(x),$$

$$g_{i\alpha}F_j^\alpha = -g_{j\alpha}F_i^\alpha, \qquad \overline{g}_{i\alpha}F_j^\alpha = -\overline{g}_{j\alpha}F_i^\alpha$$

$$F_{(i,j)}^h = q_{(i}F_{j)}^h, \tag{1.2}$$

$$F_\alpha^h F_i^\alpha = e\delta_i^h, \qquad e = 0, \pm 1, \quad i, h, j, \ldots = 1, 2, \ldots, n, \tag{1.3}$$

where Γ_{ij}^h, $\overline{\Gamma}_{ij}^h$ are the Christoffel symbols of V_n, $\overline{V}_n$, respectively; $\psi_i(x)$, $\phi_i(x)$, $q_i(x)$, $p_i(x)$ are certain covectors; $F_i^h(x)$ is affinor; brackets (i, j) denote the symmetrization with respect to the corresponding indices; comma «,» is a sign of the covariant derivative with respect to the connection of V_n.

If in (1.1) $\phi_i = 0$ and $\psi_i \neq 0$, then the quasi-geodesic mapping degenerates into a geodesic map, [8–10, 27]. For $\phi_i \neq 0$ and $\psi_i = 0$, the quasi-geodesic mapping is called *canonical.* If in (1.1) $\phi_i = 0$ and $\psi_i = 0$, the QGM is called *trivial.*

An affinor structure satisfying condition (1.3) is called [19]:

- *elliptic* if $e = -1$,
- *hyperbolic* if $e = +1$,
- *m-parabolic* when $e = 0$, $\operatorname{rank} F = m$ $(2m < n)$,
- *parabolic* when $e = 0$, $\operatorname{rank} F = m$ $(2m = n)$.

Equation (1.1) characterizes F-planar mapping which started to study Mikeš and Sinyukov [18]. These results were specified in paper [7].

1.2. We call an affinor structure F_i^h satisfying conditions (1.2) a *generalized-recurrent structure* (of elliptic, hyperbolic, or parabolic type) [15]. If in the conditions (1.2) $q_i = 0$, the affinor F_i^h defines a *K-structure* [1, 26, 28].

In [16], a recurrent-parabolic structure was introduced, which is determined by the conditions:

$$F^h_\alpha F^\alpha_i = 0, \qquad g_{i\alpha}F^\alpha_j = -g_{j\alpha}F^\alpha_i, \qquad F^h_{i,j} = q_j F^h_i. \tag{1.4}$$

The articles [12, 14, 23] are devoted to some issues that concern quasi-geodesic mappings of recurrent-parabolic spaces.

The K-structure and the recurrent-parabolic structure are the special cases of a generalized-recurrent structure.

In the context of types of recurrences and methods for extracting special spaces with structure, the papers [6, 24] are of interest.

In [15] the properties of a generalized-recurrent structure of parabolic type were studied.

We call the vector q_i in (1.2) *the generalized recurrence vector* of the structure F^h_i, and in the case $F^h_{i,j} = q_j F^h_i$, *the recurrence vector.* Note that if the vector q_i is gradient, the affinor $\tilde{F}^h_i = e^{-q}F^h_i$, where $q_i = \frac{\partial q(x)}{\partial x^i}$, defines a K-structure in the generalized-recurrent space (V_n, g_{ij}, F^h_i), and a Kählerian structure in the recurrent-parabolic space.

Shortly, in this case $(V_n, g_{ij}, \tilde{F}^h_i)$ is a parabolic Kähler space, see [17,19]. The studied mappings are holomorphically projective mappings between parabolic Kähler spaces. These problems were studied in [4, 17, 21, 22] and also in the dissertation by Shiha [25].

1.3. Let us define an operation of contraction with an affinor, which is called the *conjugation* with respect to the corresponding index and is denoted as follows:

$$T^{\cdots}_{j_1\cdots j_{k-1}\alpha j_{k+1}\cdots j_r}F^\alpha_i = T^{\cdots}_{j_1\cdots j_{k-1}\bar{i}j_{k+1}\cdots j_r}$$

$$T^{j_1\cdots j_{k-1}\alpha j_{k+1}\cdots j_r}_{\cdots}F^h_\alpha = T^{j_1\cdots j_{k-1}\bar{h}j_{k+1}\cdots j_r}_{\cdots}$$

1.4. The integrable parabolic structure F^h_i in some neighborhood of the point V_n can be reduced to the form

$$(F^h_i) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}$$

where I_m is the identity matrix of order $m = \frac{n}{2}$.

We will call such a coordinate system *adapted* to the affinor. Then under the conditions

$$g_{i\alpha}F^\alpha_j = -g_{j\alpha}F^\alpha_i, \qquad F^h_\alpha F^\alpha_i = 0,$$

the components of the metric tensor of the space V_n in the adapted coordinate system satisfy the following conditions:

$$g_{ab} = g_{ba}, \qquad g_{ab+m} = -g_{a+mb}, \qquad g_{a+mb+m} = 0,$$

for $a, b = 1, 2, \dots, m$. Further, the auxiliary tensor A^h_i, will be useful to us, which is determined in the adapted coordinate system by the matrix

$$(A^h_i) = \begin{pmatrix} P & I_m \\ -P^2 & -P \end{pmatrix}$$

where P is an arbitrary square matrix of order m.

It is easy to check that

$$F^{\beta}_{\alpha} A^{\alpha}_{\beta} = m, \qquad A^h_{\alpha} A^{\alpha}_i = 0, \qquad F^h_{\alpha} A^{\alpha}_i + A^h_{\alpha} F^{\alpha}_i = \delta^h_i. \tag{1.5}$$

1.5. In [15] we came to the conclusion that the integrable affinor structure of the generalized-recurrent space (V_n, g_{ij}, F^h_i) is characterized by the following properties:

$$F^{\alpha}_{i,\alpha} = 0, \qquad F^h_{\bar{j},i} = F^h_{j,\bar{i}} = F^{\bar{h}}_{j,i} = 0, \qquad q_{\bar{i}} = 0. \tag{1.6}$$

Note that, in contrast to hyperbolic and elliptic types, an integrable generalized-recurrent structure of parabolic type (in particular, a parabolic K-structure) need not be Kählerian, i.e. relations (1.6) do not imply that the affinor F^h_i is covariantly constant.

Further, in this paper, we consider only the integrable affinor structure.

1.6. In [15] it was proved that the image of a generalized-recurrent space under QGM is also a generalized-recurrent space, that is,

$$F^h_{(i|j)} = \tilde{q}_{(j} F^h_{i)},$$

where

$$\tilde{q}_i = q_i - \psi_i + \phi_{\bar{i}},$$

«|» is a sign of a covariant derivative with respect to the connection of $\overline{V}_n$, i.e. affinor F^h_i in the space $\overline{V}_n$ also defines a generalized-recurrent structure.

Under the condition $\tilde{q}_i = q_i$ we say that *QGM preserves the generalized recurrence vector*. In this case, the vectors ψ_i and ϕ_i in the basic QGM equations (1.1) are related as follows:

$$\psi_i = \phi_{\bar{i}} \tag{1.7}$$

and ψ_i is locally a gradient:

$$\psi(x) = \tfrac{1}{2} \ln \left| \tfrac{\overline{g}}{g} \right|.$$

In this paper, we consider quasi-geodesic mappings (QGM) of generalized-recurrent-parabolic spaces with an integrable affinor structure.

The investigation is carried out in tensor form, locally, in the class of real sufficiently smooth functions.

2. LINEAR EQUATIONS OF THE THEORY OF $QGMs$ OF GENERALIZED-RECURRENT-PARABOLIC SPACES

2.1. A generalized-recurrent-parabolic space (V_n, g_{ij}, F^h_i) admits a QGM

$$f\colon (V_n, g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i)$$

if and only if in the coordinate system (x^i) the fundamental equations of this mapping are satisfied:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_{(i}(x)\delta^h_{j)} + \phi_{(i}(x)F^h_{j)}(x), \tag{2.1}$$

$$F^h_i(x) = \overline{F}^h_i(x), \quad F^h_\alpha F^\alpha_i = 0, \tag{2.2}$$

$$\phi_{\bar{i}} = \psi_i, \tag{2.3}$$

$$g_{i\bar{j}} = -g_{j\bar{i}}, \qquad \overline{g}_{i\bar{j}} = -\overline{g}_{j\bar{i}}, \tag{2.4}$$

$$F^h_{(i,j)} = q_{(j}F^h_{i)} \quad i, h, j, \ldots = 1, 2, \ldots n \tag{2.5}$$

where q_i is the generalized recurrence vector of the structure F^h_i. In other words, the mapping $f\colon (V_n g_{ij}, F^h_i) \to (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i)$ is a QGM if and only if under conditions (2.3), (2.2), (2.4), (2.5) in the space (V_n, g_{ij}, F^h_i) the system of nonlinear differential equations in partial derivatives of the first order (2.1) with respect to the components of the tensor $\overline{g}_{ij}(x)$ and the vector $\phi_i \neq 0$ has a solution.

Using methods developed in the theory of geodesic mappings of Riemannian spaces [27], we reduce the fundamental equations (2.1)-(2.5) to a form that allows an effective study.

2.2. The following theorem holds:

Theorem 2.2.1. *A generalized-recurrent-parabolic space* (V_n, g_ij, F^h_i) *admits a non-trivial* QGM *if and only if it contains a non-singular symmetric tensor of type* $(0, 2)$ a_{ij} *satisfying the equations*

$$a_{ij,k} = \lambda_{\bar{i}}g_{jk} + \lambda_{\bar{j}}g_{ik} + \lambda_i F_{jk} + \lambda_j F_{ik}, \tag{2.6}$$

and

$$a_{i\bar{j}} = -a_{j\bar{i}}, \qquad \det(a_{ij}) \neq 0, \tag{2.7}$$

for some covector $\lambda_i \not\equiv 0$.

Proof. Assume that a generalized-recurrent-parabolic space V_n admits a QGM onto $\overline{V}_n$. Since $\overline{g}_{ij|k} = 0$ in $\overline{V}_n$, the equation (2.1) can be written in the following equivalent form:

$$\begin{gathered} \overline{g}_{ij,k} = 2\phi_{\bar{k}}\overline{g}_{ij} + \phi_{\bar{i}}\overline{g}_{jk} + \phi_{\bar{j}}\overline{g}_{ik} + \phi_i\overline{F}_{jk} + \phi_j\overline{F}_{ik}, \\ \overline{F}_{ik} = \overline{g}_{i\alpha}F^\alpha_k. \end{gathered} \tag{2.8}$$

The equations (2.8) control the existence of quasi-geodesic mappings of generalized-recurrent-parabolic spaces.

Let us introduce the following nondegenerate tensor

$$a_{ij} = e^{2\psi}\overline{g}^{\alpha\beta}g_{\alpha i}g_{\beta j}. \tag{2.9}$$

Since $\overline{g}_{i\alpha}\overline{g}^{\alpha h} = \delta_i^h$, we have that

$$\overline{g}_{i\alpha,k}\overline{g}^{\alpha h} = -\overline{g}_{i\alpha}\overline{g}^{\alpha h}_{,k}.$$

Therefore, it follows from (2.9) and (2.8) that

$$a_{ij,k} = \lambda_{\bar{i}}g_{jk} + \lambda_{\bar{j}}g_{ik} + \lambda_i F_{jk} + \lambda_j F_{ik}, \tag{2.10}$$

where

$$\lambda_i = -e^{2\psi}\phi_\gamma\overline{g}^{\gamma\alpha}g_{\alpha i}. \tag{2.11}$$

It is easy to check that in view of (2.3) and (2.4), $\lambda_{\bar{i}}$ is gradient and

$$a_{i\bar{j}} = -a_{j\bar{i}}, \qquad \det(a_{ij}) \neq 0. \tag{2.12}$$

Thus, if a pseudo-Riemannian space (V_n, g_ij, F_i^h) with a generalized-recurrent-parabolic structure F_i^h admits a non-trivial QGM on the space $(\overline{V}_n, \overline{g}_{ij}, \overline{F}_i^h)$, then it necessarily contains a nonsingular symmetric tensor a_{ij} satisfying (2.10), (2.12) for some nonzero vector λ_i.

The converse is also true. Indeed, if a_{ij} and λ_i satisfy (2.10), (2.12), then (2.8), (2.3), (2.4) hold for

$$\overline{g}_{ij} = e^{-2\psi}a^{\alpha\beta}g_{\alpha i}g_{\beta j}$$

and

$$\phi_i = -e^{-2\psi}\lambda_\alpha g^{\alpha\beta}\overline{g}_{\beta i}. \qquad \square$$

The equation (2.10) is a new linear form of the fundamental equations of the theory of QGMs of generalized-recurrent-parabolic spaces.

3. FUNDAMENTAL THEOREMS OF QGMS OF GENERALIZED-RECURRENT-PARABOLIC SPACES

3.1. Let (V_n, g_{ij}, F_i^h) be a generalized-recurrent-parabolic space, so its metric tensor $g_{ij}(x)$ and affinor $F_i^h(x)$ satisfy conditions (2.2), (2.4), (2.5). The question of the existence of a QGM of the space (V_n, g_{ij}, F_i^h) is reduced to the study of differential equations (2.10) with respect to the tensor a_{ij} and the vector λ_i satisfying conditions (2.12).

Further, we consider the integrability conditions for the equations (2.10), which, taking into account the Ricci identity and (2.5), have the following form:

$$a_{\alpha(i}R^{\alpha}_{j)kl} = g_{k(i}K_{j)l} - g_{l(i}K_{j)k} + F_{l(i}\lambda_{j),k} - F_{k(i}\lambda_{j),l} + \lambda_{(i}F_{j)[k,l]}, \tag{3.1}$$

where

$$K_{il} = \lambda_{\bar{i},l} + \lambda_\alpha F^\alpha_{i,l}. \tag{3.2}$$

Note that, in view of (2.5) it follows from (3.2) that

$$K_{\bar{i}\bar{l}} = 0. \tag{3.3}$$

Using the tensor A^h_i defined earlier let us introduce the following tensor:

$$A^{ij} = A^i_\alpha g^{\alpha j}.$$

Contract (3.1) with A^{kj} with respect to the indices k, j and conjugate with respect to the index i. Taking into account (3.3), we obtain

$$K_{i\bar{l}} = \frac{2}{n-2} a_{\alpha\beta} \tilde{R}^{\alpha\beta}_{i\bar{l}}, \tag{3.4}$$

where

$$\tilde{R}^{\alpha\beta}_{il} = \delta^\beta_{(\gamma} R^\alpha_{i)\sigma l} A^{\sigma\gamma}.$$

Contraction (3.1) with A^{kj} over indices k, j and conjugation on i gives us

$$a_{\alpha\beta}\tilde{R}^{\alpha\beta}_{\bar{i}l} = \frac{n-4}{2} K_{il} + K_{i\nu} A^{\bar{\nu}}_l - \mu F_{il} - \lambda_\alpha \left(\frac{n-2}{2} F^\alpha_{i,l} - F^\alpha_i F_{\beta[\gamma,l]} A^{\gamma\beta} \right),$$

where

$$\mu = K_{\alpha\beta} A^{\beta\alpha}.$$

Then we get from (3.4):

$$\begin{aligned} \frac{n-4}{2} K_{il} = \mu F_{il} + a_{\alpha\beta} \left(\tilde{R}^{\alpha\beta}_{\bar{i}l} - \frac{2}{n-2} \tilde{R}^{\alpha\beta}_{\bar{i}\nu} A^{\bar{\nu}}_l \right) + \\ + \lambda_\alpha \left(\frac{n-2}{2} F^\alpha_{i,l} - F^\alpha_i F_{\beta[\gamma,l]} A^{\gamma\beta} \right). \end{aligned}$$

Contracting (3.1) with A^{kj} over indices k, j in view of this equation gives

$$\lambda_{i,l} = a_{\alpha\beta} S^{\alpha\beta}_{il} + \lambda_\alpha P^\alpha_{il} + \mu Q_{il} + \nu F_{il}, \tag{3.5}$$

where

$$\begin{aligned} S^{\alpha\beta}_{il} &= \frac{4}{n(n-4)} \tilde{R}^{\alpha\beta}_{i\gamma} \left(\frac{n-4}{2} \delta^\gamma_l - \frac{n-4}{n+2} A^{\bar{\gamma}}_l \right) + \\ &\quad + \frac{4}{n(n-4)} \tilde{R}^{\alpha\beta}_{\rho\gamma} \left[\frac{n+4}{n+2} F^\rho_i A^\gamma_l + g_{\sigma i} A^{\sigma\bar{\rho}} \left(\frac{8(n-1)}{n^2-4} A^{\bar{\gamma}}_l - 2\delta^\gamma_l \right) \right], \end{aligned}$$

$$\begin{aligned} P^\alpha_{il} &= \frac{2}{n} F_{i[l,\gamma]} A^{\gamma\alpha} + \frac{2(n-2)(n+4)}{n(n-4)(n+2)} F^\alpha_{i,\gamma} A^\gamma_l - \frac{2}{n-4} F^\alpha_{\beta,l} g_{i\sigma} A^{\sigma\beta} + \\ &\quad + \frac{4}{n(n-4)} \left(2F_{\beta[\sigma,l]} g_{\gamma i} A^{\gamma\bar{\alpha}} - \frac{n-4}{2} \delta^\alpha_i F_{\beta[\sigma,l]} - \frac{n+4}{n+2} F^\alpha_i F_{\beta[\sigma,\gamma]} A^\gamma_l \right) A^{\sigma\beta}, \end{aligned}$$

$$Q_{il} = \frac{4}{n(n-4)}\left(\frac{n}{2}g_{il} + \frac{2(n+6)}{n+2}F_{\alpha i}A_l^{\alpha}\right),$$

$$\nu = \lambda_{\alpha,\beta}A^{\beta\alpha}.$$

Note, that the expression for the tensor Q_{il} means that

$$4Q_{i\bar{l}} = \frac{2(n+4)(n-6)}{n(n+2)(n-4)}F_{il}, \qquad Q_{\bar{i}l} = \frac{2}{n-4}F_{li}, \tag{3.6}$$

$$Q_{\alpha\beta}g^{\alpha\beta} = \frac{2(n-12)}{(n+2)(n-4)}, \qquad Q_{\alpha\beta}A^{\alpha\beta} = 0, \tag{3.7}$$

$$Q_{\bar{i}l,k} = 0. \tag{3.8}$$

3.2. Due to (2.10) the integrability conditions for (3.5) have the following form:

$$Q_{i[l}\mu_{,k]} + F_{i[l}\nu_{,k]} + \mu Q_{i[l,k]} + \nu F_{i[l,k]} + a_{\alpha\beta}\tilde{S}^{\alpha\beta}_{ilk} + \lambda_{\alpha}\tilde{P}^{\alpha}_{ilk} = 0 \tag{3.9}$$

where

$$\tilde{S}^{\alpha\beta}_{ilk} = S^{\alpha\beta}_{i[l,k]} + S^{\alpha\beta}_{\gamma k}P^{\gamma}_{il} - S^{\alpha\beta}_{\gamma l}P^{\gamma}_{ik},$$

$$\tilde{P}^{\alpha}_{ikl} = R^{\alpha}_{ikl} + \left(S^{\overline{\alpha}\beta}_{i[l} + S^{\beta\overline{\alpha}}_{i[l}\right)g_{k]\beta} - S^{(\overline{\alpha}\beta)}_{i[l}F_{k]\beta}.$$

Conjugating (3.9) and taking into account (3.6), (3.7), (3.8) we obtain:

$$\frac{2}{n-4}\left(F_{li}\mu_{,k} - F_{ki}\mu_{,l}\right) + a_{\alpha\beta}\tilde{S}^{\alpha\beta}_{\bar{i}lk} + \lambda_{\alpha}\tilde{P}^{\alpha}_{\bar{i}lk} = 0.$$

Comparing the result of cycling this equation by i, k, l with the original relations, we get:

$$\frac{4}{n-4}F_{kl}\mu_i + a_{\alpha\beta}\left(\tilde{S}^{\alpha\beta}_{\bar{l}ki} + \tilde{S}^{\alpha\beta}_{\bar{k}il} - \tilde{S}^{\alpha\beta}_{\bar{i}lk}\right) + \lambda_{\alpha}\left(\tilde{P}^{\alpha}_{\bar{l}ki} + \tilde{P}^{\alpha}_{\bar{k}il} - \tilde{P}^{\alpha}_{\bar{i}lk}\right) = 0.$$

Hence, after contraction with A^{kl} with respect to the indices l, k, we find that

$$\frac{2n}{n-4}\mu_{,i} = a_{\alpha\beta}\overline{S}^{\alpha\beta}_{i} + \lambda_{\alpha}\overline{P}^{\alpha}_{i}, \tag{3.10}$$

where

$$\overline{S}^{\alpha\beta}_{i} = \left(\tilde{S}^{\alpha\beta}_{\overline{\gamma}\sigma i} + \tilde{S}^{\alpha\beta}_{\overline{\sigma}i\gamma} - \tilde{S}^{\alpha\beta}_{\bar{i}\gamma\sigma}\right)A^{\sigma\gamma},$$

$$\overline{P}^{\alpha}_{i} = \left(\tilde{P}^{\alpha}_{\overline{\gamma}\sigma i} + \tilde{P}^{\alpha}_{\overline{\sigma}i\gamma} - \tilde{P}^{\alpha}_{\bar{i}\gamma\sigma}\right)A^{\sigma\gamma}.$$

Similarly, comparing the result of cycling (3.9) by i, k, l with the original relations based on the expression (3.10), we get:

$$n\nu_{,i} = a_{\alpha\beta}\overline{\overline{S}}^{\alpha\beta}_{i} + \lambda_{\alpha}\overline{\overline{P}}^{\alpha}_{i} + \mu\overline{Q}_{i} + \nu\overline{\overline{Q}}_{i}, \tag{3.11}$$

where

$$\begin{aligned}
\overline{\overline{S}}{}^{\alpha\beta}_{i} &= \left(\frac{n-4}{2n} Q_{i[\gamma} \overline{S}{}^{\alpha\beta}_{\sigma]} + \tilde{S}^{\alpha\beta}_{(i\sigma\gamma)} - 2\tilde{S}^{\alpha\beta}_{i\sigma\gamma}\right) A^{\gamma\sigma},\\
\overline{\overline{P}}{}^{\alpha}_{i} &= \left(\frac{n-4}{2n} Q_{i[\gamma} \overline{P}{}^{\alpha}_{\sigma]} + \tilde{P}^{\alpha}_{(i\sigma\gamma)} - 2\tilde{P}^{\alpha}_{i\sigma\gamma}\right) A^{\gamma\sigma},\\
\overline{Q}_i &= Q_{\sigma[\gamma,i]} + Q_{\gamma[i,\sigma]} - Q_{i[\sigma,\gamma]} A^{\gamma\sigma},\\
\overline{\overline{Q}}_i &= 2F_{\sigma\gamma,i} A^{\gamma\sigma}.
\end{aligned}$$

Equations (2.10), (3.5), (3.10) and (3.11) constitute a closed system of first order partial differential equations of Cauchy type with respect to the unknown functions a_{ij}, λ_i, μ, ν. Let us denote it by (B). In the theory of differential equations regular methods have been developed for such systems. Thus, we proved the following

Theorem 3.2.1. *A pseudo-Riemannian space $(V_n, g_{ij}(x), F^h_i(x))$ with an integrable generalized-recurrent-parabolic structure $F^h_i(x)$ admits a QGM, if and only if the system of differential equations (B) has a non-trivial solution*

$$a_{ij}(x), \qquad \lambda_i(x) \not\equiv 0, \qquad \mu(x), \qquad \nu(x),$$

satisfiing the conditions

$$a_{ij}(x) = a_{ji}(x), \qquad \det(a_{ij}(x)) \neq 0, \qquad a_{i\bar{j}} = -a_{j\bar{i}}.$$

The family of differential equations (B) is linear with coefficients of intrinsic character in V_n and independent of the choice of coordinates. If the metric tensor g and the structure tensor F of the generalized-recurrent-parabolic manifold V_n are real, then for the initial data

$$a_{ij}(x_0) = \overset{\circ}{a}_{ij}, \qquad \lambda_i(x_0) = \overset{\circ}{\lambda}_i, \qquad \mu(x_0) = \overset{\circ}{\mu}, \qquad \nu(x_0) = \overset{\circ}{\nu}.$$

The system (B) has at most one solution. Taking into account that the initial data must satisfy (2.2), (2.4) which in canonical coordinates are written as:

$$g_{ab} = g_{ba}, \qquad g_{ab+m} = -g_{a+mb}, \qquad g_{a+mb+m} = 0,$$

for $a, b = 1, 2, \ldots, m$, we see that the general solution of (B) depends on r significant parameters, where

$$r = \frac{n}{4}\left(\frac{n}{2}+1\right) + \frac{n}{4}\left(\frac{n}{2}-1\right) + n + 2 = \frac{(n+2)^2}{4} + 1.$$

The solution of (B) satisfying the given initial conditions

$$a_{ij}(x_0) = \overset{\circ}{a}_{ij}, \qquad \lambda_i(x_0) = \overset{\circ}{\lambda}_i, \qquad \mu(x_0) = \overset{\circ}{\mu}, \qquad \nu(x_0) = \overset{\circ}{\nu},$$

can be given as a Taylor series, and if necessary, computed in a neighborhood of a given point x_0 of the space.

3.3. Notice that the system (B) might not be consistent. However, this system is consistent if and only if the set of integrability conditions (B) and their differential prolongations are consistent. The integrability conditions for the first group of equations (B), taking into account (3.1), (3.2), (3.5), can be represented in the form:

$$a_{\alpha\beta} T^{\alpha\beta}_{ijkl} + \lambda_\alpha M^{\alpha}_{ijkl} + \mu N_{ijkl} = 0, \tag{3.12}$$

where

$$\begin{aligned} T^{\alpha\beta}_{ijkl} = {} & \delta^{\alpha}_{(i} R^{\beta}_{j)kl} + \frac{n-4}{2(n-2)} \tilde{R}^{\alpha\beta}_{\gamma\nu} F^{\gamma}_{(j} g_{i)[k} \left((n-4)\delta^{\nu}_{l]} + 2A^{\overline{\nu}}_{l]} \right) + \\ & + F_{l(i} S^{\alpha\beta}_{j)k} - F_{kl(i} S^{\alpha\beta}_{j)l}, \\ M^{\alpha}_{ijkl} = {} & C^{\alpha}_{ij[kl]} + F_{l(i} P^{\alpha}_{j]k} - F_{k(i} P^{\alpha}_{j]l} + \delta^{\alpha}_{(i} F_{j)[k,l]}, \\ C^{\alpha}_{ijkl} = {} & g_{k(i} \left(\frac{n-2}{2} F^{\alpha}_{j),l} - F^{\alpha}_{j)} F_{\beta[\gamma,l]} A^{\gamma\beta} \right), \\ N_{ijkl} = {} & F_{l(i} \left(Q_{j)k} - g_{j)k} \right) - F_{k(i} \left(Q_{j)l} - g_{j)l} \right). \end{aligned}$$

The integrability conditions for the second group of equations (B), taking into account (3.9), (3.10), (3.11), have the form:

$$a_{\alpha\beta} \overline{T}^{\alpha\beta}_{ikl} + \lambda_\alpha \overline{M}^{\alpha}_{ikl} + \mu \overline{N}_{ikl} + \nu \overline{L}_{ikl} = 0, \tag{3.13}$$

where

$$\begin{aligned} \overline{T}^{\alpha\beta}_{ikl} &= \frac{n-4}{2n} Q_{i[l} \overline{S}^{\alpha\beta}_{k]} + \frac{1}{n} F_{i[l} \overline{\overline{S}}^{\alpha\beta}_{k]}, \\ \overline{M}^{\alpha}_{ikl} &= \frac{n-4}{2n} Q_{i[l} \overline{P}^{\alpha}_{k]} + \frac{1}{n} F_{i[l} \overline{\overline{P}}^{\alpha}_{k]}, \\ \overline{N}_{ikl} &= Q_{i[l,k]} + \frac{1}{n} F_{i[l} \overline{Q}_{k]}, \\ \overline{L}_{ikl} &= F_{i[l,k]} + \frac{1}{n} F_{i[l} \overline{\overline{Q}}_{k]}. \end{aligned}$$

The integrability conditions for the third group of equations (B), taking into account (2.5), (3.5), (3.10), can be represented in the form:

$$a_{\alpha\beta} \overline{\overline{T}}^{\alpha\beta}_{il} + \lambda_\alpha \overline{\overline{M}}^{\alpha}_{il} + \mu \overline{\overline{N}}_{il} + \nu \overline{\overline{L}}_{il} = 0, \tag{3.14}$$

where

$$\overline{\overline{T}}^{\alpha\beta}_{il} = \overline{S}^{\alpha\beta}_{[i,l]} + S^{\alpha\beta}_{\gamma[l} \overline{P}^{\gamma}_{i]},$$

$$\overline{\overline{M}}{}^{\alpha}_{il} = \overline{P}{}^{\alpha}_{[i,l]} + P^{\alpha}_{\gamma[l}\overline{P}{}^{\gamma}_{i]} + g_{\beta[l}\left(\overline{S}{}^{(\alpha\beta)}_{i]} - \overline{S}{}^{(\overline{\alpha}\overline{\beta})}_{i]}\right),$$

$$\overline{\overline{N}}_{il} = Q_{\alpha[l}\overline{P}{}^{\alpha}_{i]}, \qquad \overline{\overline{L}}_{il} = F_{\alpha[l}\overline{P}{}^{\alpha}_{i]}.$$

Finally, the integrability conditions for the fourth group of equations (B), taking into account (2.5), (3.5), (3.10), (3.11) can be represented in the following form:

$$a_{\alpha\beta}\overset{\sim}{\overline{T}}{}^{\alpha\beta}_{il} + \lambda_{\alpha}\overset{\sim}{\overline{M}}{}^{\alpha}_{il} + \mu\overset{\sim}{\overline{N}}_{il} + \nu\overset{\sim}{\overline{\overline{L}}}_{il} = 0, \tag{3.15}$$

where

$$\overset{\sim}{\overline{T}}{}^{\alpha\beta}_{il} = \overline{\overline{S}}{}^{\alpha\beta}_{[i,l]} + S^{\alpha\beta}_{\gamma[l}\overline{\overline{P}}{}^{\gamma}_{i]} + \frac{n-4}{2n}\overline{S}{}^{\alpha\beta}_{[l}\overline{Q}_{i]} + \frac{1}{n}\overline{\overline{S}}{}^{\alpha\beta}_{[l}\overline{\overline{Q}}_{i]},$$

$$\overset{\sim}{\overline{M}}{}^{\alpha}_{il} = \overline{\overline{P}}{}^{\alpha}_{[i,l]} + P^{\alpha}_{\gamma[l}\overline{\overline{P}}{}^{\gamma}_{i]} + g_{\beta[l}\left(\overline{\overline{S}}{}^{(\alpha\beta)}_{i]} - \overline{\overline{S}}{}^{(\overline{\alpha}\overline{\beta})}_{i]}\right) + \frac{n-4}{2n}\overline{P}{}^{\alpha}_{[l}\overline{Q}_{i]} + \frac{1}{n}\overline{\overline{P}}{}^{\alpha}_{[l}\overline{\overline{Q}}_{i]},$$

$$\overset{\sim}{\overline{N}}_{il} = \overline{Q}_{[i,l]} + \frac{1}{n}\overline{Q}_{[l}\overline{\overline{Q}}_{i]} + Q_{\alpha[l}\overline{\overline{P}}{}^{\alpha}_{i]},$$

$$\overset{\sim}{\overline{\overline{L}}}_{il} = \overline{\overline{Q}}_{[i,l]} + \frac{1}{n}\overline{\overline{Q}}_{[l}\overline{\overline{Q}}_{i]} + F_{\alpha[l}\overline{\overline{P}}{}^{\alpha}_{i]}.$$

Denote the integrability conditions (3.12), (3.13), (3.14) for the system (B) by (B_0). Obviously, (B_0) is a system of linear homogeneous algebraic equations for an unknown functions a_{ij}, λ_i, μ, ν with coefficients from V_n. They must be satisfied identically for any solution of the system (B) whenever it exists.

Differentiating (B_0) covariantly and using (B) we obtain the first prolongation of (B_0) which we denote (B_1). Obviously, (B_1) is also a system of linear homogeneous algebraic equations in a_{ij}, λ_i, μ, ν with coefficients from V_n. The differential prolongations of (B_1) will be denoted (B_2) and so on.

As we see, $(B_0), (B_1), (B_2), \ldots$ is a system of linear homogeneous algebraic equations for $a_{ij}(x)$, $\lambda_i(x) \not\equiv 0$, $\mu(x)$, $\nu(x)$ with coefficients from V_n. Since the number of unknown functions is finite, there is a natural number N such that (B_N) and subsequent continuations will be consequences of $(B_0), (B_1), \ldots, (B_{N-1})$.

In accordance with the analytical theory of differential equations, the system (B) has a non-trivial solution in the neighborhood of the point M_0 if and only if the system of equations $(B_0), (B_1), \ldots, (B_{N-1})$ has a non-trivial solution at this point.

Hence, we get the following

Theorem 3.3.1. *A pseudo-Riemannian space with an integrable generalized-recurrent-parabolic structure* $(V_n, g_{ij}(x), F_i^h(x))$ *admits a QGM if and only if the system of homogeneous algebraic equations* $(B_0), (B_1), \dots, (B_{s-1})$ *has a non-trivial solution in* (V_n, g_{ij}, F_i^h)

$$a_{ij}(x), \qquad \lambda_i(x) \not\equiv 0, \qquad \mu(x), \qquad \nu(x)$$

which satisfies the conditions

$$a_{ij}(x) = a_{ji}(x), \qquad \det(a_{ij}(x)) \neq 0, \qquad a_{i\bar{j}} = -a_{j\bar{i}}.$$

Theorems 3.2.1 and 3.3.1 together supply us with a regular method enabling to decide effectively whether a generalized-recurrent-parabolic space (V_n, g_{ij}, F_i^h) admits non-trivial *QGM* or not, and in the affirmative case, we are in principle able to find all generalized-recurrent-parabolic spaces $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ that can serve as images of V_n under the mappings considered. Hence, Theorems 3.2.1 and 3.3.1 turn out to be the fundamental theorems of the theory of *QGM*s.

4. CONCLUSION

The main problem in studying any mappings

$$f\colon (V_n, g_{ij}, F_i^h) \to (\overline{V}_n, \overline{g}_{ij}, F_i^h)$$

is a possibility for a given space (V_n, g_{ij}, F_i^h) to find out whether it admits the specified mapping or not.

The question of the existence of a *QGM* of the generalized-recurrent-parabolic space (V_n, g_{ij}, F_i^h) reduces to the study of differential equations (2.10) with respect to the tensor a_{ij} and the vector λ_i satisfying conditions (2.12). Subsequently, the solution of the problem is reduced to finding non-trivial solutions

$$a_{ij}(x), \qquad \lambda_i(x) \not\equiv 0, \qquad \mu(x), \qquad \nu(x),$$

of the system of homogeneous algebraic equations in (V_n, g_{ij}, F_i^h).

Theorems 3.2.1 and 3.3.1 allow for any generalized-recurrent-parabolic space (V_n, g_{ij}, F_i^h) either to find all spaces $(\overline{V}_n, \overline{g}_{ij}, F_i^h)$ on which V_n admits a *QGM* or prove that there are no such spaces. However, for large n, the direct solution of this problem is technically rather complicated.

REFERENCES

[1] D. V. Beklemišev. Differential geometry of spaces with almost complex structure. In *Geometry 1963 (Russian)*, Itogi Nauki, pages 165–212. Akad. Nauk SSSR Inst. Naučn. Informacii, Moscow, 1965.

[2] V. E. Berezovskii and J. Mikesh. Almost geodesic mappings of spaces with affine connection. *J. Math. Sci. (N.Y.)*, 207(3):389–409, 2015. Translated from Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz. Vol. 126, Geometry, 2013. `doi:10.1007/s10958-015-2378-5`.

[3] V. E. Berezovskiĭ, Ĭ. Mikesh, G. Khuda, and E. E. Chepurnaya. Canonical almost geodesic mappings that preserve the projective curvature tensor. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (6):3–8, 2017. `doi:10.3103/s1066369x17060019`.

[4] H. Chudá, J. Mikesh, P. Peška, and M. Shiha. On holomorphically projective mappings of equidistant parabolic Kähler spaces. In *Geometry, integrability and quantization XIX*, pages 115–121. Bulgar. Acad. Sci., Sofia, 2018.

[5] D. Doikov and V. Kiosak. On the Schwarzschild model for gravitating objects of the universe. *AIP Conference Proceedings*, 2302:040001, 2020.

[6] I. Hinterleitner and V. Kiosak. Special Einstein's equations on Kähler manifolds. *Arch. Math. (Brno)*, 46(5):333–337, 2010. `doi:10.5817/am2012-5-333`.

[7] I. Hinterleitner, J. Mikesh, and P. Peška. Fundamental equations of F-planar mappings. *Lobachevskii J. Math.*, 38(4):653–659, 2017. `doi:10.1134/S1995080217040096`.

[8] V. Kiosak, A. Savchenko, and A. Kamienieva. Geodesic mappings of compact quasi-Einstein spaces with constant scalar curvature. *AIP Conference Proceedings 2302*, page 040002, 2020.

[9] V. Kiosak, A. Savchenko, and S. Khniunin. On the typology of quasi-Einstein spaces. *AIP Conference Proceedings 2302*, page 040003, 2020.

[10] V. Kiosak, A. Savchenko, and O. Latysh. Geodesic mappings of compact quasi-Einstein spaces, II. *Proc. Int. Geom. Cent.*, 14(1):80–91, 2021. `doi:10.15673/tmgc.v14i1.1936`.

[11] V. Kiosak, A. Savchenko, and T. Shevchenko. Holomorphically projective mappings of special Kähler manifolds. *AIP Conference Proceedings, 2025(080004)*, 2018.

[12] I. Kurbatova. Canonical quasi-geodesic mappings of Kähler spaces. *Proc. Intern. Geom. Center*, 7(1):53–64, 2014.

[13] I. Kurbatova. On laws of canonical quasi-geodesic mappings of parabolically Kähler spaces. *Proc. Intern. Geom. Center*, 7(2):26–35, 2014.

[14] I. Kurbatova and D. Lozienko. On canonical quasigeodesic mappings of recurrent parabolic spaces. *Proc. Intern. Geom. Center*, 10(3-4):44–57, 2017.

[15] I. Kurbatova and M. Pistruil. Quasigeodesic mappings of special pseudo-Riemannian spaces. *Proc. Int. Geom. Cent.*, 13(3):18–32, 2020. `doi:10.15673/tmgc.v13i3.1770`.

[16] I. Kurbatova and O. Sisyuk. Quasi-geodesic mappings of recurrent parabolically spaces. *Proc. Intern. Geom. Center*, 8(1):74–83, 2014.

[17] J. Mikesh. Holomorphically projective mappings and their generalizations. volume 89, pages 1334–1353. 1998. Geometry, 3. `doi:10.1007/BF02414875`.

[18] J. Mikesh and N. S. Sinyukov. Quasiplanar mappings of spaces with affine connection. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (1):55–61, 1983.

[19] J. Mikesh, A. Vanžurová, and I. Hinterleitner. *Geodesic mappings and some generalizations.* Palacký University Olomouc, Faculty of Science, Olomouc, 2009.

[20] A. Z. Petrov. Modeling of the paths of test particles in gravitation theory. *Gravitacija i Teor. Otnositel'nosti*, (6):7–21, 1969.

[21] Miloš Z. Petrović and P. Peška. Equitorsion holomorphically projective mappings of generalized m-parabolic Kähler manifolds. *Filomat*, 33(4):1047–1052, 2019. `doi:10.2298/fil1904047p`.

[22] P. Peška, J. Mikesh, H. Chudá, and M. Shiha. On holomorphically projective mappings of parabolic Kähler manifolds. *Miskolc Math. Notes*, 17(2):1011–1019, 2016. `doi:10.18514/MMN.2017.1893`.

[23] M. I. Pistruil and I. M. Kurbatova. Canonical quasi-geodesic mappings of special pseudo-Riemannian spaces. *Proc. Int. Geom. Cent.*, 15(3-4):163–176, 2022.

[24] A. Savchenko, N. Vashpanova, and N. Vasylieva. Generalized $\varphi(Ric)$-vector fields in special pseudo-Riemannian spaces. *Proc. Int. Geom. Cent.*, 14(4):231–242, 2021. `doi:10.15673/tmgc.v14i4.2155`.

[25] M. Shiha. *Geodesic and holomorphically projective mappings of parabolically Kählerian spaces*. PhD thesis, Odessa: Univ., Supervisor Mikeš, J., 1992.

[26] N. S. Sinjukov. *Geodezicheskie otobrazheniya rimanovykh prostranstv.* “Nauka”, Moscow, 1979.

[27] N. S. Sinjukov. *Geodezicheskie otobrazheniya rimanovykh prostranstv.* “Nauka”, Moscow, 1979.

[28] N. S. Sinyukov. Almost geodesic mappings of affinely connected and Riemannian spaces. In *Problems in geometry, Vol. 13*, Itogi Nauki i Tekhniki, pages 3–26, 199. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1982.

[29] M. S. Stanković, M. L. Zlatanović, and N. O. Vesić. Basic equations of G-almost geodesic mappings of the second type, which have the property of reciprocity. *Czechoslovak Math. J.*, 65(140)(3):787–799, 2015. `doi:10.1007/s10587-015-0208-z`.

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