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On the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$

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Abstract. We are motivated by [M. Arkowitz. K. Maruyama. J. Math. Soc. Japan, 66(3):735–743, 2014]: "It would be interesting to compute other Gottlieb groups of Moore spaces such as, for example, $G_{n+1}(M(A,n))$" to compute the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$ and $m \geqslant 1$.

Анотація. В статті [M. Arkowitz. K. Maruyama. J. Math. Soc. Japan, 66(3):735–743, 2014] було поставлене питання про обчислення груп Готтліба просторів Мура, наприклад, таких як $G_{n+1}(M(A,n))$. В даній роботі обчислюються групи Готтліба $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ для $k = 1, 2$ та $m \geqslant 1$.

Introduction

Given a pointed space X, its n-th Gottlieb group $G_n(X)$ with $n \geqslant 1$ was introduced and studied by Gottlieb in [8, 9], and it has been shown to have many topological applications. There have been recent results on Gottlieb groups of rational spaces [5, pp. 377–380] and on Gottlieb groups of spheres, projective and Moore spaces [7].

The paper [2] deals with $G_n(\Sigma X_1 \vee \Sigma X_2 \vee \cdots \vee \Sigma X_k)$, where ΣX_i is the suspension of the space X_i, with particular attention to the case $k = 2$ and X_i a sphere. Necessary and sufficient conditions for an element of the homotopy group $\pi_n(\Sigma X_1 \vee \cdots \vee \Sigma X_k)$ to be in $G_n(\Sigma X_1 \vee \cdots \vee \Sigma X_k)$ are presented in [2, Proposition 2.4].

We are motivated by [2, Remark 4.5]: "It would be interesting to compute other Gottlieb groups of Moore spaces such as, for example, $G_{n+1}(M(A,n))$"

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to compute the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$ and $m \geqslant 1$.

This paper is organized as follows. Section 1 fixes up some notations and definitions, and necessary known results on Gottlieb groups as well.

Section 2 generalizes some results presented in [2]. First, the groups $G_N(\bigvee_{t\in T} \mathbb{S}^{n_t})$ for $n_t \geqslant 2$ with $t \in T$ are computed provided $|T| \geqslant 2$ and $N < 2\min\{n_t\}_{t\in T} - 1$. Then [8, 9] are used to study

$$G_N\Big(\bigvee_{s\in S} \mathbb{S}^1 \vee \bigvee_{t\in T} \mathbb{S}^{n_t}\Big)$$

with $|S| \geqslant 1$, $|T| \geqslant 0$ and $n_t \geqslant 2$ for $t \in T$ provided $N < 2\min\{n_t\} - 1$.

Section 3 presents computations of Gottlieb groups of Moore spaces $M(A, n)$ for some finitely generated Abelian groups A and, in particular, it is devoted to explicit computations of Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$ and $m \geqslant 1$. The main results of this paper, as consequences of Theorems 3.9 and 3.18, say:

- *if* $n \geqslant 3$ *then* $G_{n+1}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$ *for* $m \geqslant 1$*;*
- *if* $n \geqslant 4$ *then* $G_{n+2}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$ *for* $m \geqslant 1$*.*

Conclusions on $G_{n+k}(M(A, n))$ with $k = 1, 2$ for some finitely generated Abelian groups A are derived as well.

1. Prerequisites

Throughout this paper all spaces and maps are assumed to be connected, based and of the homotopy type of CW-complexes. We use the standard terminology and notation from homotopy theory, mainly from [16]. We do not distinguish between a map and its homotopy class, write ΩX (resp. ΣX) for the loop (resp. suspension) space on a space X and $[X, Y]$ for the set of homotopy classes of maps $X \to Y$, and $X \simeq Y$ for a homotopy equivalence of spaces X and Y.

Given spaces X and Y, we use the customary notations $X \vee Y$ and $X \wedge Y = (X \times Y)/(X \vee Y)$ for the *wedge* and the *smash* product of spaces X and Y, respectively. Let ι_X be the identity map on a space X and ι_n be the identity map on the n-sphere $\mathbb{S}^n$ and write $\pi_n(X)$ with $n \geqslant 0$ for the n-th homotopy group of X.

For maps $\alpha\colon \Sigma X \to Z$ and $\beta\colon \Sigma Y \to Z$, there exists a map

$$k(\alpha, \beta)\colon \Sigma(X \wedge Y) \to Z$$

as defined in [1] and its homotopy class is the generalized Whitehead product:

$$[\alpha, \beta] = [k(\alpha, \beta)] \in [\Sigma(X \wedge Y), Z].$$

In particular, given the inclusions

$$j_1\colon \Sigma X \hookrightarrow \Sigma X \vee \Sigma Y \quad \text{and} \quad j_2\colon \Sigma Y \hookrightarrow \Sigma X \vee \Sigma Y$$

we call

$$k = k(j_1, j_2)\colon \Sigma(X \wedge Y) \to \Sigma X \vee \Sigma Y$$

the *generalized Whitehead map*. By [1, Corollary 4.3], there is a homotopy equivalence $C_k \xrightarrow{\simeq} \Sigma X \times \Sigma Y$ where C_k stands for the mapping cone of the map k.

In the sequel, we need a formula which generalizes one proved by Barcus-Barratt [3, Corollary (7.4)] in case A and B are spheres and by Rutter [15, Theorem 3.5.1] if A and B are suspensions. In fact only A needs to be a co-H-space.

Proposition 1.1 ([4, (3.4) Proposition, p. 51])**.** *Assume A is a co-H-space and B has finite dimension. Then, the Whitehead product of $\Sigma A \xrightarrow{\alpha} Z$ and $\Sigma B \xrightarrow{\gamma} \Sigma Y \xrightarrow{\beta} Z$ satisfies the formula*

$$[\alpha, \beta\gamma] = \sum_{k=1}^{\infty} [[\alpha, \beta^k] \circ (\iota_A \wedge h_k(\gamma))$$

for the James-Hopf invariant $h_k\colon [\Sigma B, \Sigma Y] \to [\Sigma B, (\Sigma Y)^{\wedge k}]$, where the symbol

$$[[\alpha, \beta^k] = [\cdots[[\alpha, \beta], \beta], \ldots, \beta]$$

denotes an iterated Whitehead product.

Remark 1.2.

(i) $h_1(\gamma) = \gamma$ for any $\gamma \in [\Sigma B, \Sigma Y]$;

(ii) if $\gamma \in [\Sigma B, \Sigma Y]$ is a suspension then it is easily seen that $h_k(\gamma) = 0$ for $k \geqslant 2$.

Now, given an Abelian group A and $n \geqslant 2$, write $M(A, n)$ for the *Moore space* of type (A, n). Notice that

$$M(A, n+1) = \Sigma M(A, n) \text{ for } n \geqslant 2,$$

and $M(A, 2) = \Sigma L(A)$ for some space $L(A)$. Certainly, $\pi_n(M(A, n)) = A$ and $M(\mathbb{Z}, n) = \mathbb{S}^n$, the n-sphere, for the group $\mathbb{Z}$ of integers. Furthermore, $M^n = \Sigma^{n-2}\mathbb{R}P^2$ with the real projective plane $\mathbb{R}P^2 = M^2$ and $n \geqslant 3$ is the Moore space of type $(\mathbb{Z}_2, n-1)$. If $\mathbf{m}\colon \mathbb{S}^n \to \mathbb{S}^n$ is a map of degree m then

$$M(\mathbb{Z}_m, n) = \mathbb{S}^n \cup_{\mathbf{m}} e^{n+1}$$

for the cyclic group $\mathbb{Z}_m$ with order m. Denote by $i_{n+1}\colon \mathbb{S}^n \hookrightarrow M(\mathbb{Z}_m, n)$ the canonical inclusion map and by $p_{n+1}\colon M(\mathbb{Z}_m, n) \to \mathbb{S}^{n+1}$ the pinching map.

Proposition 1.3 ([7, Lemma 3.15])**.** *If A and B are torsion Abelian groups whose primary components are indexed by disjoint sets of primes then the inclusion map*

$$M(A,m) \vee M(B,n) \hookrightarrow M(A,m) \times M(B,n)$$

is a homology isomorphism for $m, n \geqslant 2$, and consequently, a homotopy equivalence as well.

Notice that Proposition 1.3 implies that the space $M(A,m) \wedge M(B,n)$ is contractible for $m, n \geqslant 2$ provided that the groups A, B are Abelian torsion groups and their primary components are indexed by disjoint sets of primes. In particular, the space $M(\mathbb{Z}_k, m) \wedge M(\mathbb{Z}_l, n)$ is contractible for $m, n \geqslant 2$ provided k and l are relatively prime.

Given $m \geqslant 1$, recall that the m-th *Gottlieb group* $G_m(X)$ of a path-connected space X has been defined in [8, 9] as the subgroup of the homotopy group $\pi_m(X)$ consisting of all elements which can be represented by a map $f\colon \mathbb{S}^m \to X$ such that

$$f \vee \iota_X \colon \mathbb{S}^m \vee X \to X$$

extends (up to homotopy) to $F\colon \mathbb{S}^m \times X \to X$. Recall that $\alpha \in G_m(\Sigma X)$ if and only if the generalized Whitehead product $[\alpha, \iota_{\Sigma X}] = 0$ (see [1, Proposition 5.1]).

Given the inclusion maps

$$j_1\colon \Sigma X \hookrightarrow \Sigma X \vee \Sigma Y \quad \text{and} \quad j_2\colon \Sigma Y \hookrightarrow \Sigma X \vee \Sigma Y$$

and identifying $\Sigma X \vee \Sigma Y$ with $\Sigma(X \vee Y)$, then, by [2, Proposition 2.2], we have:

Proposition 1.4 ([2, Proposition 2.3])**.** *Let $\alpha \in \pi_n(\Sigma X \vee \Sigma Y)$. Then, $\alpha \in G_n(\Sigma X \vee \Sigma Y)$ if and only if $[\alpha, j_1] = 0 = [\alpha, j_2]$.*

A straightforward extension of the Proposition 1.4 to the wedge

$$T = \Sigma X_1 \vee \cdots \vee \Sigma X_k$$

gives that $\alpha \in G_n(T)$ if and only if $[\alpha, j_s] = 0$, for $s = 1, 2, \ldots, k$, where $j_s \in [\Sigma X_s, T]$ is the inclusion and $\alpha \in \pi_n(T)$.

Let A_p be the p-primary component of an Abelian group A. If follows from [9, Theorems 1-7 and 2-1] and Proposition 1.3 that:

Corollary 1.5.

(i) *Let A and B torsion Abelian groups whose primary components are indexed by disjoint set of primes. Then,*

$$G_k(M(A,m) \vee M(B,n)) = j_{1*}G_k(M(A,m)) \oplus j_{2*}G_k(M(B,n)),$$

for $k \geqslant 1$, $m,n \geqslant 2$. In particular, if the 2-primary component A_2 is trivial one finds that

$$G_k(M(A,m) \vee M^{n+1}) = j_{1*}G_k(M(A,m)) \oplus j_{2*}G_k(M^{n+1}),$$

for $k \geqslant 1$, $m,n \geqslant 2$.

(ii) *If $A = \bigoplus_{i=1}^{t} A_{p_i}$ is the primary decomposition of the finite Abelian group A, then*

$$G_k(M(A,m)) = \bigoplus_{i=1}^{t} j_{i*}G_k(M(A_{p_i},m)),$$

for $k \geqslant 1$, $m \geqslant 2$.

Since inclusions and projections $X_k \overset{j_k}{\hookrightarrow} X_1 \vee X_2 \xrightarrow{q_l} X_l$ satisfy

$$q_l j_k = \begin{cases} *, & \text{if } k \neq l, \\ \iota_{X_k}, & \text{if } k = l, \end{cases}$$

for $k,l = 1,2$, we can state:

Proposition 1.6. *Suppose*

$$\pi_m(X_1 \vee X_2) = j_{1*}\pi_m(X_1) \oplus j_{2*}\pi_m(X_2),$$

for some $m \geqslant 1$. Then,

$$G_m(X_1 \vee X_2) \subseteq j_{1*}G_m(X_1) \oplus j_{2*}G_m(X_2).$$

Proof. By [9, Proposition 1-4], we have

$$q_{k*}\colon G_m(X_1 \vee X_2) \to G_m(X_k).$$

If $\alpha \in G_m(X_1 \vee X_2)$, then $\alpha = j_{1*}(\alpha_1) + j_{2*}(\alpha_2)$, where $\alpha_k \in \pi_m(X_k)$ and so, $\alpha_k = q_{k*}(\alpha)$ with $\alpha_k \in G_m(X_k)$, for $k = 1,2$. This implies that

$$G_m(X_1 \vee X_2) \subseteq j_{1*}G_m(X_1) \oplus j_{2*}G_m(X_2) \approx G_m(X_1 \times X_2). \qquad \square$$

Remark 1.7. If the spaces X and Y are $(m-1)$- and $(n-1)$-connected, respectively, then $\pi_{k+1}(X \times Y, X \vee Y) = 0$ for $k+1 < m+n$. This implies that the inclusion map $X \vee Y \hookrightarrow X \times Y$ is an $(m+n-1)$-equivalence. Consequently, the inclusion map

$$M(A,m) \vee M(B,n) \hookrightarrow M(A,m) \times M(B,n)$$

implies an isomorphism

$$\pi_k(M(A,m) \vee M(B,n)) \xrightarrow{\approx} \pi_k(M(A,m)) \oplus \pi_k(M(B,n))$$

for $k < m+n-1$. In particular,

$$M(A \oplus B, n) = M(A,n) \vee M(B,n)$$

yields an isomorphism

$$\pi_k(M(A \oplus B, n)) \xrightarrow{\approx} \pi_k(M(A, n)) \oplus \pi_k(M(B, n))$$

for $k < 2n - 1$.

Let A and B be Abelian groups. By Proposition 1.6 and Remark 1.7 we get:

Corollary 1.8. *If $k < m + n - 1$, then*

$$G_k(M(A, m) \vee M(B, n)) \subseteq j_{1*}G_k(M(A, n)) \oplus j_{2*}G_k(M(B, m)),$$

for $k \geqslant 1$ and $m, n \geqslant 2$.

2. Gottlieb groups of wedge of spheres

In view of [2, Corollary 3.6], $G_N(\mathbb{S}^m \vee \mathbb{S}^n) = 0$ with $2 \leqslant m \leqslant n$ provided $N < 2m - 1$. In this section, we first compute $G_N\big(\bigvee_{t\in T} \mathbb{S}^{n_t}\big)$ with $|T| \geqslant 2$ and $n_t \geqslant 2$ for $t \in T$ provided $N < 2\min\{n_t\}_{t\in T} - 1$. Then, we make use of [8] and [9] to study $G_N\big(\bigvee_{s\in S} \mathbb{S}^1 \vee \bigvee_{t\in T} \mathbb{S}^{n_t}\big)$ with $|S| \geqslant 1$, $|T| \geqslant 1$ and $n_t \geqslant 2$ for $t \in T$ provided $N < 2\min\{n_t\} - 1$.

First recall that [2, Proposition 2.4] states:

Proposition 2.1. *Let $\alpha \in \pi_n(\Sigma X_1 \vee \cdots \vee \Sigma X_k)$. Then,*

$$\alpha \in G_n(\Sigma X_1 \vee \cdots \vee \Sigma X_n)$$

if and only if $[\alpha, j_i] = 0$ for the canonical inclusions

$$j_i \colon \Sigma X_k \hookrightarrow \Sigma X_1 \vee \cdots \vee \Sigma X_n$$

with $i = 1, \ldots, k$.

Furthermore, from [2, Theorem 3.4] it is derived:

Proposition 2.2. *Suppose that $2 \leqslant m, n$ and $N < 2\min\{m, n\} - 1$, then*

$$G_N(\mathbb{S}^m \vee \mathbb{S}^n) = 0.$$

Applying Proposition 2.1, we get a generalized version of [2, Theorem 3.4] which yields:

Proposition 2.3. *Suppose that $k \geqslant 2$ and $n_i \geqslant 2$ for $i = 1, \ldots, k$. If $N < 2 \min_{1\leqslant i\leqslant k-1}\{n_i\}$ then*

$$G_N(\mathbb{S}^{n_1} \vee \cdots \vee \mathbb{S}^{n_k}) = 0.$$

Then, we derive a generalization of Proposition 2.2:

Corollary 2.4. *Let T be a set with $|T| \geqslant 2$ and $n_t \geqslant 2$ for $t \in T$. If $N < 2\min\limits_{t\in T}\{n_t\} - 1$, then*

$$G_N\big(\bigvee_{t\in T}\mathbb{S}^{n_t}\big) = 0.$$

Proof. Let $f\colon \mathbb{S}^N \to \bigvee\limits_{t\in T}\mathbb{S}^{n_t}$. Since the image $f(\mathbb{S}^N)$ is compact, there exist $k \geqslant 1$, $t_i \in T$ with $i = 1,\ldots,k$ and a map

$$f'\colon \mathbb{S}^N \xrightarrow{f'} \bigvee_{i=1}^{k}\mathbb{S}^{n_{t_i}}$$

such that $j \circ f' = f$ for the canonical inclusion $j\colon \bigvee\limits_{i=1}^{k}\mathbb{S}^{n_{t_i}} \hookrightarrow \bigvee\limits_{t\in T}\mathbb{S}^{n_t}$.

Certainly, we may assume that $k \geqslant 2$. But, $p \circ j = \iota_{\bigvee\limits_{k=1}^{n}\mathbb{S}^{n_{t_k}}}$ for the projection map $p\colon \bigvee\limits_{t\in T}\mathbb{S}^{n_t} \to \bigvee\limits_{i=1}^{n}\mathbb{S}^{n_{t_i}}$, therefore [9, Proposition 1-4.] implies that the map

$$p_*\colon \pi_N\Big(\bigvee_{t\in T}\mathbb{S}^{n_t}\Big) \to \pi_N\Big(\bigvee_{i=1}^{n}\mathbb{S}^{n_{t_i}}\Big)$$

carries $G_N\big(\bigvee\limits_{t\in T}\mathbb{S}^{n_t}\big)$ into $G_N\big(\bigvee\limits_{i=1}^{n}\mathbb{S}^{n_{t_i}}\big)$. Consequently, in view of Proposition 2.3, we get $p_*(f) = f' = 0$ and the proof is complete. □

Next, we need:

Theorem 2.5 ([8, Theorem IV.1]). *Suppose X has the same homotopy type as a compact, connected polyhedron. Then, $G_1(X) = 0$ if the Euler-Poincaré number $\chi(X) \neq 0$.*

Since $\chi(\mathbb{S}^1 \vee \mathbb{S}^n) = (-1)^n$ for $n \geqslant 1$ and $\chi(\bigvee\limits_{i=1}^{k}\mathbb{S}^1) = 1-k$, we deduce from Theorem 2.5 that

$$G_1(\mathbb{S}^1 \vee \mathbb{S}^n) = 0 \tag{2.1}$$

for $n \geqslant 1$ and

$$G_1\Big(\bigvee_{i=1}^{k}\mathbb{S}^1\Big) = 0 \tag{2.2}$$

for $k \geqslant 2$.

Furthermore, applying (2.2) and [9, Proposition 1-4.] as in the proof of Corollary 2.4, we get:

$$G_1\Big(\bigvee_{t\in T}\mathbb{S}^1\Big) = 0 \tag{2.3}$$

for $|T| \geqslant 2$.

Then, we can state:

Proposition 2.6. *If S, T are sets with $|S|, |T| \geqslant 1$ and $n_t \geqslant 2$ for $t \in T$ then*

$$G_1\Big(\bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}\Big) = 0.$$

Proof. First, notice that the case $|S| = |T| = 1$ follows from (2.1). Next, applying the Seifert-van Kampen Theorem, we get:

$$\pi_1\big(\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}\big) = j_{1*}\pi_1\big(\mathbb{S}^1 \vee \mathbb{S}^{n_{t_0}}\big) * j_{2*}\pi_1\big(\bigvee_{t\in T\setminus\{t_0\}}\mathbb{S}^{n_t}\big)$$
$$= j_{1*}\pi_1(\mathbb{S}^1 \vee \mathbb{S}^{n_{t_0}})$$

with $|T| \geqslant 2$ for the canonical inclusions

$$j_1\colon \mathbb{S}^1 \vee \mathbb{S}^{n_{t_0}} \hookrightarrow \mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}, \qquad j_2\colon \bigvee_{t\in T\setminus\{t_0\}}\mathbb{S}^{n_t} \hookrightarrow \mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}$$

and

$$\pi_1\Big(\bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}\Big) = j_{1*}\pi_1\Big(\bigvee_{s\in S}\mathbb{S}^1\Big) * j_{2*}\pi_1\Big(\bigvee_{t\in T}\mathbb{S}^{n_t}\Big) = j_{1*}\pi_1\Big(\bigvee_{s\in S}\mathbb{S}^1\Big)$$

with $|S| \geqslant 2$ and $|T| \geqslant 1$ for the canonical inclusions

$$j_1\colon \bigvee_{s\in S}\mathbb{S}^1 \hookrightarrow \bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}, \qquad j_2\colon \bigvee_{t\in T}\mathbb{S}^{n_t} \hookrightarrow \bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}.$$

Then, Proposition 1.6 and equations (2.1), (2.3) yield that

$$G_1\Big(\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}\Big) \subseteq G_1(\mathbb{S}^1 \vee \mathbb{S}^{n_{t_0}}) = 0$$

with $|T| \geqslant 2$ and

$$G_1\Big(\bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}\Big) \subseteq G_1\Big(\bigvee_{s\in S}\mathbb{S}^1\Big) = 0$$

with $|S| \geqslant 2$ and $|T| \geqslant 1$ and the proof follows. □

Next, recall:

Theorem 2.7 ([9, Theorem 6-2]). *Let $p\colon \tilde{X} \to X$ be a covering map. If $n \geqslant 1$ then $p_*^{-1}(G_n(X) \subseteq G_n(\tilde{X}))$. In other words, if we identify $\pi_n(\tilde{X})$ with $\pi_n(X)$ under the isomorphism p_* for $n \geqslant 2$ then $G_n(\tilde{X}) \supseteq G_n(X)$.*

Now, consider the space

$$X = \bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}$$

with $|S|, |T| \geqslant 1$ and $m_t \geqslant 2$ for $t \in T$. Then, by the Seifert-van Kampen Theorem, we have that $\pi_1(X) = \pi_1\big(\bigvee_{s\in S}\mathbb{S}^1\big)$ and the space

$$\tilde{X} = \bigvee_{g\in\pi_1(X)}\Big(\bigvee_{t\in T}\mathbb{S}^{n_t}\Big)$$

is the universal covering of X.

Therefore, Corollary 2.4, Proposition 2.6 and Theorem 2.7 yield:

Proposition 2.8. *Let* $|S|, |T| \geqslant 1$ *and* $n_t \geqslant 2$ *with* $t \in T$. *If*

$$N < 2\min\{n_t\}_{t\in T} - 1$$

then $G_N\big(\bigvee_{s\in S}\mathbb{S}^1 \vee \bigvee_{t\in T}\mathbb{S}^{n_t}\big) = 0$.

3. Computations

In this section we compute Gottlieb groups $G_{n+k}(M(A,n))$ with $k = 1,2$ of Moore spaces $M(A,n)$ for some finitely generated Abelian groups A. First, we recall from [2]:

Theorem 3.1. *Let A be any finitely generated Abelian group and $n \geqslant 3$. Then,*

$$G_n(M(A,n)) = \begin{cases} 0, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd and } \operatorname{rk}(A) \neq 1,\\ 2\mathbb{Z} \subseteq \mathbb{Z} = \pi_n(\mathbb{S}^n), & \text{if } n \neq 3,7 \text{ is odd and } A = \mathbb{Z},\\ \mathbb{Z} = \pi_n(\mathbb{S}^n), & \text{if } n = 3,7 \text{ and } A = \mathbb{Z}. \end{cases}$$

In addition, by [2, Corollary 4.4] if n is odd, then $G_n(M(\mathbb{Z}\oplus A,n))$ is infinite cyclic, where A is a finite Abelian group.

We work in the sequel with a finitely generated Abelian group A whose torsion subgroup has order $|A| \equiv 2 \pmod 4$. Notice that in [7, Chapter 3] there are some results on $G_{n+1}(M(A,n))$ only for A having an odd order torsion subgroup.

Groups $G_{n+1}(M(A,n))$. First, we observe that Proposition 2.3 leads to the following:

Proposition 3.2. *If $n \geqslant 2$ and $m \geqslant 2$ then $G_{n+k}(M(\mathbb{Z}^m,n)) = 0$ for $k < n-1$.*

Because $M^n = M(\mathbb{Z}_2, n-1)$, by [7, Corollary 3.11], we have:

Proposition 3.3. $G_{n+1}(M(\mathbb{Z}_2,n)) = 0$, *for* $n \geqslant 3$.

Let

$$\eta_n = \Sigma^{n-2}\eta_2\colon \mathbb{S}^{n+1} \to \mathbb{S}^n$$

be the $(n-2)$-suspension of the Hopf map $\eta_2\colon \mathbb{S}^3 \to \mathbb{S}^2$. Now, using both propositions above, we show:

Proposition 3.4. *Let $n \geqslant 3$. Then:*

(i) $G_{n+1}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$, *for* $m \geqslant 2$;

(ii) $G_{n+1}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) \subseteq$

$$\subseteq j_{1*}(G_{n+1}(\mathbb{S}^n)) = \begin{cases} \mathbb{Z}_2\{j_1\eta_n\}, & \text{if } n = 6 \text{ or } n \equiv 3 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) Since

$$G_{n+1}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = G_{n+1}(M(\mathbb{Z}^m, n) \vee M(\mathbb{Z}_2, n)),$$

by Corollary 1.8 and Propositions 3.2 and 3.3, statement (i) follows.

(ii) First, recall from [7, (1.15)] that

$$\sharp[\iota_n, \eta_n] = \begin{cases} 1, & \text{for } n = 2, 6 \text{ or } n \equiv 3 \pmod 4, \\ 2, & \text{otherwise.} \end{cases}$$

Now, applying Corollary 1.8 and Proposition 3.3, the proof is complete. □

Remark 3.5. If A is a finite group with order $|A| \equiv 2 \pmod 4$ then the 2-primary component A_2 has order two.

Proposition 3.6. *Let A be a finite Abelian group with $|A| \equiv 2 \pmod 4$. For $n \geqslant 3$, it holds:*

(i) $G_{n+1}(M(A, n)) = 0$;

(ii) $G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = 0$ *for* $m \geqslant 2$;

(iii) $G_{n+1}(M(\mathbb{Z} \oplus A, n)) \subseteq$

$$\subseteq j_{1*}(G_{n+1}(\mathbb{S}^n)) = \begin{cases} \mathbb{Z}_2\{j_1\eta_n\}, & \text{if } n = 6 \text{ or } n \equiv 3 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (i) By Corollary 1.5 we can write

$$G_{n+1}(M(A, n)) = \bigoplus_{i=1}^{t} j_{i*}G_{n+1}(M(A_{p_i}, n))$$

for $A = \bigoplus_{i=1}^{t} A_{p_i}$. Furthermore, by [7, Proposition 3.19] we know that $G_{n+1}(M(A_{p_i}, n)) = 0$ for $p_i > 2$, and by Remark 3.5 we know that $A_2 \approx \mathbb{Z}_2$ if $|A| \equiv 2 \pmod 4$. Consequently,

$$G_{n+1}(M(A, n)) = j_{2*}G_{n+1}(M(A_2, n)) \approx G_{n+1}(M(\mathbb{Z}_2, n)) = 0,$$

where the last equality is due to Proposition 3.3.

(ii) We simply write

$$G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = G_{n+1}(M(\mathbb{Z}^m, n) \vee M(A, n)).$$

Then, by Corollary 1.8, Proposition 3.2 and previous item, the result follows.

(iii) It is similar to the proof of Proposition 3.4(ii). □

Proposition 3.4(ii) gives only two possibilities for $G_{n+1}(M(\mathbb{Z}\oplus A,n))$. In the sequel, we show that it is trivial, but first, we pose some needed results.

Write $\overline{\eta}_n\colon M^{n+2}\to\mathbb{S}^n$ for the *extension* of η_n with $n\geqslant 3$. It is the unique (up to homotopy) element satisfying $\overline{\eta}_n i_{n+2}=\eta_n$. Then,

$$\Sigma\overline{\eta}_n=\overline{\eta}_{n+1}\quad\text{and}\quad 2\overline{\eta}_n=\eta_n^2 p_{n+2},\quad (n\geqslant 3).$$

Dually, let $\tilde{\eta}_n\colon\mathbb{S}^{n+2}\to M^{n+1}$ be the *coextension* of η_n for $n\geqslant 2$. It is the unique (up to homotopy) element satisfying $p_{n+1}\tilde{\eta}_n=\eta_{n+1}$. Then,

$$\Sigma\tilde{\eta}_n=\tilde{\eta}_{n+1}\quad\text{and}\quad 2\tilde{\eta}_n=i_{n+1}\eta_n^2,\quad (n\geqslant 2). \tag{3.1}$$

By [7, Lemma 1.26], $\pi_{n+1}(M^n)=\mathbb{Z}_4\{\tilde{\eta}_{n-1}\}$ and $[M^{n+2},\mathbb{S}^n]=\mathbb{Z}_4\{\overline{\eta}_n\}$ for $n\geqslant 3$. Furthermore, from [12, Lemma 1.5] we have

$$[M^{n+1},M^n]=\mathbb{Z}_2\{i_n\overline{\eta}_{n-1}\}\oplus\mathbb{Z}_2\{\tilde{\eta}_{n-1}p_{n+1}\},\quad (n\geqslant 4). \tag{3.2}$$

Recall that

$$G_{n+1}(M(\mathbb{Z}\oplus\mathbb{Z}_2,n))\subseteq j_{1*}(G_{n+1}(\mathbb{S}^n)).$$

If $j_{1*}(G_{n+1}(\mathbb{S}^n))$ is non-trivial, we have that $j_1\eta_n\in G_{n+1}(M(\mathbb{Z}\oplus\mathbb{Z}_2,n))$ if and only if $[j_1\eta_n,j_1]=0=[j_1\eta_n,j_2]$, for the inclusions

$$j_1\colon\mathbb{S}^n\hookrightarrow M(\mathbb{Z}\oplus\mathbb{Z}_2,n)\quad\text{and}\quad j_2\colon M^{n+1}\hookrightarrow M(\mathbb{Z}\oplus\mathbb{Z}_2,n).$$

Now, $[j_1\eta_n,j_1]=j_1[\eta_n,\iota_n]=0$, since $G_{n+1}(\mathbb{S}^n)=\mathbb{Z}_2\{\eta_n\}$ and consequently $j_1\eta_n\in G_{n+1}(M(\mathbb{Z}\oplus\mathbb{Z}_2,n))$ if and only if $[j_1\eta_n,j_2]=0$.

To compute $[j_1\eta_n,j_2]$, we notice that by Remark 1.2(ii) or degree reasons $h_k(\eta_n)=0$ for $k\geqslant 2$. Then, we make use of Proposition 1.1 and obtain:

$$\begin{aligned}[j_2,j_1\eta_n]&=[j_2,j_1]\circ(\iota_{M(\mathbb{Z}_2,n-1)}\wedge\eta_n)+\sum_{k=2}^{\infty}[[j_2,j_1^k]\circ(\iota_{M(\mathbb{Z}_2,n-1)}\wedge h_k(\eta_n))\\&=[j_2,j_1]\circ(\iota_{M(\mathbb{Z}_2,n-1)}\wedge\eta_n)\end{aligned} \tag{3.3}$$

for $n\geqslant 3$, where h_k is the James-Hopf invariant and the symbol

$$[[j_2,j_1^k]=[\cdots[[j_2,j_1],j_1],\ldots,j_1]$$

denotes the iterated Whitehead product.

Now, we show a formula useful in the sequel, found in the proof of [13, Lemma 3.1]:

Lemma 3.7. *We have that*

$$\eta_2\wedge\iota_{M^2}=i_4\overline{\eta}_3+\tilde{\eta}_3p_5. \tag{3.4}$$

Proof. The cofibration $\mathbb{S}^4 \xrightarrow{i_5} M^5 \xrightarrow{p_5} \mathbb{S}^5$ yields the exact sequence

$$\cdots \to \pi_5(M^4) = \mathbb{Z}_4\{\tilde{\eta}_3\} \xrightarrow{p_5^*} [M^5, M^4] \xrightarrow{i_5^*} \pi_4(M^4) = \mathbb{Z}_2\{i_4\eta_3\} \to 0.$$

But,

$$\eta_2 \wedge \iota_{M^2} \in [M^5, M^4] = \mathbb{Z}_2\{i_4\bar{\eta}_3\} \oplus \mathbb{Z}_2\{\tilde{\eta}_3 p_5\}$$

and:

$$\eta_2 \wedge i_2 = \iota_2\eta_2 \wedge i_2\iota_1 = (\iota_2 \wedge i_2) \circ (\eta_2 \wedge \iota_1) = i_4\eta_3 = (i_4\bar{\eta}_3) \circ i_5,$$

$$\eta_2 \wedge i_2 = \eta_2\iota_3 \wedge \iota_{M^2} i_2 = (\eta_2 \wedge \iota_{M^2}) \circ (\iota_3 \wedge i_2) = (\eta_2 \wedge \iota_{M^2}) \circ i_5.$$

Hence, $i_5^*(\eta_2 \wedge \iota_{M^2} - i_4\bar{\eta}_3) = 0$ and the exact sequence above implies

$$\eta_2 \wedge \iota_{M^2} = i_4\bar{\eta}_3 + x\tilde{\eta}_3 p_5 \tag{3.5}$$

with $x = 0, 1$.

Next, consider the exact sequence

$$0 \to \pi_5(\mathbb{S}^4) = \mathbb{Z}_2\{\eta_4\} \xrightarrow{p_5^*} [M^5, \mathbb{S}^4] \xrightarrow{i_5^*} \pi_4(\mathbb{S}^4) = \mathbb{Z}\{\iota_n\} \to \cdots$$

determined by the cofibration $\mathbb{S}^4 \xrightarrow{i_5} M^5 \xrightarrow{p_5} \mathbb{S}^5$. Then, $\eta_2 \wedge p_2 \in [M^5, \mathbb{S}^4]$ and $p_5^*(\eta_4) = \eta_4 p_5 \neq 0$.

Furthermore:

$$\eta_2 \wedge p_2 = \eta_2\iota_3 \wedge \iota_2 p_2 = (\iota_2 \wedge \eta_2) \circ (\iota_3 \wedge p_2) = \eta_4 p_5 \neq 0,$$

$$\eta_2 \wedge p_2 = \iota_2\eta_2 \wedge p_2\iota_{M^2} = (\iota_2 \wedge p_2) \circ (\eta_2 \wedge \iota_{M^2}) = p_4(\eta_2 \wedge \iota_{M^2}) \neq 0.$$

Consequently, (3.5) yields

$$p_4(\eta_2 \wedge \iota_{M^2}) = p_4 i_4\bar{\eta}_3 + x p_4\tilde{\eta}_3 p_5 = x p_4\tilde{\eta}_3 p_5.$$

Since, $p_4(\eta_2 \wedge \iota_{M^2}) \neq 0$, we derive that $x = 1$ and the proof is complete. $\square$

Then, (3.2) and (3.4) lead to the following:

Corollary 3.8. $\Sigma^{n-2}(\eta_2 \wedge \iota_{M^2}) = i_{n+2}\overline{\eta}_{n+1} + \tilde{\eta}_{n+1}p_{n+3}) \neq 0$ *for* $n \geqslant 2$.

Now, we are in a position to state the main result of this section.

Theorem 3.9. *Let* $n \geqslant 3$. *Then* $G_{n+1}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) = 0$.

Proof. Writing

$$\eta_n \wedge \iota_{M(\mathbb{Z}_2, n-1)} = \Sigma^{n-2}\eta_2 \wedge \Sigma^{n-2}\iota_{M^2} = \Sigma^{2(n-2)}(\eta_2 \wedge \iota_{M^2})$$

we see, by Corollary 3.8, that

$$\eta_n \wedge \iota_{M(\mathbb{Z}_2, n-1)} \neq 0 \quad \text{for} \quad n \geqslant 2.$$

Further, by (3.3), it follows that $[j_2, j_1\eta_n] \neq 0$, for $n \geqslant 3$, since $[j_2, j_1]$ is a basic product. Then for $n \geqslant 3$

$$j_1\eta_n \notin G_{n+1}(M(\mathbb{Z}\oplus\mathbb{Z}_2, n)) \quad \text{and} \quad G_{n+1}(M(\mathbb{Z}\oplus\mathbb{Z}_2, n)) = 0. \qquad \square$$

Corollary 3.10. *Let A be a finite Abelian group with $|A| \equiv 2 \pmod 4$. Then $G_{n+1}(M(\mathbb{Z}\oplus A, n)) = 0$ for $n \geqslant 3$.*

Proof. Since A is a finite Abelian group with order $|A| \equiv 2 \pmod 4$, we apply [6, Chapter III, Theorem 15.2] to write $A = \bigoplus_{i=1}^{t} A_{p_i}$ for the primary decomposition of A with $p_1 = 2$. By Corollaries 1.5 and 1.8 we obtain

$$G_{n+1}(M(\mathbb{Z}\oplus A, n)) \subseteq$$
$$\subseteq j_{1*}G_{n+1}(M(\mathbb{Z}\oplus A_2, n)) \oplus \bigoplus_{i=2}^{t} j_{i*}G_{n+1}(M(A_{p_i}, n)).$$

As $G_{n+1}(M(A_{p_i}, n)) = 0$ for $p_i > 2$ (see [7, Proposition 3.19]) and $A_2 \approx \mathbb{Z}_2$ (cf. Remark 3.5), we get

$$G_{n+1}(M(\mathbb{Z}\oplus A, n)) \subseteq j_{1*}G_{n+1}(M(\mathbb{Z}\oplus A_2, n)) \approx$$
$$\approx G_{n+1}(M(\mathbb{Z}\oplus\mathbb{Z}_2, n)) = 0. \qquad \square$$

Recall that, by [6, Chapter III, Theorem 15.2], for any finitely generated Abelian group B there is an isomorphism $B \approx \mathbb{Z}^m \oplus A$ with $m \geqslant 0$ and a finite Abelian group A. Furthermore, by means of the universal coefficient theorem for homotopy [10, p. 30], for any Abelian group A, a pointed space X and $n \geqslant 2$, there exists a short exact sequence

$$0 \to \operatorname{Ext}(A, \pi_{n+1}(X)) \to [M(A, n), X] \to \operatorname{Hom}(A, \pi_n(X)) \to 0. \qquad (3.6)$$

Notice that by equation (3.6) the order $\sharp j_2$ of the canonical inclusion map

$$j_2 \colon M(A, n) \hookrightarrow M(\mathbb{Z}\oplus A, n)$$

is finite. If A has odd order so does $\sharp j_2$, and we conclude that

$$G_{n+1}(M(\mathbb{Z}\oplus A, n)) = j_{1*}G_{n+1}(\mathbb{S}^n),\ n \geqslant 3,$$

since $[j_2, j_1\eta_n] = 0$. Finally, we summarize the groups $G_{n+1}(M(\mathbb{Z}^m\oplus A, n))$ for a finite Abelian group A as follows:

(a) $G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = 0$ for $m \geqslant 0$, $m \neq 1$, $n \geqslant 3$ provided $|A|$ is odd;

(b) $G_{n+1}(M(\mathbb{Z}\oplus A, n)) = j_{1*}G_{n+1}(\mathbb{S}^n)$, for $n \geqslant 3$ provided $|A|$ is odd;

(c) $G_{n+1}(M(\mathbb{Z}^m\oplus A, n)) = 0$, for $m \geqslant 0$, $n \geqslant 3$ provided $|A| \equiv 2 \pmod 4$.

Groups $G_{n+2}(M(A,n))$. We compute $G_{n+2}(M(A,n))$ for $n \geqslant 4$, under the same conditions for A as before. First, we recall from [7, Corollary 3.11] that $G_{n+2}(M(\mathbb{Z}_2,n)) = 0$ for $n \geqslant 3$. Further, by [7, (1.16)],

$$\sharp[\iota_n, \eta_n^2] = \begin{cases} 1, & \text{if } n \equiv 2, 3 \pmod 4, \\ 2, & \text{otherwise.} \end{cases}$$

Analogous to Propositions 3.4 and 3.6, we can state the following two results:

Proposition 3.11. *Let $n \geqslant 4$. Then:*

(i) $G_{n+2}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$, *for* $m \geqslant 2$.

(ii) $G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) \subseteq$

$$\subseteq j_{1*}(G_{n+2}(\mathbb{S}^n)) = \begin{cases} \mathbb{Z}_2\{j_1\eta_n^2\}, & \textit{if } n \equiv 2, 3 \pmod 4, \\ 0, & \textit{otherwise.} \end{cases}$$

Proposition 3.12. *Let A be a finite Abelian group with $|A| \equiv 2 \pmod 4$. For $n \geqslant 4$, it holds:*

(i) $G_{n+2}(M(A,n)) = 0$.

(ii) $G_{n+2}(M(\mathbb{Z}^m \oplus A, n)) = 0$ *for* $m \geqslant 2$.

(iii) $G_{n+2}(M(\mathbb{Z} \oplus A, n)) \subseteq$

$$\subseteq j_{1*}(G_{n+2}(\mathbb{S}^n)) = \begin{cases} \mathbb{Z}_2\{j_1\eta_n^2\}, & \textit{if } n \equiv 2, 3 \pmod 4, \\ 0, & \textit{otherwise.} \end{cases}$$

Next, we show that the Gottlieb group from Proposition 3.11(ii) is trivial using the same technics as in Proposition 3.4(ii). First, we recall from [11, Lemma 3.7] and [14, Lemma 2.1]:

Lemma 3.13.

(i) $[M^5, M^3] = \mathbb{Z}_2\{i_3\eta_2\overline{\eta}_3\} \oplus \mathbb{Z}_2\{\tilde{\eta}_2\eta_4p_5\} \oplus \mathbb{Z}_2\{\tau p_5\} \oplus \mathbb{Z}_2\{\overline{\Sigma\lambda}_2\tilde{\eta}_3p_5\}$;

(ii) $[M^6, M^4] = \mathbb{Z}_2\{i_4\eta_3\overline{\eta}_4\} \oplus \mathbb{Z}_2\{\tilde{\eta}_3\eta_5p_6\} \oplus \mathbb{Z}_2\{\lambda_2p_6\}$;

(iii) $[M^7, M^5] = \mathbb{Z}_2\{i_5\eta_4\overline{\eta}_5\} \oplus \mathbb{Z}_2\{\tilde{\eta}_4\eta_6p_7\} \oplus \mathbb{Z}_2\{i_5\nu_4p_7\}$.

Lemma 3.14. *For $n \geqslant 6$ we have that*

$$[M^{n+2}, M^n] = \mathbb{Z}_2\{i_n\eta_{n-1}\overline{\eta}_n\} \oplus \mathbb{Z}_2\{\tilde{\eta}_{n-1}\eta_{n+1}p_{n+2}\} \oplus \mathbb{Z}_2\{i_n\nu_{n-1}p_{n+2}\},$$

Proof. The cofibration

$$\mathbb{S}^{n+1} \xrightarrow{i_{n+2}} M^{n+2} \xrightarrow{p_{n+2}} \mathbb{S}^{n+2}$$

leads to the short exact sequence of homotopy groups

$$0 \to \pi_{n+2}(M^n) \xrightarrow{p^*_{n+2}} [M^{n+2}, M^n] \xrightarrow{i^*_{n+2}} 2\pi_{n+1}(M^n) \to 0,$$

for $n \geqslant 6$. By [7, Lemmas 1.26 and 1.27], we know that

$$\pi_{n+1}(M^n) = \mathbb{Z}_4\{\tilde{\eta}_{n-1}\}$$

and

$$\pi_{n+2}(M^n) = \mathbb{Z}_2\{\tilde{\eta}_{n-1}\eta_{n+1}\} \oplus \mathbb{Z}_2\{i_n\nu_{n-1}\}.$$

The sequence above is therefore

$$0 \to \mathbb{Z}_2\{\tilde{\eta}_{n-1}\eta_{n+1}\} \oplus \mathbb{Z}_2\{i_n\nu_{n-1}\} \xrightarrow{p^*_{n+2}} [M^{n+2}, M^n] \xrightarrow{i^*_{n+2}} \mathbb{Z}_2\{2\tilde{\eta}_{n-1}\} \to 0,$$

for $n \geqslant 6$, which splits by means of

$$\theta\colon \mathbb{Z}_2\{2\tilde{\eta}_{n-1}\} \to [M^{n+2}, M^n], \qquad \theta(2\tilde{\eta}_{n-1}) = i_n\eta_{n-1}\overline{\eta}_n.$$

Thus,

$$(i^*_{n+2}\theta)(2\tilde{\eta}_{n-1}) = i_n\eta_{n-1}\overline{\eta}_n i_{n+2}.$$

From (3.1) we conclude that

$$(i^*_{n+2}\theta)(2\tilde{\eta}_{n-1}) = 2\tilde{\eta}_{n-1},$$

that is, $i^*_{n+2}\theta$ is the identity homomorphism and the proof follows. □

Remark 3.15. It follows from Lemmas 3.13 and 3.14 that the suspension map

$$\Sigma_*\colon [M^{n+2}, M^n] \to [M^{n+3}, M^{n+1}]$$

is an isomorphism, for $n \geqslant 5$.

From [16, Proposition 5.6] we know that, the 2-primary component of $\pi_6(\mathbb{S}^3)$ is $\pi^3_6 = \mathbb{Z}_4\{\nu'\}$ and $\pi^4_7 = \mathbb{Z}\{\nu_4\} \oplus \mathbb{Z}_4\{\Sigma\nu'\}$.

By [14, (2.1) and (2.2)], the following relations hold:

$$\begin{aligned} &\pi_6(M^4) = \mathbb{Z}_4\{\lambda_2\} \oplus \mathbb{Z}_4\{\tilde{\eta}_3\eta_5\}, && 2\lambda_2 = i_4\nu', \\ &\pi_7(M^5) = \mathbb{Z}_4\{i_5\nu_4\} \oplus \mathbb{Z}_4\{\tilde{\eta}_4\eta_6\}, && \Sigma\lambda_2 = 2(i_5\nu_4) \in \pi_7(M^5), \\ &[\iota_{M^4}, i_4] = \lambda_2 p_6, && \pm\nu' = \overline{\eta}_3\tilde{\eta}_4. \end{aligned} \tag{3.7}$$

Next, recall from (3.4) that $\eta_2 \wedge \iota_{M^2} = i_4\overline{\eta}_3 + \tilde{\eta}_3 p_5$. Then, for $\eta_2^2 \wedge \iota_{M^2}$, we have the following:

Lemma 3.16. $\eta_2^2 \wedge \iota_{M^2} = i_4\eta_3\overline{\eta}_4 + \tilde{\eta}_3\eta_5 p_6$.

Proof. In view of (3.4), we have

$$\begin{aligned} \eta_2^2 \wedge \iota_{M^2} &= (\eta_2\eta_3) \wedge \iota_{M^2} \\ &= (\eta_2 \wedge \iota_{M^2})(\eta_3 \wedge \iota_{M^2}) \end{aligned}$$

$$\begin{aligned}
&= (\eta_2 \wedge \iota_{M^2})(\Sigma(\eta_2 \wedge \iota_{M^2}))\\
&= (i_4\overline{\eta}_3 + \tilde{\eta}_3 p_5)(\Sigma(\eta_2 \wedge \iota_{M^2})\\
&= (i_4\overline{\eta}_3)(\Sigma(\eta_2 \wedge \iota_{M^2})) + (\tilde{\eta}_3 p_5)(\Sigma(\eta_2 \wedge \iota_{M^2}))\\
&= (i_4\overline{\eta}_3)(i_5\overline{\eta}_4 + \tilde{\eta}_4 p_6) + (\tilde{\eta}_3 p_5)(i_5\overline{\eta}_4 + \tilde{\eta}_4 p_6)\\
&= i_4\overline{\eta}_3 i_5\overline{\eta}_4 + i_4\overline{\eta}_3\tilde{\eta}_4 p_6 + \tilde{\eta}_3 p_5 i_5\overline{\eta}_4 + \tilde{\eta}_3 p_5\tilde{\eta}_4 p_6\\
&= i_4\overline{\eta}_3 i_5\overline{\eta}_4 + i_4\overline{\eta}_3\tilde{\eta}_4 p_6 + \tilde{\eta}_3 p_5\tilde{\eta}_4 p_6
\end{aligned}$$

since $p_5 i_5 = 0$.

Also, due to relations (3.7),

$$\eta_2^2 \wedge \iota_{M^2} = i_4\eta_3\overline{\eta}_4 + i_4(\pm\nu')p_6 + \tilde{\eta}_3\eta_5 p_6 = i_4\eta_3\overline{\eta}_4 \pm i_4\nu' p_6 + \tilde{\eta}_3\eta_5 p_6.$$

Next, consider the sequence of maps

$$M^6 \xrightarrow{p_6} \mathbb{S}^6 \xrightarrow{\nu'} \mathbb{S}^3 \xrightarrow{i_4} M^4.$$

Again by (3.7), $i_4\nu' = 2\lambda_2$ and $[\iota_{M^4}, i_4] = \lambda_2 p_6$, and then

$$\begin{aligned}
i_4\nu' p_6 &= (2\lambda_2)p_6 = (\lambda_2 + \lambda_2)(\Sigma p_5)\\
&= \lambda_2(\Sigma p_5) + \lambda_2(\Sigma p_5) = 2(\lambda_2 p_6)\\
&= 2[\iota_{M^4}, i_4] = [\iota_{M^4}, 2i_4]\\
&= [\iota_{M^4}, 0] = 0,
\end{aligned}$$

and the proof follows. □

Making use of Lemmas 3.13 and 3.14, the formula stated in Lemma 3.16 yields a non trivial suspended element:

Lemma 3.17. $\Sigma^{n-2}(\eta_2^2 \wedge \iota_{M^2}) = i_{n+2}\eta_{n+1}\overline{\eta}_{n+2} + \tilde{\eta}_{n+1}\eta_{n+3}p_{n+4}$ *for* $n \geqslant 2$.

To improve Proposition 3.11(2) in the next theorem, first we observe that $j_1\eta_n^2 \in G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n))$ if and only if $[j_1\eta_n^2, j_2] = 0$, where

$$j_1\colon \mathbb{S}^n \hookrightarrow M(\mathbb{Z} \oplus \mathbb{Z}_2, n) \qquad \text{and} \qquad j_2\colon M^{n+1} \hookrightarrow M(\mathbb{Z} \oplus \mathbb{Z}_2, n)$$

are the inclusions. Further, by Proposition 1.1, we have for $n \geqslant 3$:

$$\begin{aligned}
[j_2, j_1\eta_n^2] &= [j_2, j_1] \circ (\iota_{M(\mathbb{Z}_2, n-1)} \wedge \eta_n^2) + \sum_{k=2}^{\infty} [[j_2, j_1^k] \circ (\iota_{M(\mathbb{Z}_2, n-1)} \wedge h_k(\eta_n^2))\\
&= [j_2, j_1] \circ (\iota_{M(\mathbb{Z}_2, n-1)} \wedge \eta_n^2).
\end{aligned} \tag{3.8}$$

Then, the main result of this subsection now comes:

Theorem 3.18. *If* $n \geqslant 4$ *then* $G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) = 0$.

Proof. Writing

$$\eta_n^2 \wedge \iota_{M(\mathbb{Z}_2,n-1)} = \Sigma^{n-2}\eta_2^2 \wedge \Sigma^{n-2}\iota_{M^2} = \Sigma^{2(n-2)}(\eta_2^2 \wedge \iota_{M^2}),$$

Lemma 3.17 implies that $\eta_n^2 \wedge \iota_{M(\mathbb{Z}_2,n-1)} \neq 0$ for $n \geqslant 2$. Then, by (3.8), we conclude that $[j_2, j_1\eta_n^2] \neq 0$ for $n \geqslant 3$. Therefore $j_1\eta_n^2 \notin G_{n+2}(M(\mathbb{Z}\oplus\mathbb{Z}_2, n))$, for $n \geqslant 4$ and consequently $G_{n+2}(M(\mathbb{Z}\oplus\mathbb{Z}_2, n)) = 0$. □

Corollary 3.19. *Let A be a finite Abelian group with $|A| \equiv 2 \pmod 4$. Then $G_{n+2}(M(\mathbb{Z}\oplus A, n)) = 0$, for $n \geqslant 4$.*

We close the paper summarizing the groups as before:

(a) $G_{n+2}(M(\mathbb{Z}^m \oplus A, n)) = 0$ for $m \geqslant 0$, $m \neq 1$, $n \geqslant 4$ provided $|A|$ is odd;

(b) $G_{n+2}(M(\mathbb{Z}\oplus A, n)) = j_{1*}G_{n+1}(\mathbb{S}^n)$, for $n \geqslant 4$ provided $|A|$ is odd;

(c) $G_{n+2}(M(\mathbb{Z}^m\oplus A, n)) = 0$, for $m \geqslant 0$, $n \geqslant 4$ provided $|A| \equiv 2 \pmod 4$.

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