On the Gottlieb groups
$G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$

Marek Golasiński, Thiago de Melo, Rodrigo Bononi

Abstract. We are motivated by [M. Arkowitz. K. Maruyama. J. Math. Soc. Japan, 66(3):735–743, 2014]: “It would be interesting to compute other Gottlieb groups of Moore spaces such as, for example, $G_{n+1}(M(A, n))$” to compute the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$ and $m \geq 1$.


INTRODUCTION

Given a pointed space $X$, its $n$-th Gottlieb group $G_n(X)$ with $n \geq 1$ was introduced and studied by Gottlieb in [8, 9], and it has been shown to have many topological applications. There have been recent results on Gottlieb groups of rational spaces [5, pp. 377–380] and on Gottlieb groups of spheres, projective and Moore spaces [7].

The paper [2] deals with $G_n(\sum X_1 \vee \sum X_2 \vee \cdots \vee \sum X_k)$, where $\sum X_i$ is the suspension of the space $X_i$, with particular attention to the case $k = 2$ and $X_i$ a sphere. Necessary and sufficient conditions for an element of the homotopy group $\pi_n(\sum X_1 \vee \cdots \vee \sum X_k)$ to be in $G_n(\sum X_1 \vee \cdots \vee \sum X_k)$ are presented in [2, Proposition 2.4].

We are motivated by [2, Remark 4.5]: “It would be interesting to compute other Gottlieb groups of Moore spaces such as, for example, $G_{n+1}(M(A, n))$”
to compute the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$ and $m \geq 1$.

This paper is organized as follows. Section 1 fixes up some notations and definitions, and necessary known results on Gottlieb groups as well.

Section 2 generalizes some results presented in [2]. First, the groups $G_N(\vee_{t \in T} S^*_{n_t})$ for $n_t \geq 2$ with $t \in T$ are computed provided $|T| \geq 2$ and $N < 2 \min\{n_t\}_t - 1$. Then [8,9] are used to study

$$G_N(\vee_{s \in S} S^1 \vee_{t \in T} S^{n_t})$$

with $|S| \geq 1$, $|T| \geq 0$ and $n_t \geq 2$ for $t \in T$ provided $N < 2 \min\{n_t\}_t - 1$.

Section 3 presents computations of Gottlieb groups of Moore spaces $M(A, n)$ for some finitely generated Abelian groups $A$ and, in particular, it is devoted to explicit computations of Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$ and $m \geq 1$. The main results of this paper, as consequences of Theorems 3.9 and 3.18, say:

- if $n \geq 3$ then $G_{n+1}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$ for $m \geq 1$;
- if $n \geq 4$ then $G_{n+2}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$ for $m \geq 1$.

Conclusions on $G_{n+k}(M(A, n))$ with $k = 1, 2$ for some finitely generated Abelian groups $A$ are derived as well.

1. PREREQUISITES

Throughout this paper all spaces and maps are assumed to be connected, based and of the homotopy type of CW-complexes. We use the standard terminology and notation from homotopy theory, mainly from [16]. We do not distinguish between a map and its homotopy class, write $\Omega X$ (resp. $\Sigma X$) for the loop (resp. suspension) space on a space $X$ and $[X, Y]$ for the set of homotopy classes of maps $X \to Y$, and $X \simeq Y$ for a homotopy equivalence of spaces $X$ and $Y$.

Given spaces $X$ and $Y$, we use the customary notations $X \vee Y$ and $X \wedge Y = (X \times Y)/(X \vee Y)$ for the wedge and the smash product of spaces $X$ and $Y$, respectively. Let $\iota_X$ be the identity map on a space $X$ and $\iota_n$ be the identity map on the $n$-sphere $S^n$ and write $\pi_n(X)$ with $n \geq 0$ for the $n$-th homotopy group of $X$.

For maps $\alpha: \Sigma X \to Z$ and $\beta: \Sigma Y \to Z$, there exists a map

$$k(\alpha, \beta): \Sigma(X \wedge Y) \to Z$$

as defined in [1] and its homotopy class is the generalized Whitehead product:

$$[\alpha, \beta] = [k(\alpha, \beta)] \in [\Sigma(X \wedge Y), Z].$$
In particular, given the inclusions
\[ j_1 : \Sigma X \hookrightarrow \Sigma X \vee \Sigma Y \quad \text{and} \quad j_2 : \Sigma Y \hookrightarrow \Sigma X \vee \Sigma Y \]
we call
\[ k = k(j_1, j_2) : \Sigma(X \wedge Y) \to \Sigma X \vee \Sigma Y \]
the generalized Whitehead map. By [1, Corollary 4.3], there is a homotopy equivalence \( C_k \cong \Sigma X \wedge \Sigma Y \) where \( C_k \) stands for the mapping cone of the map \( k \).

In the sequel, we need a formula which generalizes one proved by Barcus-Barratt [3, Corollary (7.4)] in case \( A \) and \( B \) are spheres and by Rutter [15, Theorem 3.5.1] if \( A \) and \( B \) are suspensions. In fact only \( A \) needs to be a co-\( H \)-space.

**Proposition 1.1** ([4, (3.4) Proposition, p. 51]). Assume \( A \) is a co-\( H \)-space and \( B \) has finite dimension. Then, the Whitehead product of \( \Sigma A \overset{\alpha}{\to} Z \) and \( \Sigma B \overset{\gamma}{\to} \Sigma Y \overset{\beta}{\to} Z \) satisfies the formula
\[ [\alpha, \beta \gamma] = \sum_{k=1}^{\infty} [[\alpha, \beta^k] \circ (\iota_A \wedge h_k(\gamma))] \]
for the James-Hopf invariant \( h_k : [\Sigma B, \Sigma Y] \to [\Sigma B, (\Sigma Y)^{\wedge k}] \), where the symbol
\[ [[\alpha, \beta^k] = [\cdots [[\alpha, \beta], \beta], \ldots, \beta] \]
denotes an iterated Whitehead product.

**Remark 1.2.**
(i) \( h_1(\gamma) = \gamma \) for any \( \gamma \in [\Sigma B, \Sigma Y] \);
(ii) if \( \gamma \in [\Sigma B, \Sigma Y] \) is a suspension then it is easily seen that \( h_k(\gamma) = 0 \) for \( k \geq 2 \).

Now, given an Abelian group \( A \) and \( n \geq 2 \), write \( M(A, n) \) for the Moore space of type \((A, n)\). Notice that
\[ M(A, n + 1) = \Sigma M(A, n) \quad \text{for} \ n \geq 2, \]
and \( M(A, 2) = \Sigma L(A) \) for some space \( L(A) \). Certainly, \( \pi_n(M(A, n)) = A \) and \( M(\mathbb{Z}, n) = S^n \), the \( n \)-sphere, for the group \( \mathbb{Z} \) of integers. Furthermore, \( M^n = \Sigma^{n-2} \mathbb{R}P^2 \) with the real projective plane \( \mathbb{R}P^2 = M^2 \) and \( n \geq 3 \) is the Moore space of type \((\mathbb{Z}_2, n - 1)\). If \( m : S^n \to S^n \) is a map of degree \( m \) then
\[ M(\mathbb{Z}_m, n) = S^n \cup_m e^{n+1} \]
for the cyclic group \( \mathbb{Z}_m \) with order \( m \). Denote by \( i_{n+1} : S^n \hookrightarrow M(\mathbb{Z}_m, n) \) the canonical inclusion map and by \( p_{n+1} : M(\mathbb{Z}_m, n) \to S^{n+1} \) the pinching map.
Proposition 1.3 ([7, Lemma 3.15]). If $A$ and $B$ are torsion Abelian groups whose primary components are indexed by disjoint sets of primes then the inclusion map

$$M(A, m) \cup M(B, n) \hookrightarrow M(A, m) \times M(B, n)$$

is a homology isomorphism for $m, n \geq 2$, and consequently, a homotopy equivalence as well.

Notice that Proposition 1.3 implies that the space $M(A, m) \cup M(B, n)$ is contractible for $m, n \geq 2$ provided that the groups $A, B$ are Abelian torsion groups and their primary components are indexed by disjoint sets of primes. In particular, the space $M(\mathbb{Z}_k, m) \cup M(\mathbb{Z}_l, n)$ is contractible for $m, n \geq 2$ provided $k$ and $l$ are relatively prime.

Given $m \geq 1$, recall that the $m$-th Gottlieb group $G_m(X)$ of a path-connected space $X$ has been defined in [8,9] as the subgroup of the homotopy group $\pi_m(X)$ consisting of all elements which can be represented by a map $f: S^m \to X$ such that

$$f \circ \iota_X: S^m \vee X \to X$$

extends (up to homotopy) to $F: S^m \times X \to X$. Recall that $\alpha \in G_m(\Sigma X)$ if and only if the generalized Whitehead product $[\alpha, \iota_{\Sigma X}] = 0$ (see [1, Proposition 5.1]).

Given the inclusion maps

$$j_1: \Sigma X \hookrightarrow \Sigma X \vee \Sigma Y \quad \text{and} \quad j_2: \Sigma Y \hookrightarrow \Sigma X \vee \Sigma Y$$

and identifying $\Sigma X \vee \Sigma Y$ with $\Sigma(X \vee Y)$, then, by [2, Proposition 2.2], we have:

Proposition 1.4 ([2, Proposition 2.3]). Let $\alpha \in \pi_n(\Sigma X \vee \Sigma Y)$. Then, $\alpha \in G_n(\Sigma X \vee \Sigma Y)$ if and only if $[\alpha, j_1] = 0 = [\alpha, j_2]$.

A straightforward extension of the Proposition 1.4 to the wedge

$$T = \Sigma X_1 \vee \cdots \vee \Sigma X_k$$

gives that $\alpha \in G_n(T)$ if and only if $[\alpha, j_s] = 0$, for $s = 1, 2, \ldots, k$, where $j_s \in [\Sigma X_s, T]$ is the inclusion and $\alpha \in \pi_n(T)$.

Let $A_p$ be the $p$-primary component of an Abelian group $A$. If follows from [9, Theorems 1-7 and 2-1] and Proposition 1.3 that:

Corollary 1.5.

(i) Let $A$ and $B$ torsion Abelian groups whose primary components are indexed by disjoint set of primes. Then,

$$G_k(M(A, m) \vee M(B, n)) = j_{1*}G_k(M(A, m)) \oplus j_{2*}G_k(M(B, n)),$$
for \( k \geq 1, m, n \geq 2 \). In particular, if the 2-primary component \( A_2 \) is trivial one finds that
\[
G_k(M(A, m) \vee M^{n+1}) = j_{1*}G_k(M(A, m)) \oplus j_{2*}G_k(M^{n+1}),
\]
for \( k \geq 1, m, n \geq 2 \).

(ii) If \( A = \bigoplus_{i=1}^t A_{p_i} \) is the primary decomposition of the finite Abelian group \( A \), then
\[
G_k(M(A, m)) = \bigoplus_{i=1}^t j_{i*}G_k(M(A_{p_i}, m)),
\]
for \( k \geq 1, m \geq 2 \).

Since inclusions and projections \( X_k \xrightarrow{j_k} X_1 \vee X_2 \xrightarrow{q_l} X_l \) satisfy
\[
q_lj_k = \begin{cases} 
*, & \text{if } k \neq l, \\
\iota_{X_k}, & \text{if } k = l,
\end{cases}
\]
for \( k, l = 1, 2 \), we can state:

**Proposition 1.6.** Suppose
\[
\pi_m(X_1 \vee X_2) = j_{1*}\pi_m(X_1) \oplus j_{2*}\pi_m(X_2),
\]
for some \( m \geq 1 \). Then,
\[
G_m(X_1 \vee X_2) \subseteq j_{1*}G_m(X_1) \oplus j_{2*}G_m(X_2).
\]

**Proof.** By [9, Proposition 1-4], we have
\[
q_{k*} : G_m(X_1 \vee X_2) \to G_m(X_k).
\]
If \( \alpha \in G_m(X_1 \vee X_2) \), then \( \alpha = j_{1*}(\alpha_1) + j_{2*}(\alpha_2) \), where \( \alpha_k \in \pi_m(X_k) \) and so, \( \alpha_k = q_{k*}(\alpha) \) with \( \alpha_k \in G_m(X_k) \), for \( k = 1, 2 \). This implies that
\[
G_m(X_1 \vee X_2) \subseteq j_{1*}G_m(X_1) \oplus j_{2*}G_m(X_2) \approx G_m(X_1 \times X_2). \quad \blacksquare
\]

**Remark 1.7.** If the spaces \( X \) and \( Y \) are \( (m-1) \)- and \( (n-1) \)-connected, respectively, then \( \pi_{k+1}(X \times Y, X \vee Y) = 0 \) for \( k+1 < m+n \). This implies that the inclusion map \( X \vee Y \hookrightarrow X \times Y \) is an \( (m+n-1) \)-equivalence. Consequently, the inclusion map
\[
M(A, m) \vee M(B, n) \hookrightarrow M(A, m) \times M(B, n)
\]
implies an isomorphism
\[
\pi_k(M(A, m) \vee M(B, n)) \xrightarrow{\cong} \pi_k(M(A, m)) \oplus \pi_k(M(B, n))
\]
for \( k < m+n-1 \). In particular,
\[
M(A \oplus B, n) = M(A, n) \vee M(B, n)
\].
On the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$

yields an isomorphism

$$\pi_k(M(A \oplus B, n)) \cong \pi_k(M(A, n)) \oplus \pi_k(M(B, n))$$

for $k < 2n - 1$.

Let $A$ and $B$ be Abelian groups. By Proposition 1.6 and Remark 1.7 we get:

**Corollary 1.8.** If $k < m + n - 1$, then

$$G_k(M(A, m) \vee M(B, n)) \subseteq j_1*G_k(M(A, n)) \oplus j_2*G_k(M(B, m)),$$

for $k \geq 1$ and $m, n \geq 2$.

2. **Gottlieb groups of wedge of spheres**

In view of [2, Corollary 3.6], $G_N(S^m \vee S^n) = 0$ with $2 \leq m \leq n$ provided $N < 2m - 1$. In this section, we first compute $G_N(\bigvee_{t \in T}^s S^{n_t})$ with $|T| \geq 2$ and $n_t \geq 2$ for $t \in T$ provided $N < 2 \min\{n_t\}_{t \in T} - 1$. Then, we make use of [8] and [9] to study $G_N(\bigvee_{s \in S}^T \bigvee_{t \in T}^s S^{n_t})$ with $|S| \geq 1$, $|T| \geq 1$ and $n_t \geq 2$ for $t \in T$ provided $N < 2 \min\{n_t\} - 1$.

First recall that [2, Proposition 2.4] states:

**Proposition 2.1.** Let $\alpha \in \pi_n(\Sigma X_1 \vee \cdots \vee \Sigma X_k)$. Then,

$$\alpha \in G_n(\Sigma X_1 \vee \cdots \vee \Sigma X_n)$$

if and only if $[\alpha, j_i] = 0$ for the canonical inclusions

$$j_i: \Sigma X_k \hookrightarrow \Sigma X_1 \vee \cdots \vee \Sigma X_n$$

with $i = 1, \ldots, k$.

Furthermore, from [2, Theorem 3.4] it is derived:

**Proposition 2.2.** Suppose that $2 \leq m, n$ and $N < 2 \min\{m, n\} - 1$, then

$$G_N(S^m \vee S^n) = 0.$$

Applying Proposition 2.1, we get a generalized version of [2, Theorem 3.4] which yields:

**Proposition 2.3.** Suppose that $k \geq 2$ and $n_i \geq 2$ for $i = 1, \ldots, k$. If $N < 2 \min_{1 \leq i \leq k-1} \{n_i\}$ then

$$G_N(S^{n_1} \vee \cdots \vee S^{n_k}) = 0.$$

Then, we derive a generalization of Proposition 2.2:
Corollary 2.4. Let $T$ be a set with $|T| \geq 2$ and $n_t \geq 2$ for $t \in T$. If $N < 2 \min\{n_t\} - 1$, then

$$G_N\left( \bigvee_{t \in T} S^{n_t} \right) = 0.$$ 

Proof. Let $f : S^N \to \bigvee_{t \in T} S^n$. Since the image $f(S^N)$ is compact, there exist $k \geq 1$, $t_i \in T$ with $i = 1, \ldots, k$ and a map

$$f' : S^N \xrightarrow{f'} \bigvee_{i=1}^k S^{n_{t_i}}$$

such that $j \circ f' = f$ for the canonical inclusion $j : \bigvee_{i=1}^k S^{n_{t_i}} \to \bigvee_{t \in T} S^n$.

Certainly, we may assume that $k \geq 2$. But, $p \circ j = \iota_n \bigvee_{k=1}^n S^{nt_k}$ for the projection map $p : \bigvee_{t \in T} S^n \to \bigvee_{i=1}^n S^{n_{t_i}}$, therefore [9, Proposition 1-4.] implies that the map

$$p_* : \pi_N\left( \bigvee_{t \in T} S^n \right) \to \pi_N\left( \bigvee_{i=1}^n S^{n_{t_i}} \right)$$

carries $G_N\left( \bigvee_{t \in T} S^n \right)$ into $G_N\left( \bigvee_{i=1}^n S^{n_{t_i}} \right)$. Consequently, in view of Proposition 2.3, we get $p_*(f) = f' = 0$ and the proof is complete. \qed

Next, we need:

Theorem 2.5 ([8, Theorem IV.1]). Suppose $X$ has the same homotopy type as a compact, connected polyhedron. Then, $G_1(X) = 0$ if the Euler-Poincaré number $\chi(X) \neq 0$.

Since $\chi(S^1 \lor S^n) = (-1)^n$ for $n \geq 1$ and $\chi\left( \bigvee_{i=1}^k S^1 \right) = 1 - k$, we deduce from Theorem 2.5 that

$$G_1(S^1 \lor S^n) = 0$$

data (2.1)

for $n \geq 1$ and

$$G_1\left( \bigvee_{i=1}^k S^1 \right) = 0$$

data (2.2)

for $k \geq 2$.

Furthermore, applying (2.2) and [9, Proposition 1-4.] as in the proof of Corollary 2.4, we get:

$$G_1\left( \bigvee_{t \in T} S^1 \right) = 0$$

data (2.3)

for $|T| \geq 2$.

Then, we can state:
Proposition 2.6. If $S, T$ are sets with $|S|, |T| \geq 1$ and $n_t \geq 2$ for $t \in T$ then

$$G_1 \left( \bigvee_{s \in S} S^1 \vee \bigvee_{t \in T} S^{nt} \right) = 0.$$ 

Proof. First, notice that the case $|S| = |T| = 1$ follows from (2.1). Next, applying the Seifert-van Kampen Theorem, we get:

$$\pi_1 \left( \bigvee_{s \in S} S^{nt} \right) = j_1 \pi_1 \left( \bigvee_{s \in S} S^1 \right) \ast j_2 \pi_1 \left( \bigvee_{t \in T \setminus \{t_0\}} S^{nt} \right)$$

$$= j_1 \pi_1 \left( \bigvee_{s \in S} S^{nt} \right)$$

with $|T| \geq 2$ for the canonical inclusions

$$j_1 : S^1 \vee S^{n_0} \hookrightarrow S^1 \vee \bigvee_{t \in T} S^{nt}, \quad j_2 : \bigvee_{t \in T\setminus\{t_0\}} S^{nt} \hookrightarrow S^1 \vee \bigvee_{t \in T} S^{nt}$$

and

$$\pi_1 \left( \bigvee_{s \in S} S^1 \vee \bigvee_{t \in T} S^{nt} \right) = j_1 \pi_1 \left( \bigvee_{s \in S} S^1 \right) \ast j_2 \pi_1 \left( \bigvee_{t \in T} S^{nt} \right) = j_1 \pi_1 \left( \bigvee_{s \in S} S^1 \right)$$

with $|S| \geq 2$ and $|T| \geq 1$ for the canonical inclusions

$$j_1 : \bigvee_{s \in S} S^1 \hookrightarrow \bigvee_{s \in S} S^1 \vee \bigvee_{t \in T} S^{nt}, \quad j_2 : \bigvee_{t \in T} S^{nt} \hookrightarrow \bigvee_{s \in S} S^1 \vee \bigvee_{t \in T} S^{nt}.$$

Then, Proposition 1.6 and equations (2.1), (2.3) yield that

$$G_1 \left( \bigvee_{t \in T} S^{nt} \right) \subseteq G_1 \left( S^1 \vee S^{n_0} \right) = 0$$

with $|T| \geq 2$ and

$$G_1 \left( \bigvee_{s \in S} S^1 \vee \bigvee_{t \in T} S^{nt} \right) \subseteq G_1 \left( \bigvee_{s \in S} S^1 \right) = 0$$

with $|S| \geq 2$ and $|T| \geq 1$ and the proof follows. \qed

Next, recall:

Theorem 2.7 ([9, Theorem 6-2]). Let $p: \tilde{X} \to X$ be a covering map. If $n \geq 1$ then $p_*^{-1}(G_n(X)) \subseteq G_n(\tilde{X})$. In other words, if we identify $\pi_n(\tilde{X})$ with $\pi_n(X)$ under the isomorphism $p_*$ for $n \geq 2$ then $G_n(\tilde{X}) \supseteq G_n(X)$.

Now, consider the space

$$X = \bigvee_{s \in S} S^1 \vee \bigvee_{t \in T} S^{nt}$$

with $|S|, |T| \geq 1$ and $n_t \geq 2$ for $t \in T$. Then, by the Seifert-van Kampen Theorem, we have that $\pi_1(X) = \pi_1 \left( \bigvee_{s \in S} S^1 \right)$ and the space

$$\tilde{X} = \bigvee_{g \in \pi_1(X)} \left( \bigvee_{t \in T} S^{nt} \right)$$
is the universal covering of \( X \).

Therefore, Corollary 2.4, Proposition 2.6 and Theorem 2.7 yield:

**Proposition 2.8.** Let \(|S|, |T| \geq 1 \) and \( n_t \geq 2 \) with \( t \in T \). If
\[
N < 2 \min\{n_t\}_{t \in T} - 1
\]
then \( G_N\left( \bigvee_{s \in S} S^1 \lor \bigvee_{t \in T} S^{n_t} \right) = 0 \).

3. Computations

In this section we compute Gottlieb groups \( G_{n+k}(M(A, n)) \) with \( k = 1, 2 \) of Moore spaces \( M(A, n) \) for some finitely generated Abelian groups \( A \). First, we recall from [2]:

**Theorem 3.1.** Let \( A \) be any finitely generated Abelian group and \( n \geq 3 \). Then,
\[
G_n(M(A, n)) = \begin{cases} 
0, & \text{if } n \text{ is even}, \\
0, & \text{if } n \text{ is odd and } \text{rk}(A) \neq 1, \\
2\mathbb{Z} \subseteq \mathbb{Z} = \pi_n(S^n), & \text{if } n \neq 3, 7 \text{ is odd and } A = \mathbb{Z}, \\
\mathbb{Z} = \pi_n(S^n), & \text{if } n = 3, 7 \text{ and } A = \mathbb{Z}.
\end{cases}
\]

In addition, by [2, Corollary 4.4] if \( n \) is odd, then \( G_n(M(\mathbb{Z} \oplus A, n)) \) is infinite cyclic, where \( A \) is a finite Abelian group.

We work in the sequel with a finitely generated Abelian group \( A \) whose torsion subgroup has order \(|A| \equiv 2 \pmod{4}\). Notice that in [7, Chapter 3] there are some results on \( G_{n+1}(M(A, n)) \) only for \( A \) having an odd order torsion subgroup.

**Groups \( G_{n+1}(M(A, n)) \).** First, we observe that Proposition 2.3 leads to the following:

**Proposition 3.2.** If \( n \geq 2 \) and \( m \geq 2 \) then \( G_{n+k}(M(\mathbb{Z}^m, n)) = 0 \) for \( k < n - 1 \).

Because \( M^n = M(\mathbb{Z}_2, n - 1) \), by [7, Corollary 3.11], we have:

**Proposition 3.3.** \( G_{n+1}(M(\mathbb{Z}_2, n)) = 0 \), for \( n \geq 3 \).

Let
\[
\eta_n = \Sigma^{n-2}\eta_2: S^{n+1} \to S^n
\]
be the \((n - 2)\)-suspension of the Hopf map \( \eta_2: S^3 \to S^2 \). Now, using both propositions above, we show:
Proposition 3.4. Let $n \geq 3$. Then:
(i) $G_{n+1}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$, for $m \geq 2$;
(ii) $G_{n+1}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) \subseteq$
    $\subseteq j_1^\ast(G_{n+1}(S^n)) = \begin{cases} \mathbb{Z}_2\{j_1 \eta_n\}, & \text{if } n = 6 \text{ or } n \equiv 3 \pmod{4}, \\ 0, & \text{otherwise}. \end{cases}$

Proof. (i) Since
$$G_{n+1}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = G_{n+1}(M(\mathbb{Z}^m, n) \vee M(\mathbb{Z}_2, n)),$$
by Corollary 1.8 and Propositions 3.2 and 3.3, statement (i) follows.
(ii) First, recall from [7, (1.15)] that
$$\sharp[t_n, \eta_n] = \begin{cases} 1, & \text{for } n = 2, 6 \text{ or } n \equiv 3 \pmod{4}, \\ 2, & \text{otherwise}. \end{cases}$$
Now, applying Corollary 1.8 and Proposition 3.3, the proof is complete. □

Remark 3.5. If $A$ is a finite group with order $|A| \equiv 2 \pmod{4}$ then the 2-primary component $A_2$ has order two.

Proposition 3.6. Let $A$ be a finite Abelian group with $|A| \equiv 2 \pmod{4}$. For $n \geq 3$, it holds:
(i) $G_{n+1}(M(A, n)) = 0$;
(ii) $G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = 0$ for $m \geq 2$;
(iii) $G_{n+1}(M(\mathbb{Z} \oplus A, n)) \subseteq$
    $\subseteq j_1^\ast(G_{n+1}(S^n)) = \begin{cases} \mathbb{Z}_2\{j_1 \eta_n\}, & \text{if } n = 6 \text{ or } n \equiv 3 \pmod{4}, \\ 0, & \text{otherwise}. \end{cases}$

Proof. (i) By Corollary 1.5 we can write
$$G_{n+1}(M(A, n)) = \bigoplus_{i=1}^t j_i^\ast G_{n+1}(M(A_{p_i}, n))$$
for $A = \bigoplus_{i=1}^t A_{p_i}$. Furthermore, by [7, Proposition 3.19] we know that
$G_{n+1}(M(A_{p_i}, n)) = 0$ for $p_i > 2$, and by Remark 3.5 we know that $A_2 \cong \mathbb{Z}_2$ if $|A| \equiv 2 \pmod{4}$. Consequently,
$$G_{n+1}(M(A, n)) = j_2^\ast G_{n+1}(M(A_2, n)) \cong G_{n+1}(M(\mathbb{Z}_2, n)) = 0,$$
where the last equality is due to Proposition 3.3.
(ii) We simply write
$$G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = G_{n+1}(M(\mathbb{Z}^m, n) \vee M(A, n)).$$
Then, by Corollary 1.8, Proposition 3.2 and previous item, the result follows.

(iii) It is similar to the proof of Proposition 3.4(ii). □

Proposition 3.4(ii) gives only two possibilities for $G_{n+1}(M(Z \oplus A, n))$. In the sequel, we show that it is trivial, but first, we pose some needed results.

Write $\tilde{\eta}_n : M^{n+2} \to S^n$ for the extension of $\eta_n$ with $n \geq 3$. It is the unique (up to homotopy) element satisfying $\tilde{\eta}_n i_{n+2} = \eta_n$. Then,

$$\Sigma \tilde{\eta}_n = \eta_{n+1} \quad \text{and} \quad 2\tilde{\eta}_n = \eta_n^2 p_{n+2}, \quad (n \geq 3).$$

Dually, let $\bar{\eta}_n : S^{n+2} \to M^{n+1}$ be the coextension of $\eta_n$ for $n \geq 2$. It is the unique (up to homotopy) element satisfying $p_{n+1} \bar{\eta}_n = \eta_{n+1}$. Then,

$$\Sigma \bar{\eta}_n = \bar{\eta}_{n+1} \quad \text{and} \quad 2\bar{\eta}_n = i_{n+1} \eta_n^2, \quad (n \geq 2). \quad (3.1)$$

By [7, Lemma 1.26], $\pi_{n+1}(M^n) = Z_4\{\tilde{\eta}_{n-1}\}$ and $[M^{n+2}, S^n] = Z_4\{\bar{\eta}_n\}$ for $n \geq 3$. Furthermore, from [12, Lemma 1.5] we have

$$[M^{n+1}, M^n] = Z_2\{i_n \tilde{\eta}_{n-1}\} \oplus Z_2\{\bar{\eta}_{n-1} p_{n+1}\}, \quad (n \geq 4). \quad (3.2)$$

Recall that

$$G_{n+1}(M(Z \oplus Z_2, n)) \subseteq j_1^*(G_{n+1}(S^n)).$$

If $j_1^*(G_{n+1}(S^n))$ is non-trivial, we have that $j_1 \eta_n \in G_{n+1}(M(Z \oplus Z_2, n))$ if and only if $[j_1 \eta_n, j_1] = 0 = [j_1 \eta_n, j_2]$, for the inclusions

$$j_1 : S^n \hookrightarrow M(Z \oplus Z_2, n) \quad \text{and} \quad j_2 : M^{n+1} \hookrightarrow M(Z \oplus Z_2, n).$$

Now, $[j_1 \eta_n, j_1] = j_1[\eta_n, \tau_n] = 0$, since $G_{n+1}(S^n) = Z_2\{\eta_n\}$ and consequently $j_1 \eta_n \in G_{n+1}(M(Z \oplus Z_2, n))$ if and only if $[j_1 \eta_n, j_2] = 0$.

To compute $[j_1 \eta_n, j_2]$, we notice that by Remark 1.2(ii) or degree reasons $h_k(\eta_n) = 0$ for $k \geq 2$. Then, we make use of Proposition 1.1 and obtain:

$$[j_2, j_1 \eta_n] = [j_2, j_1] \circ (\iota_{M(Z_2, n-1)} \wedge \eta_n) + \sum_{k=2}^\infty [[j_2, j_1^k] \circ (\iota_{M(Z_2, n-1)} \wedge h_k(\eta_n))]$$

$$= [j_2, j_1] \circ (\iota_{M(Z_2, n-1)} \wedge \eta_n) \quad (3.3)$$

for $n \geq 3$, where $h_k$ is the James-Hopf invariant and the symbol

$$[[j_2, j_1^k] = [\cdots [[j_2, j_1], j_1], \ldots, j_1]$$

denotes the iterated Whitehead product.

Now, we show a formula useful in the sequel, found in the proof of [13, Lemma 3.1]:

**Lemma 3.7.** We have that

$$\eta_2 \wedge \iota_{M^2} = i_4 \tilde{\eta}_3 + \bar{\eta}_3 p_5. \quad (3.4)$$
On the Gottlieb groups $G_{n+k}(M(Z^m \oplus Z_2, n))$ for $k = 1, 2$

Proof. The cofibration $S^4 \overset{i_5}{\to} M^5 \overset{p_5}{\to} S^5$ yields the exact sequence

$$\cdots \to \pi_5(M^4) = Z_4\{\tilde{\eta}_3\} \overset{p_5^*}{\longrightarrow} [M^5, M^4] \overset{i_5^*}{\longrightarrow} \pi_4(M^4) = Z_2\{i_4\tilde{\eta}_3\} \to 0.$$ 

But,

$$\eta_2 \wedge \iota_{M^2} \in [M^5, M^4] = Z_2\{i_4\tilde{\eta}_3\} \oplus Z_2\{\tilde{\eta}_3p_5\}$$

and:

$$\eta_2 \wedge i_2 = \iota_2\eta_2 \wedge i_2\iota_1 = (\iota_2 \wedge i_2) \circ (\eta_2 \wedge \iota_1) = i_4\eta_3 = (i_4\tilde{\eta}_3) \circ i_5,$$

$$\eta_2 \wedge i_2 = \eta_2i_3 \wedge \iota_{M^2}i_2 = (\eta_2 \wedge \iota_{M^2}) \circ (i_3 \wedge i_2) = (\eta_2 \wedge \iota_{M^2}) \circ i_5.$$ 

Hence, $i_5^*(\eta_2 \wedge \iota_{M^2} - i_4\tilde{\eta}_3) = 0$ and the exact sequence above implies

$$\eta_2 \wedge \iota_{M^2} = i_4\tilde{\eta}_3 + x\tilde{\eta}_3p_5 \tag{3.5}$$

with $x = 0, 1$.

Next, consider the exact sequence

$$0 \to \pi_5(S^4) = Z_2\{\eta_4\} \overset{p_5^*}{\longrightarrow} [M^5, S^4] \overset{i_5^*}{\longrightarrow} \pi_4(S^4) = Z\{\iota_n\} \to \cdots$$

determined by the cofibration $S^4 \overset{i_5}{\to} M^5 \overset{p_5}{\to} S^5$. Then, $\eta_2 \wedge p_2 \in [M^5, S^4]$ and $p_5^*(\eta_4) = \eta_4p_5 \neq 0$.

Furthermore:

$$\eta_2 \wedge p_2 = \eta_2\iota_3 \wedge \iota_2p_2 = (\iota_2 \wedge \eta_2) \circ (\iota_3 \wedge p_2) = \eta_4p_5 \neq 0,$$

$$\eta_2 \wedge p_2 = \iota_2\eta_2 \wedge p_2\iota_{M^2} = (\iota_2 \wedge p_2) \circ (\eta_2 \wedge \iota_{M^2}) = p_4(\eta_2 \wedge \iota_{M^2}) \neq 0.$$ 

Consequently, (3.5) yields

$$p_4(\eta_2 \wedge \iota_{M^2}) = p_4i_4\tilde{\eta}_3 + x p_4\tilde{\eta}_3p_5 = x p_4\tilde{\eta}_3p_5.$$ 

Since, $p_4(\eta_2 \wedge \iota_{M^2}) \neq 0$, we derive that $x = 1$ and the proof is complete. □

Then, (3.2) and (3.4) lead to the following:

**Corollary 3.8.** $\Sigma^{n-2}(\eta_2 \wedge \iota_{M^2}) = \iota_{n+2}\tilde{\eta}_{n+1} + \tilde{\eta}_{n+1}p_{n+3} \neq 0$ for $n \geq 2$.

Now, we are in a position to state the main result of this section.

**Theorem 3.9.** Let $n \geq 3$. Then $G_{n+1}(M(Z \oplus Z_2, n)) = 0$.

Proof. Writing

$$\eta_n \wedge \iota_{M(Z \oplus Z_2, n-1)} = \Sigma^{n-2}\eta_2 \wedge \Sigma^{n-2}\iota_{M^2} = \Sigma^{2(n-2)}(\eta_2 \wedge \iota_{M^2})$$

we see, by Corollary 3.8, that

$$\eta_n \wedge \iota_{M(Z \oplus Z_2, n-1)} \neq 0 \text{ for } n \geq 2.$$
Further, by (3.3), it follows that \([j_2, j_1 \eta_n] \neq 0\), for \(n \geq 3\), since \([j_2, j_1]\) is a basic product. Then for \(n \geq 3\)
\[j_1 \eta_n \notin G_{n+1}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) \quad \text{and} \quad G_{n+1}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) = 0. \quad \square\]

**Corollary 3.10.** Let \(A\) be a finite Abelian group with \(|A| \equiv 2 \pmod{4}\). Then \(G_{n+1}(M(\mathbb{Z} \oplus A, n)) = 0\) for \(n \geq 3\).

**Proof.** Since \(A\) is a finite Abelian group with order \(|A| \equiv 2 \pmod{4}\), we apply [6, Chapter III, Theorem 15.2] to write \(A = \bigoplus_{i=1}^{t} A_{p_i}\) for the primary decomposition of \(A\) with \(p_1 = 2\). By Corollaries 1.5 and 1.8 we obtain
\[G_{n+1}(M(\mathbb{Z} \oplus A, n)) \subseteq \]
\[\subseteq j_1 G_{n+1}(M(\mathbb{Z} \oplus A_2, n)) \oplus \bigoplus_{i=2}^{t} j_i G_{n+1}(M(A_{p_i}, n)).\]

As \(G_{n+1}(M(A_{p_i}, n)) = 0\) for \(p_i > 2\) (see [7, Proposition 3.19]) and \(A_2 \cong \mathbb{Z}_2\) (cf. Remark 3.5), we get
\[G_{n+1}(M(\mathbb{Z} \oplus A, n)) \subseteq j_1 G_{n+1}(M(\mathbb{Z} \oplus A_2, n)) \cong \]
\[\cong G_{n+1}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) = 0. \quad \square\]

Recall that, by [6, Chapter III, Theorem 15.2], for any finitely generated Abelian group \(B\) there is an isomorphism \(B \cong \mathbb{Z}^m \oplus A\) with \(m \geq 0\) and a finite Abelian group \(A\). Furthermore, by means of the universal coefficient theorem for homotopy [10, p. 30], for any Abelian group \(A\), a pointed space \(X\) and \(n \geq 2\), there exists a short exact sequence
\[0 \to \Ext(A, \pi_{n+1}(X)) \to [M(A, n), X] \to \Hom(A, \pi_n(X)) \to 0. \quad (3.6)\]

Notice that by equation (3.6) the order \(\sharp j_2\) of the canonical inclusion map
\[j_2: M(A, n) \hookrightarrow M(\mathbb{Z} \oplus A, n)\]
is finite. If \(A\) has odd order so does \(\sharp j_2\), and we conclude that
\[G_{n+1}(M(\mathbb{Z} \oplus A, n)) = j_1 G_{n+1}(\mathbb{S}^n), \quad n \geq 3,\]
since \([j_2, j_1 \eta_n] = 0\). Finally, we summarize the groups \(G_{n+1}(M(\mathbb{Z}^m \oplus A, n))\) for a finite Abelian group \(A\) as follows:
(a) \(G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = 0\) for \(m \geq 0\), \(m \neq 1\), \(n \geq 3\) provided \(|A|\) is odd;
(b) \(G_{n+1}(M(\mathbb{Z} \oplus A, n)) = j_1 G_{n+1}(\mathbb{S}^n), \) for \(n \geq 3\) provided \(|A|\) is odd;
(c) \(G_{n+1}(M(\mathbb{Z}^m \oplus A, n)) = 0\), for \(m \geq 0\), \(n \geq 3\) provided \(|A| \equiv 2 \pmod{4}\).
On the Gottlieb groups $G_{n+k}(M(Z^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2, 3$.

**Groups** $G_{n+2}(M(A, n))$. We compute $G_{n+2}(M(A, n))$ for $n \geq 4$, under the same conditions for $A$ as before. First, we recall from [7, Corollary 3.11] that $G_{n+2}(M(\mathbb{Z}_2, n)) = 0$ for $n \geq 3$. Further, by [7, (1.16)],

$$
\# [t_n, \eta^2_n] = \begin{cases} 
1, & \text{if } n \equiv 2, 3 \pmod{4}, \\
2, & \text{otherwise}.
\end{cases}
$$

Analogous to Propositions 3.4 and 3.6, we can state the following two results:

**Proposition 3.11.** Let $n \geq 4$. Then:

(i) $G_{n+2}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n)) = 0$, for $m \geq 2$.

(ii) $G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) \subseteq j_1*(G_{n+2}(\mathbb{S}^n)) = \begin{cases} 
\mathbb{Z}_2 \{j_1 \eta^2_n\}, & \text{if } n \equiv 2, 3 \pmod{4}, \\
0, & \text{otherwise}.
\end{cases}$

**Proposition 3.12.** Let $A$ be a finite Abelian group with $|A| = 2 \pmod{4}$. For $n \geq 4$, it holds:

(i) $G_{n+2}(M(A, n)) = 0$.

(ii) $G_{n+2}(M(\mathbb{Z}^m \oplus A, n)) = 0$ for $m \geq 2$.

(iii) $G_{n+2}(M(\mathbb{Z} \oplus A, n)) \subseteq j_1*(G_{n+2}(\mathbb{S}^n)) = \begin{cases} 
\mathbb{Z}_2 \{j_1 \eta^2_n\}, & \text{if } n \equiv 2, 3 \pmod{4}, \\
0, & \text{otherwise}.
\end{cases}$

Next, we show that the Gottlieb group from Proposition 3.11(ii) is trivial using the same technics as in Proposition 3.4(ii). First, we recall from [11, Lemma 3.7] and [14, Lemma 2.1]:

**Lemma 3.13.**

(i) $[M^5, M^3] = \mathbb{Z}_2 \{i_3 \eta_2 \bar{\eta}_3\} \oplus \mathbb{Z}_2 \{\bar{\eta}_2 \eta_4 p_5\} \oplus \mathbb{Z}_2 \{\tau p_5\} \oplus \mathbb{Z}_2 \{\Sigma_2 \eta_3 p_5\}$;

(ii) $[M^6, M^4] = \mathbb{Z}_2 \{i_4 \eta_3 \bar{\eta}_4\} \oplus \mathbb{Z}_2 \{\bar{\eta}_3 \eta_5 p_6\} \oplus \mathbb{Z}_2 \{\lambda_2 p_6\}$;

(iii) $[M^7, M^5] = \mathbb{Z}_2 \{i_5 \eta_4 \bar{\eta}_5\} \oplus \mathbb{Z}_2 \{\bar{\eta}_4 \eta_6 p_7\} \oplus \mathbb{Z}_2 \{i_5 \nu_4 p_7\}$.

**Lemma 3.14.** For $n \geq 6$ we have that

$[M^{n+2}, M^n] = \mathbb{Z}_2 \{i_n \eta_{n-1} \bar{\eta}_n\} \oplus \mathbb{Z}_2 \{\bar{\eta}_{n-1} \eta_{n+1} p_{n+2}\} \oplus \mathbb{Z}_2 \{i_n \nu_{n-1} p_{n+2}\}$.

**Proof.** The cofibration

$$
\mathbb{S}^{n+1} \xrightarrow{i_{n+2}} M^{n+2} \xrightarrow{p_{n+2}} \mathbb{S}^{n+2}
$$
leads to the short exact sequence of homotopy groups
\[ 0 \to \pi_{n+2}(M^n) \xrightarrow{p^*_{n+2}} [M^{n+2}, M^n] \xrightarrow{i^*_{n+2}} 2\pi_{n+1}(M^n) \to 0, \]
for \( n \geq 6 \). By [7, Lemmas 1.26 and 1.27], we know that
\[ \pi_{n+1}(M^n) = \mathbb{Z}_4 \{ \eta_{n-1} \} \]
and
\[ \pi_{n+2}(M^n) = \mathbb{Z}_2 \{ \tilde{\eta}_{n-1} \eta_{n+1}, i_n \nu_{n-1} \}. \]
The sequence above is therefore
\[ 0 \to \mathbb{Z}_2 \{ \tilde{\eta}_{n-1} \eta_{n+1} \} \oplus \mathbb{Z}_2 \{ i_n \nu_{n-1} \} \xrightarrow{p^*_{n+2}} [M^{n+2}, M^n] \xrightarrow{i^*_{n+2}} \mathbb{Z}_2 \{ 2\tilde{\eta}_{n-1} \} \to 0, \]
for \( n \geq 6 \), which splits by means of
\[ \theta: \mathbb{Z}_2 \{ 2\tilde{\eta}_{n-1} \} \to [M^{n+2}, M^n], \quad \theta(2\tilde{\eta}_{n-1}) = i_n \eta_{n-1} \eta_n. \]
Thus,
\[ (i^*_{n+2}\theta)(2\tilde{\eta}_{n-1}) = i_n \eta_{n-1} \eta_n i_{n+2}. \]
From (3.1) we conclude that
\[ (i^*_{n+2}\theta)(2\tilde{\eta}_{n-1}) = 2\tilde{\eta}_{n-1}, \]
that is, \( i^*_{n+2}\theta \) is the identity homomorphism and the proof follows. \( \square \)

**Remark 3.15.** It follows from Lemmas 3.13 and 3.14 that the suspension map
\[ \Sigma_*: [M^{n+2}, M^n] \to [M^{n+3}, M^{n+1}] \]
is an isomorphism, for \( n \geq 5 \).

From [16, Proposition 5.6] we know that, the 2-primary component of \( \pi_6(S^3) \) is \( \pi_6^3 = \mathbb{Z}_4 \{ \nu' \} \) and \( \pi_7^4 = \mathbb{Z} \{ \nu_4 \} \oplus \mathbb{Z}_4 \{ \Sigma \nu' \} \).

By [14, (2.1) and (2.2)], the following relations hold:
\[
\begin{align*}
\pi_6(M^4) &= \mathbb{Z}_4 \{ \lambda_2 \} \oplus \mathbb{Z}_4 \{ \tilde{\eta}_3 \eta_5 \}, \quad 2\lambda_2 = i_4 \nu', \\
\pi_7(M^5) &= \mathbb{Z}_4 \{ i_5 \nu_4 \} \oplus \mathbb{Z}_4 \{ \tilde{\eta}_4 \eta_6 \}, \quad \Sigma \lambda_2 = 2(i_5 \nu_4) \in \pi_7(M^5), \quad (3.7) \\
\langle \iota_{M^4}, i_4 \rangle &= \lambda_2 p_6, \quad \pm \nu' = \tilde{\eta}_3 \eta_4.
\end{align*}
\]

Next, recall from (3.4) that \( \eta_2 \wedge \iota_{M^4} = i_4 \tilde{\eta}_3 + \tilde{\eta}_3 \eta_5 p_6 \). Then, for \( \eta_2^2 \wedge \iota_{M^2} \), we have the following:

**Lemma 3.16.** \( \eta_2^2 \wedge \iota_{M^2} = i_4 \eta_3 \eta_4 + \tilde{\eta}_3 \eta_5 p_6. \)

**Proof.** In view of (3.4), we have
\[
\eta_2^2 \wedge \iota_{M^2} = (\eta_2 \eta_3) \wedge \iota_{M^2} \\
= (\eta_2 \wedge \iota_{M^2})(\eta_3 \wedge \iota_{M^2})
\]
On the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$

$$= (\eta_2 \wedge \iota_{M^2})(\Sigma(\eta_2 \wedge \iota_{M^2}))$$

$$= (i_4\eta_3 + \tilde{\eta}_3p_5)(\Sigma(\eta_2 \wedge \iota_{M^2}))$$

$$= (i_4\eta_3)(\Sigma(\eta_2 \wedge \iota_{M^2})) + (\tilde{\eta}_3p_5)(\Sigma(\eta_2 \wedge \iota_{M^2}))$$

$$= (i_4\eta_3)(i_5\eta_4 + \tilde{\eta}_4p_6) + (\tilde{\eta}_3p_5)(i_5\eta_4 + \tilde{\eta}_4p_6)$$

$$= i_4\eta_3i_5\eta_4 + i_4\eta_3\tilde{\eta}_4p_6 + \tilde{\eta}_3p_5\tilde{\eta}_4p_6$$

$$= i_4\eta_3i_5\eta_4 + i_4\eta_3\tilde{\eta}_4p_6 + \tilde{\eta}_3p_5\tilde{\eta}_4p_6$$

since $p_5i_5 = 0$.

Also, due to relations (3.7),

$$\eta_2^2 \wedge \iota_{M^2} = i_4\eta_3\eta_4 + i_4(\pm \nu')p_6 + \tilde{\eta}_3\eta_5p_6 = i_4\eta_3\eta_4 \pm i_4\nu'p_6 + \tilde{\eta}_3\eta_5p_6.$$  

Next, consider the sequence of maps

$$M^6 \xrightarrow{p_6} S^6 \xrightarrow{\nu'} S^3 \xrightarrow{i_4} M^4.$$

Again by (3.7), $i_4\nu' = 2\lambda_2$ and $[\iota_{M^4}, i_4] = \lambda_2p_6$, and then

$$i_4\nu'p_6 = (2\lambda_2)p_6 = (\lambda_2 + \lambda_2)(\Sigma p_5)$$

$$= \lambda_2(\Sigma p_5) + \lambda_2(\Sigma p_5) = 2(\lambda_2p_6)$$

$$= 2[\iota_{M^4}, i_4] = [\iota_{M^4}, 2i_4]$$

$$= [\iota_{M^4}, 0] = 0,$$

and the proof follows.

Making use of Lemmas 3.13 and 3.14, the formula stated in Lemma 3.16 yields a non trivial suspended element:

**Lemma 3.17.** $\Sigma^{n-2}(\eta_2^2 \wedge \iota_{M^2}) = i_{n+2}\eta_{n+1}\eta_{n+2} + \tilde{\eta}_{n+1}\eta_{n+3}p_{n+4}$ for $n \geq 2$.

To improve Proposition 3.11(2) in the next theorem, first we observe that $j_1\eta_2^2 \in G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n))$ if and only if $[j_1\eta_2^2, j_2] = 0$, where

$$j_1: S^n \hookrightarrow M(\mathbb{Z} \oplus \mathbb{Z}_2, n) \quad \text{and} \quad j_2: M^{n+1} \hookrightarrow M(\mathbb{Z} \oplus \mathbb{Z}_2, n)$$

are the inclusions. Further, by Proposition 1.1, we have for $n \geq 3$:

$$[j_2, j_1\eta_2^2] = [j_2, j_1] \circ (\iota_{M(\mathbb{Z}_2, n-1)} \wedge \eta_2^2) + \sum_{k=2}^{\infty} [j_2, j_1^k] \circ (\iota_{M(\mathbb{Z}_2, n-1)} \wedge h_k(\eta_2^2))$$

$$= [j_2, j_1] \circ (\iota_{M(\mathbb{Z}_2, n-1)} \wedge \eta_2^2).  \quad (3.8)$$

Then, the main result of this subsection now comes:

**Theorem 3.18.** If $n \geq 4$ then $G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) = 0$.  

Proof. Writing
\[ \eta_n^2 \wedge \iota_M(\mathbb{Z}_2, n-1) = \Sigma^{n-2} \eta_2^2 \wedge \Sigma^{n-2} \iota_{M^2} = \Sigma^{2(n-2)} (\eta_2^2 \wedge \iota_{M^2}), \]
Lemma 3.17 implies that \( \eta_n^2 \wedge \iota_M(\mathbb{Z}_2, n-1) \neq 0 \) for \( n \geq 2 \). Then, by (3.8), we conclude that \( [j_2, j_1 \eta_n^2] \neq 0 \) for \( n \geq 3 \). Therefore \( j_1 \eta_n^2 \notin G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) \), for \( n \geq 4 \) and consequently \( G_{n+2}(M(\mathbb{Z} \oplus \mathbb{Z}_2, n)) = 0 \).

**Corollary 3.19.** Let \( A \) be a finite Abelian group with \( |A| \equiv 2 \) (mod 4). Then \( G_{n+2}(M(\mathbb{Z} \oplus A, n)) = 0 \), for \( n \geq 4 \).

We close the paper summarizing the groups as before:

(a) \( G_{n+2}(M(\mathbb{Z}^m \oplus A, n)) = 0 \) for \( m \geq 0 \), \( m \neq 1 \), \( n \geq 4 \) provided \( |A| \) is odd;

(b) \( G_{n+2}(M(\mathbb{Z} \oplus A, n)) = j_1^* G_{n+1}(S^n) \), for \( n \geq 4 \) provided \( |A| \) is odd;

(c) \( G_{n+2}(M(\mathbb{Z}^m \oplus A, n)) = 0 \), for \( m \geq 0 \), \( n \geq 4 \) provided \( |A| \equiv 2 \) (mod 4).

**Acknowledgments.** The authors are deeply indebted to Professor Juno Mukai for his valuable and fruitful discussion on the description of \( \eta_2^2 \wedge \iota_{M^2} \). Furthermore, they greatly appreciate the anonymous referee for a careful reading of the manuscript last version and his/her insightful comments and suggestions.

**References**


On the Gottlieb groups $G_{n+k}(M(\mathbb{Z}^m \oplus \mathbb{Z}_2, n))$ for $k = 1, 2$


Received: July 12, 2023, accepted: September 27, 2023.

Marek Golasiński

Faculty of Mathematics and Computer Science, University of Warmia and Mazury, Słoneczna 54 Street, 10-710 Olsztyn, Poland

Email: marekg@matman.uwm.edu

ORCID: 0000-0001-6969-8986

Thiago de Melo

São Paulo State University (UNESP), Institute of Geosciences and Exact Sciences, Av. 24A, 1515, Bela Vista. CEP 13.506–900. Rio Claro–SP, Brazil

Email: thiago.melo@unesp.br

ORCID: 0000-0002-4031-2805

Rodrigo Bononi

São Paulo State University (UNESP), Institute of Biosciences, Letters and Exact Sciences, R. Cristóvão Colombo, 2265, Jardim Nazareth. CEP 15054-000. São José do Rio Preto–SP, Brazil

Email: rodrigo.bononi@unesp.br

ORCID: 0000-0003-0452-0276