Fundamental theorems
of quasi-geodesic mappings
of generalized-recurrent-parabolic spaces

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Abstract. In previous papers we studied mappings of pseudo-Riemannian spaces being mutually quasi-geodesic and almost geodesic of the 2nd type. As a result, we arrived at the quasi-geodesic mapping

\[ f: (V_n, g_{ij}, F^h_i) \to (\nabla_n, \tilde{g}_{ij}, \tilde{F}^h_i) \]

of spaces with an affine structure, which was called generalized-recurrent. Quasi-geodesic mappings are divided into two types: general and canonical. In this article, the fundamental issues of the theory of quasi-geodesic mappings of generalized-recurrent-parabolic spaces are considered. First, the fundamental equations of quasi-geodesic mappings are reduced to a form that allows effective investigation. Then, using a new form of the fundamental equations, we prove theorems that allow for any generalized-recurrent-parabolic space \((V_n, g_{ij}, F^h_i)\) or to find all spaces \((\nabla_n, \tilde{g}_{ij}, \tilde{F}^h_i)\) onto which \(V_n\) admits a quasi-geodesic mapping of the general form, or prove that there are no such spaces.

Keywords: affine structure; quasi-geodesic mapping; pseudo-Riemannian space.

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1. Introduction

1.1. We continue to study diffeomorphisms of pseudo-Riemannian spaces which are also quasi-geodesic mappings (QGM) [5, 12, 13, 15, 16, 20, 26] with the reciprocity condition and almost-geodesic mappings of the second type [2, 3, 11, 19, 27–29].

We mean that QGM

\[ f: V_n(g_{ij}, F^h_i) \rightarrow \overline{V}_n(\overline{g}_{ij}, \overline{F}^h_i) \]

satisfies the reciprocity condition if the reverse mapping \( f^{-1} \) is also QGM. We have obtained the fundamental equations of such a mapping

\[ f: (V_n, g_{ij}, F^h_i) \rightarrow (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i) \]

in the common coordinate system \((x^i)\) with respect to the mapping \( f [15]:\)

\[
\begin{align*}
\Gamma^h_{ij}(x) &= \Gamma^h_{ij}(x) + \phi_i(x)\delta^h_j + \phi_j(x)F^h_i(x), \\
F^h_i(x) &= \overline{F}^h_i(x), \\
g_{i\alpha}F^\alpha_j &= -g_{ja}F^\alpha_i, \\
\overline{g}_{i\alpha}F^\alpha_j &= -\overline{g}_{ja}F^\alpha_i, \\
F^h_{(i,j)} &= q_iF^h_j, \\
F^h_iF^\alpha_i &= e\delta^h_i, & e &= 0, \pm 1, & i, h, j, \ldots &= 1, 2, \ldots, n,
\end{align*}
\]

(1.1)

(1.2)

(1.3)

where \( \Gamma^h_{ij}, \Gamma^h_{ij} \) are the Christoffel symbols of \( V_n, \overline{V}_n \), respectively; \( \psi_i(x), \phi_i(x), q_i(x), p_i(x) \) are certain covectors; \( F^h_i(x) \) is affinor; brackets \((i, j)\) denote the symmetrization with respect to the corresponding indices; comma «,» is a sign of the covariant derivative in respect to the connection of \( V_n \).

If in (1.1) \( \phi_i = 0 \) and \( \psi_i \neq 0 \), then the quasi-geodesic mapping degenerates into a geodesic [8–10, 27]. For \( \phi_i \neq 0 \) and \( \psi_i = 0 \), the quasi-geodesic mapping is called canonical. If in (1.1) \( \phi_i = 0 \) and \( \psi_i = 0 \), the QGM is called trivial.

An affinor structure that satisfies condition (1.3) is called [19]:

- **elliptic** if \( e = -1 \),
- **hyperbolic** if \( e = +1 \),
- **m-parabolic** when \( e = 0, \) rank \( F = m \) \((2m < n)\),
- **parabolic** when \( e = 0, \) rank \( F = m \) \((2m = n)\).

Equation (1.1) characterizes \( F \)-planar mapping which started to study Mikeš and Sinyukov [18]. These results were specified in paper [7].

1.2. We call an affinor structure \( F^h_i \) that satisfies conditions (1.2) a **generalized-recurrent structure** (of elliptic, hyperbolic, or parabolic type) [15]. If in conditions (1.2) \( q_i = 0 \), the affinor \( F^h_i \) defines a \( K \)-structure [1, 26, 28].
In [16], a recurrent-parabolic structure was introduced, which is determined by the conditions:

\[ F^h_{\alpha F^\alpha_i} = 0, \quad g_{\alpha F^\alpha_j} = -g_{j\alpha F^\alpha_i}, \quad F^h_{i,j} = q_j F^h_i. \] (1.4)

The articles [12, 14, 23] are devoted to some issues that concern quasi-geodesic mappings of recurrent-parabolic spaces.

The \( K \)-structure and the recurrent-parabolic structure are the special cases of generalized-recurrent structure.

In the context of types of recurrences and methods for extracting special spaces with structure, the papers [6, 24] are of interest.

In [15] the properties of a generalized-recurrent structure of parabolic type were studied.

We call the vector \( q_i \) in (1.2) the generalized recurrence vector of the structure \( F^h_i \), and in the case \( F^h_{i,j} = q_j F^h_i \), the recurrence vector. Note that under the condition that the vector \( q_i \) is gradient, the affinor \( \tilde{F}^h_i = e^{-q_i F^h_i} \), where \( q_i = \frac{\partial q(x)}{\partial x^i} \), defines a \( K \)-structure in the generalized-recurrent space \((V_n, g_{ij}, F^h_i)\), and a Kählerian structure in the recurrent-parabolic space.

Shortly, in this case \((V_n, g_{ij}, \tilde{F}^h_i)\) is a parabolic Kähler space, see [17, 19]. The studied mappings are holomorphically projective mappings between parabolic Kähler spaces. These problems were studied in papers [4, 17, 21, 22] and also in the dissertation by Shiha [25].

1.3. Let us define an operation of contraction with an affinor, which is called conjugation with respect to the corresponding index and is denoted as follows:

\[
T_{j_1...j_{k-1}\alpha j_{k+1}...j_r} F^\alpha_i = T_{j_1...j_{k-1}\alpha j_{k+1}...j_r} F^\alpha_i
\]

\[
T_{j_1...j_{k-1}\alpha j_{k+1}...j_r} F^h_{\alpha F^\alpha_i} = T_{j_1...j_{k-1}\alpha j_{k+1}...j_r} F^h_{\alpha F^\alpha_i}
\]

1.4. The integrable parabolic structure \( F^h_i \) in some neighborhood of the point \( V_n \) can be reduced to the form

\[
(F^h_i) = \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix}
\]

where \( I_m \) is the identity matrix of order \( m = \frac{n}{2} \).

We will call such a coordinate system adapted to the affinor. Then under the conditions

\[ g_{\alpha F^\alpha_j} = -g_{j\alpha F^\alpha_i}, \quad F^h_{\alpha F^\alpha_i} = 0, \]

the components of the metric tensor of the space \( V_n \) in the adapted coordinate system satisfy the conditions:

\[ g_{ab} = g_{ba}, \quad g_{ab+m} = -g_{a+mb}, \quad g_{a+mb+m} = 0 \]
for $a, b = 1, 2, \ldots, m$. Further, the auxiliary tensor $A^h_i$, will be useful to us, which is determined in the adapted coordinate system by the matrix

$$(A^h_i) = \begin{pmatrix} P & I_m \\ -P^2 & -P \end{pmatrix}$$

where $P$ is an arbitrary square matrix of order $m$.

It is easy to check that

$$F^\beta_\alpha A^\alpha_\beta = m, \quad A^h_\alpha A^\alpha_i = 0, \quad F^h_\alpha A^\alpha_i + A^h_\alpha F^\alpha_i = \delta^h_i. \quad (1.5)$$

1.5. In [15] we came to the conclusion that the integrable affinor structure of the generalized-recurrent space $(V_n, g_{ij}, F^h_i)$ is characterized by the following properties:

$$F^\alpha_i, \alpha = 0, \quad F^h_j, i \equiv F^h_j, i = F^h_j, i = 0, \quad q_i = 0. \quad (1.6)$$

Note that, in contrast to hyperbolic and elliptic types, an integrable generalized-recurrent structure of parabolic type (in particular, a parabolic $K$-structure) need not be Kählerian, i.e. relations (1.6) do not imply that the affinor $F^h_i$ is covariantly constant.

Further, in this paper, we consider only the integrable affinor structure.

1.6. In [15] it was proved that the image of a generalized-recurrent space under $QGM$ is also a generalized-recurrent space, that is,

$$F^h_{(i|j)} = \tilde{q}(j^h_{i})$$

where

$$\tilde{q}_i = q_i - \psi_i + \phi_i,$$  

«|$» is a sign of a covariant derivative in respect to the connection of $\overline{V}_n$, i.e. affinor $F^h_i$ in the space $\overline{V}_n$ also defines a generalized-recurrent structure.

Under the condition $\tilde{q}_i = q_i$ we say that $QGM$ preserves the generalized recurrence vector. In this case, the vectors $\psi_i$ and $\phi_i$ in the basic $QGM$ equations (1.1) are related in this way

$$\psi_i = \phi_i$$  \hspace{1cm} (1.7)

and $\psi_i$ is locally a gradient:

$$\psi(x) = \frac{1}{2} \ln|\frac{g_{ij}}{g}|.$$  

In this paper, we consider the quasi-geodesic mapping ($QGM$) of generalized-recurrent-parabolic spaces with an integrable affinor structure.

The investigation is carried out in tensor form, locally, in the class of real sufficiently smooth functions.
2. Linear equations of the theory of QGMs of generalized-recurrent-parabolic spaces

2.1. A generalized-recurrent-parabolic space \((V_n, g_{ij}, F^h_i)\) admits a QGM
\[ f: (V_n, g_{ij}, F^h_i) \rightarrow (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i) \]
if and only if in the coordinate system \((x^i)\) the fundamental equations of this mapping are satisfied:
\[
\begin{align*}
\Gamma^h_{ij}(x) &= \Gamma^h_{ij}(x) + \psi_i(x)\delta^h_j + \phi_i(x)F^h_{ji}(x), \\
F^h_i(x) &= F^h_i(x), \\
\phi^h_i &= \psi_i, \\
g_{ij} &= g_{ij}, \\
F^h_{(i,j)} &= q_{(j}F^h_{i)} \\
\end{align*}
\]  \(\text{(2.1)-(2.5)}\)

where \(q_i\) is the generalized recurrence vector of the structure \(F^h_i\). In other words, the mapping \( f: (V_n, g_{ij}, F^h_i) \rightarrow (\overline{V}_n, \overline{g}_{ij}, \overline{F}^h_i) \) is a QGM if and only if under conditions (2.3), (2.2), (2.4), (2.5) in the space \((V_n, g_{ij}, F^h_i)\) the system of nonlinear differential equations in partial derivatives of the first order (2.1) with respect to the components of the tensor \(\overline{g}_{ij}(x)\) and the vector \(\phi_i \neq 0\) has a solution.

Using methods that developed in the theory of geodesic mappings of Riemannian spaces [27], we reduce the fundamental equations (2.1)-(2.5) to a form that allows an effective study.

2.2. The following theorem holds:

**Theorem 2.2.1.** A generalized-recurrent-parabolic space \((V_n, g_{ij}, F^h_i)\) admits a non-trivial QGM if and only if it contains a non-singular symmetric tensor of type \((0,2)\) \(a_{ij}\) that satisfies the equations
\[
a_{ij,k} = \lambda_1 g_{jk} + \lambda_2 g_{ik} + \lambda_3 F_{jk} + \lambda_4 F_{ik},
\]
and
\[
a_{ij} = -a_{ji}, \quad \det(a_{ij}) \neq 0
\]  \(\text{(2.6)-(2.7)}\)

for some covector \(\lambda_i \neq 0\).

**Proof.** Assume that a generalized-recurrent-parabolic space \(V_n\) admits a QGM onto \(\overline{V}_n\). Since \(\overline{g}_{ij|k} = 0\) in \(\overline{V}_n\), equation (2.1) can be written in the following equivalent form:
\[
\overline{g}_{ij,k} = 2\phi_k\overline{g}_{ij} + \phi_j\overline{g}_{ik} + \phi_i\overline{F}_{jk} + \phi_j\overline{F}_{ik},
\]
where
\[
\overline{F}_{ik} = \overline{g}_{ia}F^a_k.
\]  \(\text{(2.8)}\)
The equations (2.8) control the existence of quasi-geodesic mappings of generalized-recurrent-parabolic spaces.

Let us introduce the nondegenerate tensor 

\[ a_{ij} = e^{2\psi}g^{\alpha\beta}g_{\alpha i}g_{\beta j}. \]  

(2.9)

Since \( \bar{g}_{ia\alpha}g^{\alpha h} = \delta_i^h \), we have that

\[ \bar{g}_{ia,k}g^{\alpha h} = -\bar{g}_{i\alpha\alpha}g^{\alpha h}. \]

Therefore, it follows from (2.9) and (2.8) that

\[ a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} + \lambda_i F_{jk} + \lambda_j F_{ik}, \]  

(2.10)

where

\[ \lambda_i = -e^{2\psi}\phi_i g_{\alpha i}. \]  

(2.11)

It is easy to check that in view of (2.3) and (2.4), \( \lambda_i \) is gradient and

\[ a_{ij} = -a_{ji}, \quad \det(a_{ij}) \neq 0. \]  

(2.12)

Thus, if a pseudo-Riemannian space \( (V_n, g_{ij}, F^h_i) \) with a generalized-recurrent-parabolic structure \( F^h_i \) admits a non-trivial QGM on \( (V_n, \bar{g}_{ij}, \bar{F}^h_i) \), then it necessarily contains a nonsingular symmetric tensor \( a_{ij} \) that satisfies (2.10), (2.12) for some nonzero vector \( \lambda_i \).

The converse is also true. Indeed, if \( a_{ij} \) and \( \lambda_i \) satisfy (2.10), (2.12), then (2.8), (2.3), (2.4) hold for

\[ \bar{g}_{ij} = e^{-2\psi}a^{\alpha\beta}g_{\alpha i}g_{\beta j} \]

and

\[ \phi_i = -e^{-2\psi}\lambda_\alpha g^{\alpha\beta}\bar{g}_{\beta i}. \]

The equation (2.10) is a new linear form of the fundamental equations of the theory of QGMs of generalized-recurrent-parabolic spaces.

3. Fundamental theorems of QGMs of generalized-recurrent-parabolic spaces

3.1. Let \( (V_n, g_{ij}, F^h_i) \) be a generalized-recurrent-parabolic space, so its metric tensor \( g_{ij}(x) \) and affinor \( F^h_i(x) \) satisfy conditions (2.2), (2.4), (2.5) are known. The question of the existence of a QGM of the space \( (V_n, g_{ij}, F^h_i) \) is reduced to the study of differential equations (2.10) with respect to the tensor \( a_{ij} \) and the vector \( \lambda_i \), which satisfy conditions (2.12).

Further, we consider the integrability conditions for the equations (2.10), which, taking into account the Ricci identity and (2.5), have the following form:

\[ a_\alpha(iR^\alpha_j)_{kl} = g_{k(i}K_{j)l} - g_{l(i}K_{j)k} + F_{l(i}\lambda_{j)k} - F_{k(i}\lambda_{j)l} + \lambda_{(i}F_{j)k,l}, \]

(3.1)
where

\[ K_{il} = \lambda_{i,l} + \lambda_\alpha F_{\alpha i,l}. \]  \hspace{1cm} (3.2)

Note that from (3.2), in view of (2.5), it follows:

\[ K_{il} = 0. \]  \hspace{1cm} (3.3)

Using the tensor \( A_i^h \) we defined earlier and introduce the following tensor

\[ A_{ij} = A_{i\alpha} g_{\alpha j}. \]

Contract (3.1) with \( A_{kj} \) with respect to the indices \( k,j \) and conjugate with respect to the index \( i \). Taking into account (3.3), we obtain

\[ K_{il} = 2 n \left[ 4 \frac{F_{\alpha i,l} F_{\beta i,l} A_{\gamma \sigma} A_{\gamma l}}{n^2 - 4} \right], \]  \hspace{1cm} (3.4)

where

\[ \left( 4 \frac{F_{\alpha i,l} F_{\beta i,l} A_{\gamma \sigma} A_{\gamma l}}{n^2 - 4} \right) = 2 n \left[ 4 \frac{F_{\alpha i,l} F_{\beta i,l} A_{\gamma \sigma} A_{\gamma l}}{n^2 - 4} \right]. \]

Contraction (3.1) with \( A_{kj} \) over indices \( k,j \) and conjugation on \( i \) gives us

\[ a_{\alpha \beta} R_{il}^{\alpha \beta} = \frac{n-4}{2} K_{il} + K_{i\nu} A_{l}^\nu - \mu F_{il} - \lambda_\alpha \left( \frac{n-2}{2} F_{i,l}^\alpha - F_{i}^\alpha F_{[\gamma,l]} A_{\gamma \beta} \right), \]

where

\[ \mu = K_{\alpha \beta} A_{\beta \alpha}. \]

Then based on (3.4):

\[ \frac{n-4}{2} K_{il} = \mu F_{il} + a_{\alpha \beta} \left( \tilde{R}_{il}^{\alpha \beta} - \frac{2}{n-2} \tilde{R}_{iw}^{\alpha \beta} A_{l}^w \right) + \lambda_\alpha \left( \frac{n-2}{2} F_{i,l}^\alpha - F_{i}^\alpha F_{[\gamma,l]} A_{\gamma \beta} \right). \]

Contraction (3.1) with \( A_{kj} \) over indices \( k,j \) in view of this equation gives

\[ \lambda_{i,l} = a_{\alpha \beta} S_{il}^{\alpha \beta} + \lambda_\alpha P_{il}^{\alpha} + \mu Q_{il} + \nu F_{il}, \]  \hspace{1cm} (3.5)

where

\[ S_{il}^{\alpha \beta} = \frac{4}{n(n-4)} \tilde{R}_{il}^{\alpha \beta} \left( \frac{n-4}{2} g_{l}^{\gamma} - \frac{n-4}{n+2} A_{l}^{\gamma} \right) + \frac{4}{n(n-4)} \tilde{R}_{\nu l}^{\alpha \beta} \left[ \frac{n+4}{n+2} F_{i}^{\gamma} A_{l}^{\gamma} + g_{\sigma i} A_{\sigma}^{\gamma} \left( \frac{8(n-1)}{n^2 - 4} A_{l}^{\gamma} - 2 g_{l}^{\gamma} \right) \right], \]

\[ P_{il}^{\alpha} = \frac{2}{n} F_{i[l,\gamma]} A_{\gamma}^{\alpha} + \frac{2(n-2)(n+4)}{n(n-4)(n+2)} F_{i,\gamma}^{\alpha} A_{l}^{\gamma} - \frac{2}{n-4} F_{\beta,\gamma}^{\alpha} g_{i\sigma} A_{\sigma}^{\beta} + \frac{4}{n(n-4)} \left( 2 F_{\beta,\gamma}^{\alpha} g_{i\sigma} A_{\gamma}^{\alpha} - \frac{n-4}{2} g_{i}^{\alpha} F_{\beta,\gamma}^{\alpha} - \frac{n+4}{n+2} F_{i}^{\alpha} F_{\beta,\gamma}^{\alpha} A_{l}^{\gamma} \right) A^{\beta}. \]
\[ Q_{i\ell} = \frac{4}{n(n-4)} \left( \frac{n}{2} g_{i\ell} + \frac{2(n+6)}{n+2} F_{\alpha i} A^\alpha_{\ell} \right), \]

\[ \nu = \lambda_{\alpha,\beta} A^{\beta\alpha}. \]

Note, that the expression for the tensor \( Q_{i\ell} \) means that

\[ 4 Q_{i\ell} = 2(n+4)(n-6) F_{i\ell}, \quad Q_{i\ell} = \frac{2}{n-4} F_{i\ell}, \quad (3.6) \]

\[ Q_{\alpha\beta} g^{\alpha\beta} = 2(n-12) \left( \frac{n}{2} g_{i\ell} + 2(n+6) \right), \quad Q_{\alpha\beta} A^{\alpha\beta} = 0, \quad (3.7) \]

\[ Q_{i\ell,k} = 0. \quad (3.8) \]

3.2. Taking into account (2.10) the integrability conditions for (3.5) have the following form:

\[ Q_{i[l,k]} + F_{i[l} \nu_{,k]} + \mu Q_{i[l,k]} + \nu F_{i[l,k]} + a_{\alpha\beta} \tilde{S}_{\alpha\beta i\ell k} + \lambda_{\alpha} \tilde{P}_{\alpha i\ell k} = 0 \quad (3.9) \]

where

\[ \tilde{S}_{\alpha\beta i\ell k} = S_{\alpha\beta i[l,k]} + S_{\gamma\sigma i[l} P_{\sigma_{\gamma}i\ell k]} - S_{\gamma\sigma i[l} P_{\sigma_{\gamma}i\ell k]}, \]

\[ \tilde{P}_{\alpha i\ell k} = R_{\alpha i\ell k} + \left( S_{\alpha i[l} + S_{\alpha i[l]} \right) g_{k][\beta} - S_{i[l}^{[\alpha\beta]} F_{k][\beta]. \]

Conjugate (3.9) taking into account (3.6), (3.7), (3.8):

\[ \frac{2}{n-4} \left( F_{i\ell\mu,k} - F_{k\ell\mu,l} \right) + a_{\alpha\beta} \tilde{S}_{\alpha\beta i\ell k} + \lambda_{\alpha} \tilde{P}_{\alpha i\ell k} = 0. \]

Comparing the result of cycling this equation by \( i, k, l \) with the original relations, we get:

\[ \frac{4}{n-4} F_{i\ell\mu,k} + a_{\alpha\beta} \left( \tilde{S}_{\alpha\beta i\ell k} + \tilde{S}_{\alpha\beta k\ell i} - \tilde{S}_{\alpha\beta i\ell k} \right) + \lambda_{\alpha} \left( \tilde{P}_{\alpha i\ell k} + \tilde{P}_{\alpha k\ell i} - \tilde{P}_{\alpha i\ell k} \right) = 0. \]

From here, after contraction with \( A^{k\ell} \) with respect to the indices \( l,k \), we find that

\[ \frac{2n}{n-4} \mu_{,i} = a_{\alpha\beta} \tilde{S}_{\alpha\beta i} + \lambda_{\alpha} \tilde{P}_{\alpha i}, \quad (3.10) \]

where

\[ \tilde{S}_{\alpha\beta i} = \left( \tilde{S}_{\alpha\beta i} + \tilde{S}_{\alpha\beta i} - \tilde{S}_{\alpha\beta i} \right) A^{\gamma\sigma}, \]

\[ \tilde{P}_{\alpha i} = \left( \tilde{P}_{\alpha i} + \tilde{P}_{\alpha i} - \tilde{P}_{\alpha i} \right) A^{\gamma\sigma}. \]

Similarly, comparing the result of cycling (3.9) by \( i, k, l \) with the original relations based on the expression (3.10), we get:

\[ n \nu_{,i} = a_{\alpha\beta} \tilde{S}_{\alpha\beta i} + \lambda_{\alpha} \tilde{P}_{\alpha i} + \mu Q_{i} + \nu \tilde{Q}_{i}, \quad (3.11) \]
where
\[
\bar{S}_i^{\alpha\beta} = \left( \frac{n - 4}{2n} Q_i[\gamma] S_{\sigma}^{\alpha\beta} + \tilde{S}_{(i\sigma\gamma)}^{\alpha\beta} - 2\tilde{S}_{i\sigma\gamma}^{\alpha\beta} \right) A^{\gamma\sigma},
\]
\[
\bar{P}_i^\alpha = \left( \frac{n - 4}{2n} Q_i[\gamma] P_{\sigma}^{\alpha} + \tilde{P}_{(i\sigma\gamma)}^{\alpha} - 2\tilde{P}_{i\sigma\gamma}^{\alpha} \right) A^{\gamma\sigma},
\]
\[
\bar{Q}_i = Q_{\sigma[\gamma,i]} + Q_{\gamma[i,\sigma]} - Q_{i[\sigma,\gamma]} A^{\gamma\sigma},
\]
\[
\bar{Q}_i = 2 F_{\sigma\gamma,i} A^{\gamma\sigma}.
\]

Equations (2.10), (3.5), (3.10) and (3.11) form a closed system of the first order partial differential equations of Cauchy type with respect to the unknown functions \(a_{ij}, \lambda_i, \mu, \nu\). Let us denote it by (B). In the theory of differential equations regular methods have been developed for such systems. Thus, we proved the following

**Theorem 3.2.1.** A pseudo-Riemannian space \((V_n, g_{ij}(x), F^h_i(x))\) with an integrable generalized-recurrent-parabolic structure \(F^h_i(x)\) admits a QGM, if and only if the system of differential equations (B) has a non-trivial solution
\[
a_{ij}(x), \quad \lambda_i(x) \neq 0, \quad \mu(x), \quad \nu(x)
\]
which satisfies the conditions
\[
a_{ij}(x) = a_{ji}(x), \quad \det(a_{ij}(x)) \neq 0, \quad a_{i\bar{j}} = -a_{j\bar{i}}.
\]

The family of differential equations (B) is linear with coefficients of intrinsic character in \(V_n\) and independent of the choice of coordinates. If the metric tensor \(g\) and the structure tensor \(F\) of the generalized-recurrent-parabolic manifold \(V_n\) are real then for the initial data
\[
a_{ij}(x_0) = \hat{a}_{ij}, \quad \lambda_i(x_0) = \hat{\lambda}_i, \quad \mu(x_0) = \hat{\mu}, \quad \nu(x_0) = \hat{\nu}.
\]

The system (B) has at most one solution. Accounting that the initial data must satisfy (2.2), (2.4) which in read, canonical coordinates,
\[
g_{ab} = g_{ba}, \quad g_{ab+m} = -g_{a+mb}, \quad g_{a+mb+m} = 0,
\]
for \(a, b = 1, 2, \ldots, m\), it follows that the general solution of (B) depends on \(r\) significant parameters, where
\[
r = \frac{n}{4} \left( \frac{n}{2} + 1 \right) + \frac{n}{4} \left( \frac{n}{2} - 1 \right) + n + 2 = \frac{(n + 2)^2}{4} + 1.
\]

The solution of (B) satisfying the given initial conditions
\[
a_{ij}(x_0) = \hat{a}_{ij}, \quad \lambda_i(x_0) = \hat{\lambda}_i, \quad \mu(x_0) = \hat{\mu}, \quad \nu(x_0) = \hat{\nu},
\]
can be given as a Taylor series, and if necessary, enumerated in a neighborhood of a given point $x_0$ of the space.

3.3. Of course, the system $(B)$ might not be consistent. However, this system is consistent if and only if the set of integrability conditions $(B)$ and their differential prolongations are consistent. The integrability conditions for the first group of equations $(B)$, taking into account (3.1), (3.2), (3.5), can be represented in the form:

$$a_{\alpha\beta}T_{ijkl}^{\alpha\beta} + \lambda_\alpha M_{ijkl}^\alpha + \mu N_{ijkl} = 0,$$

(3.12)

where

$$T_{ijkl}^{\alpha\beta} = \delta_{(i}^{\alpha} R_{jkl)}^{\beta} + \frac{n - 4}{2(n - 2)} \tilde{R}_{\gamma\nu}^{\alpha\beta} F_{(j}^{\gamma} g_{i[k]} \left((n - 4) \delta_{l]}^{\nu} + 2 A_{[l]}^\nu\right) + F_{l(i} S_{j)k_]}^{\alpha\beta} - F_{kl(i} S_{j]l}^{\alpha\beta},$$

$$M_{ijkl}^\alpha = C_{ij[kl]}^\alpha + F_{l(i} P_{j)k}^\alpha - F_{k(i} P_{j)l}^\alpha + \delta_{(i} F_{j)l[k]},$$

$$C_{ijkl}^\alpha = g_{k(i} \left(\frac{n - 2}{2} F_{j,l}^{\alpha} - F_{j)l}^{\alpha} F_{\beta[\gamma,]} A_{}\gamma\beta\right),$$

$$N_{ijkl} = F_{l(i} Q_{j)k} - g_{j)k]} - F_{k(i} Q_{j)l} - g_{j)l]}.$$

The integrability conditions for the second group of equations $(B)$, taking into account (3.9), (3.10), (3.11), have the form:

$$a_{\alpha\beta} \overline{T}_{ikl}^{\alpha\beta} + \lambda_\alpha \overline{M}_{ikl}^\alpha + \mu \overline{N}_{ikl} + \nu \overline{L}_{ikl} = 0,$$

(3.13)

where

$$\overline{T}_{ikl}^{\alpha\beta} = \frac{n - 4}{2n} Q_{i[l} \overline{S}_{k]}^{\alpha\beta} + \frac{1}{n} F_{i[l} \overline{S}_{k]}^{\alpha\beta},$$

$$\overline{M}_{ikl}^\alpha = \frac{n - 4}{2n} Q_{i[l} \overline{P}_{k]}^{\alpha} + \frac{1}{n} F_{i[l} \overline{P}_{k]}^{\alpha},$$

$$\overline{N}_{ikl} = Q_{i[l,]} + \frac{1}{n} F_{i[l} \overline{Q}_{k]},$$

$$\overline{L}_{ikl} = F_{i[l,]} + \frac{1}{n} F_{i[l} \overline{Q}_{k]}.$$

The integrability conditions for the third group of equations $(B)$, taking into account (2.5), (3.5), (3.10), can be represented in the form:

$$a_{\alpha\beta} \overline{T}_{il}^{\alpha\beta} + \lambda_\alpha \overline{M}_{il}^\alpha + \mu \overline{N}_{il} + \nu \overline{L}_{il} = 0,$$

(3.14)

where

$$\overline{T}_{il}^{\alpha\beta} = \overline{S}_{[i,l]}^{\alpha\beta} + \overline{S}_{[i]l}^{\alpha\beta} \overline{P}_{i]}^{\gamma},$$
\[
\overline{M}_{il} = P_{[i,l]}^\alpha + P_{\gamma[l]}^\alpha P_{i}^\gamma + g_{\beta[l]} \left( S_{i}^{(\alpha\beta)} - S_{i}^{(\alpha\beta)} \right),
\]
\[
\overline{N}_{il} = Q_{\alpha[l]} P_{i}^\alpha, \quad \overline{L}_{il} = F_{\alpha[l]} P_{i}^\alpha.
\]

Finally, the integrability conditions for the fourth group of equations \((B)\), taking into account \((2.5), (3.5), (3.10), (3.11)\) can be represented in the following form:

\[
a_{\alpha\beta} \overline{T}_{il} + \lambda_\alpha \overline{M}_{il} + \mu \overline{N}_{il} + \nu \overline{L}_{il} = 0, \tag{3.15}
\]

where

\[
\overline{T}_{il} = S_{[i,l]}^{\alpha\beta} + S_{\gamma[l]} S_{i}^\alpha + \frac{n-4}{2n} S_{[l]}^\alpha Q_{i} + \frac{1}{n} \overline{S}_{[l]} Q_{i},
\]

\[
\overline{M}_{il} = \overline{P}_{[i,l]}^\alpha + P_{\gamma[l]}^\alpha \overline{P}_{i}^\gamma + g_{\beta[l]} \left( \overline{S}_{i}^{(\alpha\beta)} - \overline{S}_{i}^{(\alpha\beta)} \right) + \frac{n-4}{2n} \overline{P}_{[l]} Q_{i} + \frac{1}{n} \overline{P}_{[l]} Q_{i},
\]

\[
\overline{N}_{il} = \overline{Q}_{[i,l]} + \frac{1}{n} \overline{Q}_{[l]} \overline{Q}_{i} + Q_{\alpha[l]} \overline{P}_{i}^\alpha,
\]

\[
\overline{L}_{il} = \overline{Q}_{[i,l]} + \frac{1}{n} \overline{Q}_{[l]} \overline{Q}_{i} + F_{\alpha[l]} \overline{P}_{i}^\alpha.
\]

Denote the integrability conditions for the system \((B)\) \((3.12), (3.13), (3.14)\) by \((B_0)\). Obviously, \((B_0)\) is a system of linear homogeneous algebraic equations for an unknown functions \(a_{ij}, \lambda_i, \mu, \nu\) with coefficients from \(V_n\). They must be satisfied identically for any solution of the system \((B)\) whenever it exists.

Differentiating \((B_0)\) covariantly and using \((B)\) we obtain the first prolongation of \((B_0)\) which we denote \((B_1)\). Obviously, \((B_1)\) is also a system of linear homogeneous algebraic equations in \(a_{ij}, \lambda_i, \mu, \nu\) with coefficients from \(V_n\). The differential prolongations of \((B_1)\) will be denoted \((B_2)\) and so on.

As we see, \((B_0), (B_1), (B_2), \ldots\) is a system of linear homogeneous algebraic equations for \(a_{ij}(x), \lambda_i(x) \neq 0, \mu(x), \nu(x)\) with coefficients from \(V_n\). Since the number of unknown functions is finite, there is a natural number \(N\) such that \((B_N)\) and subsequent continuations will be consequences of \((B_0), (B_1), \ldots, (B_{N-1})\).

In accordance with the analytical theory of differential equations, the system \((B)\) has a non-trivial solution in the neighborhood of the point \(M_0\) if and only if the system of equations \((B_0), (B_1), \ldots, (B_{N-1})\) has a non-trivial solution at this point.

Hence, we get the following
Theorem 3.3.1. A pseudo-Riemannian space with an integrable generalized-recurrent-parabolic structure \((V_n, g_{ij}(x), F_i^h(x))\) admits a QGM, if and only if the system of homogeneous algebraic equations \((B_0), (B_1), \ldots, (B_{s-1})\) has a non-trivial solution in \((V_n, g_{ij}, F_i^h)\)

\[
a_{ij}(x), \quad \lambda_i(x) \neq 0, \quad \mu(x), \quad \nu(x)
\]

which satisfies the conditions

\[
a_{ij}(x) = a_{ji}(x), \quad \det(a_{ij}(x)) \neq 0, \quad a_{i\bar{j}} = -a_{j\bar{i}}.
\]

Theorems 3.2.1 and 3.3.1 together supply us with a regular method enabling to decide effectively whether a generalized-recurrent-parabolic space \((V_n, g_{ij}, F_i^h)\) admits non-trivial QGM or not, and in the affirmative case, we are in principle able to find all generalized-recurrent-parabolic spaces \((\overline{V}_n, \overline{g}_{ij}, F_i^h)\) that can serve as images of \(V_n\) under the mappings considered. Hence, Theorems 3.2.1 and 3.3.1 turn out to be the fundamental theorems of the theory of QGMs.

4. Conclusion

The main problem in studying any mappings

\[f: (V_n, g_{ij}, F_i^h) \rightarrow (\overline{V}_n, \overline{g}_{ij}, F_i^h)\]

is a possibility for a given space \((V_n, g_{ij}, F_i^h)\) find out if it admits the specified mapping or not.

The question of the existence of a QGM of the generalized-recurrent-parabolic space \((V_n, g_{ij}, F_i^h)\) reduces to the study of differential equations \((2.10)\) with respect to the tensor \(a_{ij}\) and the vector \(\lambda_i\) satisfying conditions \((2.12)\). Subsequently, the solution of the problem is reduced to finding non-trivial solutions

\[
a_{ij}(x), \quad \lambda_i(x) \neq 0, \quad \mu(x), \quad \nu(x),
\]

of the system of homogeneous algebraic equations in \((V_n, g_{ij}, F_i^h)\).

Theorems 3.2.1 and 3.3.1 allow for any generalized-recurrent-parabolic space \((V_n, g_{ij}, F_i^h)\) either to find all spaces \((\overline{V}_n, \overline{g}_{ij}, F_i^h)\) on which \(V_n\) admits a QGM or prove that there are no such spaces. However, for large \(n\), the direct solution of this problem is technically rather complicated.

References

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