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Cantorvals as sets of subsums for a series related with trigonometric functions

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Abstract. We study properties of the set of subsums for convergent series $k_1 \sin x + \cdots + k_m \sin x + \cdots + k_1 \sin x^{\left[\frac{n-1}{m}+1\right]} + \cdots + k_m \sin x^{\left[\frac{n-1}{m}+1\right]} + \ldots$, where $k_1, k_2, \ldots, k_m$ are fixed positive integers and $0 < x < 1$. It is proved that depending on the parameter x this set can be a finite union of closed intervals or Cantor-type set or even Cantorval.

Анотація. В роботі вивчаються властивості множини підсум збіжного ряду $k_1 \sin x + \cdots + k_m \sin x + \cdots + k_1 \sin x^{\left[\frac{n-1}{m}+1\right]} + \cdots + k_m \sin x^{\left[\frac{n-1}{m}+1\right]} + \ldots$, де $k_1, k_2, \ldots, k_m$ додатні цілі числа і $0 < x < 1$. Показано, що в залежності від параметру x ця множина може бути або об'єднанням скінченного числа замкнених інтервалів, або множиною типу Кантора, або канторвалом.

1. Introduction

Definition 1.1. Let $M \subseteq \mathbb{N}$ be a subset of the set of natural numbers. Then the number

$$x = x(M) = \sum_{n \in M \subseteq \mathbb{N}} u_n = \sum_{n=1}^{\infty} \varepsilon_n u_n, \quad \varepsilon_n = \begin{cases} 1, & \text{for } n \in M, \\ 0, & \text{for } n \notin M, \end{cases} \tag{1.1}$$

is called the *subsum* of the series $\sum u_n$. We will denote by $E(u_n)$ the set of all subsums (1.1) for this series. This mathematical object is also defined by some scientists as an *achievement set* of the sequence (u_n).

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Ключові слова: множина неповних сум, мультигеометричні ряди, канторвал

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Investigation of subsums of a convergent positive, absolutely convergent, divergent, complex series has a very deep history. In the paper [6] Soichi Kakeya studied basic properties of subsums and proved that the set $E(u_n)$ for a positive convergent series is:

- a perfect set;
- a *union of finitely many mutually disjoint closed intervals* whenever the relation $u_n \leqslant r_n = \sum_{i=n+1}^{\infty} u_i$ holds for all sufficiently large n, and an *interval* if $u_n \leqslant r_n$ for any $n \in \mathbb{N}$;
- homeomorphic to the classic *Cantor set* (or shortly Cantor-type set) if the relation $u_n > r_n = \sum_{i=n+1}^{\infty} u_i$ holds for all sufficiently large n.

In the same paper Kakeya formulated a hypothesis that for any positive convergent series the set $E(u_n)$ of all subsums is either a Cantor-type set or a finite union of closed intervals. However, there exists a third type of the set of subsums. The simplest example was presented by mathematicians J. Guthrie and J. Nymann [5]. Namely, the set T of subsums for Guthrie-Nymann's series

$$\frac{3}{4} + \frac{2}{4} + \frac{3}{4^2} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{2}{4^3} + \cdots + \frac{3}{4^n} + \frac{2}{4^n} + \cdots \tag{1.2}$$

contains the interval $[2/3, 1]$, but it is not a finite union of closed intervals. That set T is called Cantorval and it has properties simultaneously similar to a nowhere dense set and to an infinite union of closed intervals. Every Cantorval is homeomorphic to the set T^* which is defined as follows:

$$T^* \equiv C \cup \bigcup_{n=1}^{\infty} G_{2n-1} = [0,1] \setminus \bigcup_{n=1}^{\infty} G_{2n},$$

where C is the Cantor ternary set, G_n is the union of the 2^{n-1} open middle thirds which are removed from $[0,1]$ at the n-th step in the construction of C. As a result of articles [5] and [8], we have the following theorem.

Theorem 1.2. *The set of subsums of a convergent positive series is one of the following three types:*

(1) *a finite union of closed intervals;*
(2) *homeomorphic to the Cantor set;*
(3) *homeomorphic to the set T (Cantorval).*

2. Multigeometric sequences and related series

A necessary and sufficient condition that the set of subsums is a Cantorval or homeomorphic to the Cantor set is still unknown. Despite essential progress for some series the problem is quite difficult in general. In this context scientists focus on series such that their terms are elements of some sequence [9] with some condition of homogeneity (depend on finite numbers of parameters and defined by a formula for the general term or by a recurrence relation).

Many papers are devoted to the study of topological and metric properties of the set of subsums of the series $\sum\limits_{n=1}^{\infty} v_n$ given by

$$k_1 + \cdots + k_m + k_1 q + \cdots + k_m q + \cdots + k_1 q^{\left[\frac{n-1}{m}\right]} + \cdots + k_m q^{\left[\frac{n-1}{m}\right]} + \ldots, \quad (2.1)$$

where $k_1, k_2, \ldots, k_m$ are fixed positive integers, $q \in (0,1)$. Terms of this series are elements of a multigeometric sequence. In the paper [2] some conditions on $E(v_n)$ to be a Cantorval or a Cantor-type set are given.

Theorem 2.1 ([2]). *Let $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$ and $K = \sum\limits_{i=1}^{m} k_i$. Assume that there exist positive integers n_0 and n^* such that each of the numbers*

$$n_0,\ n_0+1,\ n_0+2,\ \ldots,\ n_0+n^*$$

can be obtained by summing up the numbers $k_1, k_2, \ldots, k_m$. Then the set $E(v_n)$ satisfies the following conditions:

- *if $q \geqslant 1/(n^*+1)$, then $E(v_n)$ contains an interval;*
- *if $q < k_m/(K+k_m)$, then $E(v_n)$ is not a finite union of closed intervals;*
- *if $1/(n^*+1) \leqslant q < k_m/(K+k_m)$, then $E(v_n)$ is a Cantorval.*

Theorem 2.2 ([2]). *Define the following set Σ by*

$$\Sigma = \left\{ \sum_{i=1}^{m} \varepsilon_i k_i \mid (\varepsilon_i)_{i=1}^{m} \in \{0,1\}^m \right\}.$$

If $q < 1/\operatorname{card}(\Sigma)$, then $E(v_n)$ is a Cantor-type set.

This result was generalized by J. Ferdinands and T. Ferdinands in [4]. Moreover, in the same paper the authors also described another interesting family of Cantorvals generated by a multigeometric sequence.

Theorem 2.3 ([4]). *Consider the following numbers*

$$k_1 = a + 2nd, \qquad k_2 = a + (2n-2)d, \qquad \ldots,$$
$$k_{m-2} = a + 2d, \qquad k_{m-1} = a, \qquad k_m = d,$$

where $2nd < a < (2n+2)d$ *and* $n \geqslant 4$*. If*

$$\frac{1}{2n+2} \leqslant q < \min\left\{\frac{d}{a}, \frac{a-d}{(n+2)a+(n^2+n)d}\right\}$$

then $E(v_n)$ *is a Cantorval.*

Investigation of sets with complicated local structure, such as Cantorvals or fractals, is a trend in the modern mathematical research. In particular, for Guthrie-Nymann's Cantorval and its modifications were solved some topological, metric and probabilistic problems [3, 10]. The paper [1] is devoted to study of some self-similar sets having the same complicated properties as $E(v_n)$.

3. Trigonometric functions and connected series

Consider the series $\sum\limits_{n=1}^{\infty} a_n$ having the form

$$k_1 \sin x + \cdots + k_m \sin x + \cdots + \\ + k_1 \sin x^{\left[\frac{n-1}{m}+1\right]} + \cdots + k_m \sin x^{\left[\frac{n-1}{m}+1\right]} + \cdots , \tag{3.1}$$

where $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$ are fixed positive integers and $x \in (0,1)$.

Put $K = k_1 + k_2 + \cdots + k_m$. Notice that for $0 < x < 1$ the series (3.1) is convergent, since the inequality $\sin x < x$ implies that

$$\sum_{n=1}^{\infty} a_n = K \sum_{n=1}^{\infty} \sin x^n < K \sum_{n=1}^{\infty} x^n = \frac{Kx}{1-x}.$$

We will study properties of the set $E(a_n)$ for the series (3.1) depending on the initial parameters $k_1, \ldots, k_m, x$. The aim of our article is to prove that there exists a non-multigeometric series whose set of subsums is a Cantorval.

Theorem 3.1. *If the series* (3.1) *satisfies the condition*

$$x \geqslant \frac{d}{K+d},$$

where

$$d = \max_{1 \leqslant j \leqslant m} d_j, \qquad d_j = \frac{\pi}{2} k_j - k_{j+1} - k_{j+2} - \cdots - k_m,$$

then $E(a_n)$ *is an interval.*

Proof. For any $n = im + j$, where $i \in \mathbb{N}_0$ and $j = \overline{1,m}$, we have that

$$a_{im+j} = k_j \cdot \sin x^{i+1},$$

$$r_{im+j} = k_{j+1}\sin x^{i+1} + k_{j+2}\sin x^{i+1} + \cdots + k_m \sin x^{i+1} + K\sum_{p=i+2}^{\infty}\sin x^p.$$

Taking into account Jordan's inequality [7]

$$\frac{2x}{\pi} < \sin x < x \quad \text{for} \quad 0 < x < \frac{\pi}{2},$$

we get

$$a_{im+j} < k_j x^{i+1},$$

$$r_{im+j} > \frac{2}{\pi}\left(K\frac{x^{i+2}}{1-x} + k_{j+1}x^{i+1} + \cdots + k_m x^{i+1}\right).$$

Thus, for $x \geqslant d_j/(K+d_j)$, where $d_j = \frac{\pi}{2}k_j - k_{j+1} - k_{j+2} - \cdots - k_m$, the following inequality holds

$$a_{im+j} < k_j x^{i+1} \leqslant \frac{2}{\pi}\left(K\frac{x^{i+2}}{1-x} + k_{j+1}x^{i+1} + \cdots + k_m x^{i+1}\right) < r_{im+j}.$$

Moreover, if $x \geqslant d/(K+d)$, where $d = \max\limits_{1\leqslant j\leqslant m} d_j$, then the inequality $a_n < r_n$ holds for any $n \in \mathbb{N}$. Hence, due to Kakeya's result, the set $E(a_n)$ is an interval. □

Example 3.2. For the case $k_1 = 2$ and $k_2 = 1$ we get that $K = 3$, $d_1 = \pi - 1$, and $d_2 = \pi/2$. Then, according to Theorem 3.1, for any values of parameter x satisfying the inequality

$$x \geqslant \frac{(\pi-1)}{3+(\pi-1)},$$

the set $E(a_n)$ is an interval. Consequently, for the series

$$2\sin\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) + \cdots + 2\sin\left(\frac{1}{2}\right)^{\left[\frac{n+1}{2}\right]} + \sin\left(\frac{1}{2}\right)^{\left[\frac{n+1}{2}\right]} + \cdots,$$

where $x = 1/2$, the set of its subsums is an interval.

Theorem 3.3. *If the series* (3.1) *satisfies the condition*

$$x \leqslant \frac{2k_m}{K\pi + 2k_m},$$

then $E(a_n)$ is not a finite union of closed intervals.

Proof. We will show that $a_n > r_n$ holds for infinitely many numbers of n. According to Jordan's inequality, for any $i \in \mathbb{N}$ we have that

$$r_{mi} = K\sum_{p=i+1}^{\infty}\sin x^p < K\sum_{p=i+1}^{\infty}x^p = \frac{Kx^{i+1}}{1-x},$$

$$a_{mi} = k_m \sin x^i > \frac{2k_m x^i}{\pi}.$$

It is easy to see that for $x \leqslant 2k_m/(K\pi + 2k_m)$ the following inequality holds

$$r_{mi} < \frac{Kx^{i+1}}{1-x} \leqslant \frac{2k_m x^i}{\pi} < a_{mi}.$$

So $a_{mi} > r_{mi}$ for any $i \in \mathbb{N}$, whence, due to Kakeya's result and its generalization, $E(a_n)$ is not a finite union of intervals. □

Theorem 3.4. *Suppose that for the series* (3.1) *there exist numbers* n_0 *and* n^* *such that every number* $n_0, n_0+1, n_0+2, \ldots, n_0+n^*$ *can be obtained by summing up the numbers* $k_1, k_2, \ldots, k_m$*, and parameter* x *satisfies the inequality*

$$x \geqslant \frac{\pi}{2n^* + \pi}.$$

Then $E(a_n)$ *contains an interval.*

Proof. According to the statement for any $h \in \{0, 1, 2, \ldots, n^*\}$ there exists sequence $(c_i)_{i=1}^m$ of 0 and 1 such that

$$n_0 + h = \sum_{i=1}^{m} c_i k_i.$$

From it follows that every number t having the representation

$$t = \sum_{i=1}^{\infty} (n_0 + h_i) \sin x^i = n_0 \sum_{i=1}^{\infty} \sin x^i + \sum_{i=1}^{\infty} h_i \sin x^i, \quad h_i \in \{0, 1, \ldots, n^*\},$$

is contained on $E(a_n)$. The set of all such numbers t is an arithmetic sum of a constant and the set of subsums for the series

$$\sum_{n=1}^{\infty} b_n = \underbrace{\sin x + \cdots + \sin x}_{n^*} + \cdots + \underbrace{\sin x^{\left[\frac{n-1}{n^*}+1\right]} + \cdots + \sin x^{\left[\frac{n-1}{n^*}+1\right]}}_{n^*} + \ldots,$$

where every term repeats n^* times. It is obvious, that $\sum\limits_{p=n+1}^{\infty} b_p = r_n^* > b_n$ for $n \neq in^*$ and $i \in \mathbb{N}$. Taking into account Jordan's inequality, we have

$$b_{n^*i} = \sin x^i < x^i,$$

$$r^*_{n^*i} = n^*(\sin x^{i+1} + \sin x^{i+2} + \cdots) > n^* \left(\frac{2x}{\pi} + \frac{2x^2}{\pi} + \cdots \right) = \frac{2n^* x}{\pi(1-x)}.$$

It is easy to show that for $x \geqslant \pi/(2n^* + \pi)$ the following inequality holds

$$b_{mi} < x^i \leqslant \frac{2n^* x^{i+1}}{\pi(1-x)} < r^*_{mi}.$$

Thus, in this case $b_n < r_n^*$ for any $n \in \mathbb{N}$. Then, due to Kakeya's result, the set $E(b_n)$ is an interval. Since

$$n_0 \sum_{i=1}^{\infty} \sin x^i + E(b_n) \subset E(a_n),$$

we obtain that the set $E(a_n)$ contains an interval. □

Corollary 3.5. *Suppose that for the series* (3.1) *there exist numbers n_0 and n^* such that every number $n_0, n_0+1, n_0+2, \ldots, n_0+n^*$ can be obtained by summing up the numbers $k_1, k_2, \ldots, k_m$ and parameter x satisfies the inequality*

$$\frac{\pi}{2n^* + \pi} \leqslant x \leqslant \frac{2k_m}{K\pi + 2k_m}.$$

Then the set $E(a_n)$ is a Cantorval.

Example 3.6. For $k_1 = 8, k_2 = 7, k_3 = 6, k_4 = 5, k_5 = 4$ we have that $K = 30$ and $n_0 = 4, n^* = 22$. Then, according to the Corollary 3.5, for all x satisfying the condition

$$\frac{\pi}{44 + \pi} \leqslant x \leqslant \frac{8}{30\pi + 8}$$

the set $E(a_n)$ is a Cantorval. As a consequence for the series

$$8\sin\frac{1}{15} + 7\sin\frac{1}{15} + 6\sin\frac{1}{15} + 5\sin\frac{1}{15} + 4\sin\frac{1}{15} + \cdots + 8\sin\frac{1}{15^{\left[\frac{n+4}{5}\right]}} +$$
$$+ 7\sin\frac{1}{15^{\left[\frac{n+4}{5}\right]}} + 6\sin\frac{1}{15^{\left[\frac{n+4}{5}\right]}} + 5\sin\frac{1}{15^{\left[\frac{n+4}{5}\right]}} + 4\sin\frac{1}{15^{\left[\frac{n+4}{5}\right]}} + \cdots,$$

when $x = 1/15$, the set of subsums is a Cantorval.

Theorem 3.7. *Suppose that the series* (3.1) *satisfies the following condition*

$$x \leqslant \frac{l}{K + l},$$

where

$$l = \min_{1 \leqslant j \leqslant m} l_j, \qquad l_j = \frac{2}{\pi} k_j - k_{j+1} - k_{j+2} - \cdots - k_m.$$

Then $E(a_n)$ is a Cantor-type set.

Proof. For any $n = im + j$, where $i \in \mathbb{N}_0$, $j = \overline{1, m}$ we have

$$a_{im+j} = k_j \cdot \sin x^{i+1},$$

$$r_{im+j} = k_{j+1} \sin x^{i+1} + k_{j+2} \sin x^{i+1} + \cdots + k_m \sin x^{i+1} + K \sum_{p=i+2}^{\infty} \sin x^p.$$

Taking into account Jordan's inequality, we have

$$a_{im+j} > \frac{2}{\pi} k_j x^{i+1},$$

$$r_{im+j} < K\frac{x^{i+2}}{1-x} + k_{j+1}x^{i+1} + \cdots + k_m x^{i+1}.$$

Thus, for $x \leqslant l_j/(K+l_j)$, where $l_j = \frac{2}{\pi}k_j - k_{j+1} - k_{j+2} - \cdots - k_m$, the following inequality holds:

$$a_{im+j} > \frac{2}{\pi} k_j x^{i+1} \geqslant K\frac{x^{i+2}}{1-x} + k_{j+1}x^{i+1} + \cdots + k_m x^{i+1} > r_{im+j}.$$

Moreover, if $x \leqslant l/(K+l)$, where $l = \min_{1 \leqslant j \leqslant m} l_j$, then $a_n > r_n$ for any $n \in \mathbb{N}$. Thus, due to Kakeya's result, the set $E(a_n)$ is a Cantor-type set. □

Example 3.8. For $k_1 = 4$ and $k_2 = 1$ we have that $K = 5$ and $l_1 = 8/\pi - 1$, $l_2 = 2/\pi$. Then, according to the Theorem 3.7, for any value x satisfying the condition

$$x \leqslant \frac{2/\pi}{5 + 2/\pi}$$

the set $E(a_n)$ is a Cantor-type set. Consequently, for the series

$$4\sin\left(\frac{1}{10}\right) + \sin\left(\frac{1}{10}\right) + \cdots + 4\sin\left(\frac{1}{10}\right)^{\left[\frac{n+1}{2}\right]} + \sin\left(\frac{1}{10}\right)^{\left[\frac{n+1}{2}\right]} + \cdots,$$

where $x = 1/10$, the set of subsums is a Cantor-type set.

4. Open problems and hypothesis

As a result of the paper we find a certain sufficient condition for $E(a_n)$ to be an interval, Cantor-type set or Cantorval. However, there still exist a few open problems:

- for some values of the parameter x the topological type of $E(a_n)$ is still unknown (e.g. for $2k_m/(\pi K + 2k_m) < x < d/(K+d)$);
- it is not clear how to compute the Lebesgue measure of Cantorvals generated by main series;
- we can't still find the necessary and sufficient condition for $E(a_n)$ to have a zero Lebesgue measure;
- t is also not clear how to compute the Hausdorff-Besicovitch dimension of $E(a_n)$ in the case when it has zero Lebesgue measure.

We also described some properties of the set of subsums for the series (3.1) in the Section 3 by means of properties of its terms

$$k_j \frac{2x^n}{\pi} < k_j \sin x^n < k_j x^n$$

for any $n \in \mathbb{N}$, $j \in \{1, 2, \ldots, m\}$ and $0 < x < 1$. Let us note that terms of the series (3.1) are located between terms of two multigeometric series (2.1) with $q_1 = 2x/\pi$ and $q_2 = x$. These observations lead to the following main hypothesis.

Conjecture 4.1. *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be convergent positive series with the same types of the set of subsums (a finite union of intervals, a Cantor-type set or a Cantorval). If* $\sum_{n=1}^{\infty} c_n$ *satisfies the conditions*

$$a_n < c_n < b_n \qquad \text{and} \qquad \left[\begin{array}{l} b_n \leqslant r_n^a = \sum_{n=1}^{\infty} a_n, \\ \sum_{i=n+1}^{\infty} b_i = r_n^b < a_n, \end{array}\right. \tag{4.1}$$

for any $n \in \mathbb{N}$, *then* $E(a_n)$, $E(b_n)$, $E(c_n)$ *have the same topological type.*

It is easy to prove this statement for case when $E(a_n)$ and $E(b_n)$ are both a finite union of closed intervals. We also can prove the hypothesis for case when inequalities $a_n \leqslant r_n^a$ and $b_n \leqslant r_n^b$ hold only for finite numbers of n. In general case this hypothesis is not obvious and not so easy to prove.

References

[1] Taras Banakh, Artur Bartoszewicz, Małgorzata Filipczak, and Emilia Szymonik. Topological and measure properties of some self-similar sets. *Topol. Methods Nonlinear Anal.*, 46(2):1013–1028, 2015. doi:10.12775/TMNA.2015.075.

[2] Artur Bartoszewicz, Mał gorzata Filipczak, and Emilia Szymonik. Multigeometric sequences and Cantorvals. *Cent. Eur. J. Math.*, 12(7):1000–1007, 2014. doi:10.2478/s11533-013-0396-4.

[3] Wojciech Bielas, Szymon Plewik, and Marta Walczyńska. On the center of distances. *Eur. J. Math.*, 4(2):687–698, 2018. doi:10.1007/s40879-017-0199-4.

[4] J. Ferdinands and T. Ferdinands. A family of Cantorvals. *Open Math.*, 17(1):1468–1475, 2019. doi:10.1515/math-2019-0109.

[5] J. A. Guthrie and J. E. Nymann. The topological structure of the set of subsums of an infinite series. *Colloq. Math.*, 55(2):323–327, 1988. doi:10.4064/cm-55-2-323-327.

[6] S. Kakeya. On the partial sums of an infinite series. *The Science Reports of the Tôhocu University*, 3:159–164, 1914. doi:10.11429/PTMPS1907.7.14_250.

[7] Da-Wei Niu, Jian Cao, and Feng Qi. Generalizations of Jordan's inequality and concerned relations. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.*, 72(3):85–98, 2010. URL: https://www.scientificbulletin.upb.ro/rev_docs_arhiva/full3105.pdf.

[8] J. E. Nymann and R. A. Sáenz. On a paper of Guthrie and Nymann on subsums of infinite series. *Colloq. Math.*, 83(1):1–4, 2000. doi:10.4064/cm-83-1-1-4.

[9] Mykola Pratsiovytyi and Dmytro Karvatsky. Jacobsthal-Lucas series and their applications. *Algebra Discrete Math.*, 24(1):169–180, 2017. URL: https://admjournal.luguniv.edu.ua/index.php/adm/article/download/297/pdf.

[10] Mykola Pratsiovytyi and Dmytro Karvatsky. The set of subsums for modified Guthrie-Nymann's series. *Bukovinian Math. Journal*, 10(2):195–203, 2022. (in Ukrainian). doi: 10.31861/bmj2022.02.15.

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