Proceedings of the
International Geometry Center
Vol. 16, no. 3-4 (2023) pp. 231–243

Topological structure of functions with isolated critical points on a 3-manifold

Bohdana Hladysh, Maria Loseva, Alexandr Prishlyak

Abstract. To each isolated critical point of a smooth function on a 3-manifold we put in correspondence a tree (graph without cycles). We will prove that two such functions are topologically equivalent in some neighborhoods of critical points if and only if the corresponding trees are isomorphic. A complete topological invariant of functions with four critical points, on a closed 3-manifold, was constructed. A criterion for the topological equivalence of functions with a finite number of critical points on 3-manifolds is given.

Анотація. Кожній ізольованій критичній точці гладкої функції на 3-многовиді поставлено у відповідність дерево (граф без циклів). В роботі доведено, що дві такі функції на 3-многовиді топологічно еквівалентні в околі критичних точок тоді і тільки тоді, коли відповідні дерева ізоморфні. Побудовано повний топологічний інваріант функцій з ізольованими критичними точками на замкнутому 3-многовиді, та встановлено критерій топологічної еквівалентності функцій з довільним числом критичних точок на тривимірних многовидах.

1. Introduction

There are many papers focused on topological properties of functions defined on manifolds. The first ones on this topic were Kronrod's [7] and Reeb's [16] papers. Let M be a smooth 3-manifold. Two smooth functions $f, g\colon M \to \mathbb{R}$ are called *topologically equivalent*, if there are homeomorphisms $h\colon M \to M$ and $k\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ h = k \circ g$.

We say that f and g are *topologically conjugated*, if they are topologically equivalent and homeomorphism k preserves orientation. In this case h and k will be called by *conjugated homeomorphisms.*

2020 Mathematics Subject Classification: 37C10, 37C15, 37C20.
Keywords: topological equivalence; critical point; 3-manifold.
DOI: http://dx.doi.org/10.15673/pigc.v16i3.2512

Notice that functions without critical points can be topologically equivalent to function with critical points. For example, consider the following two functions

$$f, g : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x, y) = x^2 + y, \qquad g(x, y) = x^2 + y^3.$$

Then we have a topological equivalence between f and g given by the homeomorphisms $h(x, y) = (x, y^3)$ and $k = \mathrm{id}_{\mathbb{R}}$, and f has no critical points, while g has a critical point $(0, 0)$.

A critical point of a function f is *removable* if the function f is topologically equivalent to the function without critical points in some neighborhoods of this point.

Differentiable equivalence is studied in the theory of singularities and means a topological equivalence in which the conjugated homeomorphisms are diffeomorphisms.

The problem of topological classification of Morse functions was solved in [1, 2, 4–6, 9, 10, 17] for manifolds of different dimensions. The same result for arbitrary functions with isolated critical point on closed 2-manifolds was obtained in [12]. The relevance of this problem is contributed by the close connection with the Hamiltonian dynamical system's classification in dimensions 2 and 4. In another way, the structure of Morse functions can be described by setting the structure of its gradient field, which is the Morse-Smale field, and the order of the values of the function at critical points [11, 13–15].

In this paper we give a local topological classification of functions with isolated critical points and a global topological classification of smooth functions with four critical points on closed 3-manifolds.

Takens [18, Lemma 3.2] has proven that an isolated critical point of a smooth function on 3-manifold has conic type. We use this result for local topological classification of functions.

In Section 2, we construct a certain graph Gf_p for a critical point p of a function f, being the dual graph to the set of circles embedded in the 2-sphere. Here the sphere is the boundary of a regular neighborhood of the critical point, and the circles are the intersection of the critical level with it.

The main statement of that section is Theorem 2.7, which states that functions are topologically equivalent in some neighborhoods of isolated critical points, except extreme points, if and only if their graphs at these points are equivalent.

To prove such a result, we use a gradient-like vector field and construct a canonical neighborhood of the critical point. Using graph isomorphisms, we construct homemorphisms of areas on the boundary of the canonical neighborhood, and that homeomorphisms defined the correspondences between trajectories. The restriction of a function to each trajectory yields a homeomorphism to the image of that trajectory, and this is used to construct homeomorphisms of the corresponding trajectories, which together gives the desired homeomorphism between neighborhoods.

In Section 3 we consider functions with four critical points. An existence of such functions on any 3-manifold follows from [18, Theorem 3.3]. We use the obtained local classification to construct an analogue of Heegaard diagrams for such functions and use these diagrams for the global classification.

In this case the diagram consists of a regular surface of the function level between the second and third critical value. There are two sets of nested circles on it, which correspond to circles on regular neighborhoods of the second and third critical points. This correspondence is specified by movement along the trajectories of a gradient field or a gradient-like field. Since these fields are constructed ambiguously, this ambiguity leads to semi-isotopy diagrams (when the nested circles of the first set remain fixed, and the second ones are replaced by isotopic ones). A complete invariant of a function is a diagram, which, in addition to the diagram, includes graphs, one of which is responsible for gluing regular neighborhoods of the first and second critical points, and the second graph for the third and fourth points. In addition, the diagram includes a correspondence between the circles of the diagram and some vertices of the graph. Main Theorem 3.6 of this section states that functions are topologically conjugate if and only if their circuits are semi-isotopic.

In Section 4 we give a criterion for the topological equivalence of functions with an arbitrary number of critical points. To do this, we generalize a handle decomposition, which is constructed for Morse functions, to the case of functions with isolated critical points. Main Theorem 4.1 of this section states that functions are topologically equivalent if they have isomorphicm handle decompositions.

2. Topological structure of a neighborhood of a critical point

Let f be a smooth function on a smooth 3-manifold M. It is known [18, Lemma 3.2] that if p is an isolated critical point and $y = f(p)$, then there exists a closed neighborhood $U(p)$ and a homeomorphism

$$f^{-1}(y) \cap U(p) \;\cong\; \mathrm{Con}\left(\bigcup S_i^1\right),$$

where $\mathrm{Con}(\cup S_i^1)$ is a cone over a disjoint union of circles S_i^1, that is the union of two-dimensional disks, whose centers are pasted together into the point p.

In order to describe the behavior of a function in a neighborhood of a critical point p we will construct a graph Gf_p. Let $U(p)$ be the neighborhood described above, whose boundary is a sphere S^2 and

$$\partial\left(f^{-1}(y)\cap U(p)\right)=\bigcup S_i^1$$

is the union of the embedded circles. To each component D_j of $S^2\backslash\left(\bigcup S_i^1\right)$ we put in correspondence a vertex v_j of the graph Gf_p and to each circle S_i^1 we associate an edge e_i. The vertex v_j is incident to e_i if and only if the boundary of D_j contains S_i^1. Thus, v_i and v_j are connected by an edge if D_i and D_j are neighbor.

Definition 2.1. *A graph of a function f in a point p* is the graph Gf_p consructed as above. So the graph of a function f in a point p is the dual graph to the set of $\bigcup_i S_i^1$ in $\partial U(p)$.

Example 2.2. The function $f(x,y,z)=x^2+y^2-z^2$ has $k=2$ circles, while the function

$$f(x,y,z)=(x^2+y^2-z^2)(x^2+y^2-4z^2)(x^2-4y^2+z^2)$$

has $k=6$ circles at the critical point p being the origin of $\mathbb{R}^3$. The location of these circles on the sphere, as well as the corresponding graphs Gf_p is shown in Figure 2.1. (The sphere is regarded as a plane with a point at infinity).

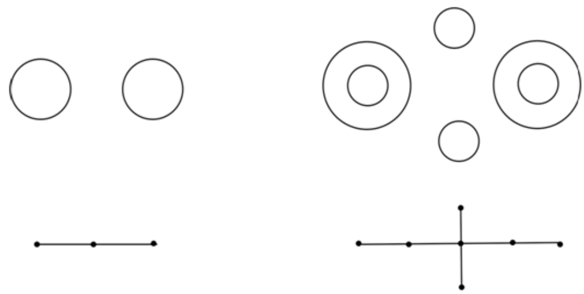

FIGURE 2.1.

Lemma 2.3. *The graph Gf_p is a tree. If p is a local extreme, then Gf_p is a trivial graph, i.e. it has one vertex and no edges.*

Proof. Since, according to Jordan curve theorem, every closed curve on a sphere splits it into two parts, then the corresponding edge on the dual graph splits the graph. If the graph had a non-trivial cycle, then the edges of this cycle would not split the graph. Therefore, Gf_p is a tree.

The second statement of lemma is evident. □

For a function f and its isolated critical point p we define a neighborhood W_p of a point p in $f^{-1}(f(p))$, which is homeomorphic to Con $\left(\bigcup S_i^1\right)$. Following Takens [18, Lemma 3.2] let

$$W_p(\varepsilon) = \left\{x \in M : |f(x) - y_0| < \varepsilon,\ \mathrm{cl}(\gamma(x)) \cap W_p \neq \varnothing\right\}$$

be a neighborhood $W_p(\varepsilon)$ of p in $\mathbb{R}^3$ for $\varepsilon > 0$, where $\gamma(x)$ is the integral trajectory of the gradient field of f containing the point x.

Definition 2.4. The above neighborhood $W_p(\varepsilon)$ will be called *canonical.*

Let $W_p(\varepsilon)$ be a canonical neighborhood. Then its boundary $\partial W_p(\varepsilon)$ is a union:

$$\partial W_p(\varepsilon) = V_- \cup V_0 \cup V_+,$$

where

$$\begin{aligned}
V_- &= \partial W_p(\varepsilon) \cap f^{-1}(f(p) - \varepsilon),\\
V_+ &= \partial W_p(\varepsilon) \cap f^{-1}(f(p) + \varepsilon),\\
V_- &= \mathrm{cl}\big(\partial W_p(\varepsilon) \backslash (V_- \cup V_+)\big) = \mathrm{cl}\big(\partial W_p(\varepsilon) \cap f^{-1}(f(p) - \varepsilon, f(p) + \varepsilon)\big).
\end{aligned}$$

We will say that V_+ and V_- are the *upper* and the *lower bases*, accordingly, and V_0 is the *side wall* of the neighborhood $W_p(\varepsilon)$.

The side wall is a union of closed neighborhoods of circles S_i^1. Therefore,

$$V_0 = \bigcup S_i^1 \times [-\varepsilon, \varepsilon].$$

Thus, for every i and $t \in [-\varepsilon, \varepsilon]$ we have that

$$S_i^1 \times \{t\} \subset f^{-1}(f(p) + t),$$

and for each $s \in S_i^1$:

$$s \times [-\varepsilon, \varepsilon] \subset \gamma(s, 0).$$

On each cylinder $S_i^1 \times [-\varepsilon, \varepsilon]$ the level lines of the function and the integral curves define a structure of a direct product.

Denote by D_i' the connected component of V_+ or V_- belonging to D_i. Then

$$D_i' = D_i \backslash \cup S_i^1 \times (-\varepsilon, \varepsilon).$$

Let $W_i = Con(S_i^1) \subset f^{-1}(f(p))$ and D_i'' be the set of those points from D_i' whose integral trajectories have p as a limit point, i.e.

$$D_i'' = \{x \in D' | \gamma(x) \cap W = \varnothing\}.$$

Then D_i'' is a deformation retract of both D_i' and D_i see Example 2.8.

For definiteness, let $f(D_i'') = f(p) + \varepsilon$. We will construct a new vector field X' on the set

$$U_i' = \left\{x \in \mathrm{cl}(W_p\varepsilon) : \gamma(x) \cap W \neq \varnothing, f(x) \geqslant f(p)\right\} \ \cong\ (0,1] \times S_i^1 \times [0, \varepsilon].$$

In order to do that we will consider coordinates (u, s, t) on U_i'. The coordinate t of a point x is equal to $f(x) - f(p)$. Since $\mathrm{Con}(S_i^1)\backslash\{p\}$ is homeomorphic to $(0, 1] \times S_i^1$, coordinates u and s at $t = 0$ are defined by that homeomorphism. For an arbitrary point $x \in U_i'$ we will choose coordinates u and s to be equal to the coordinates u and s of the point $\gamma(x) \cap \mathrm{cl}(W_p)$. The existence of such coordinates follows from the tube theorem for flows (or from the theorem of rectification of a vector field). Since the integral curves of the vector field ∇f coincide with coordinate lines, it has coordinates $(0, 0, v(x))$, where $v(x) > 0$ for each point $x \in U_i$.

Define the following vector field X':

$$X' = \begin{cases} \left(\dfrac{u \cdot v(x)}{\sqrt{u^2 + t^2}}, \dfrac{t \cdot v(x)}{\sqrt{u^2 + t^2}}, 0\right), & \text{if } t \geqslant 2u\varepsilon, \\ \left(\dfrac{(1-u) \cdot v(x)}{\sqrt{(1-u)^2 + (2\varepsilon - t)^2}}, \dfrac{(2\varepsilon - t) \cdot v(x)}{\sqrt{(1-u)^2 + (2\varepsilon - t)^2}}, 0\right), & \text{if } t \leqslant 2\varepsilon u. \end{cases}$$

Definition 2.5. The vector field X' that is constructed in a such way will be called an *inclined* vector field.

Lemma 2.6. *An inclined vector field X' has a following properties:*
- *X' is a gradient-like vector field;*
- *X' coincides with X at points with coordinate $u = 1$ and at $u \to 0$;*
- *the set $D_i'\backslash D_i''$ for X' consists of points with coordinate $u > 1/2$;*
- *all inclined vector fields of a function f are topologically equivalent.*

Proof. The first three properties follow directly from the formula defining the field X'. The proof of the last property follows from the proof of the following theorem. ☐

Theorem 2.7. *Let p and q be isolated critical points of smooth functions $f\colon \mathbb{R}^3 \to \mathbb{R}^1$ and $g\colon \mathbb{R}^3 \to \mathbb{R}^1$ correspondingly. Then there are neighborhoods U of p and V of q and homeomorphisms $h\colon U \to V$ and $k\colon \mathbb{R} \to \mathbb{R}$ such that $f \circ h = k \circ g$ if and only if the graphs Gf_p and Gg_q are isomorphic.*

Proof. Necessity. It follows from the construction of the graphs that the restriction of a homeomorphism h to the boundaries of the canonical neighborhoods will determine the required isomorphism of the corresponding graphs.

Sufficiency. Fix an isomorphism $i\colon Gf_p \to Gg_q$. Let $W_p(\varepsilon)$ be a canonical neighborhood of point p and $\pi\colon W_p(\varepsilon) \to W_p$ be a map given by the formula

$$\pi(x) = \begin{cases} p, & \text{if } \gamma(x) \cap f^{-1}(f(p)) = \varnothing, \\ \gamma(x) \cap f^{-1}(f(p)), & \text{if } \gamma(x) \cap f^{-1}(f(p)) \neq \varnothing. \end{cases}$$

For q and its canonical neighborhood $W_q(\varepsilon)$ define π in a similar way. We will now construct a homeomorphism of the boundary spheres

$$H\colon \partial W_p(\varepsilon) \to \partial W_q(\varepsilon)$$

such that at each point $x \in W_p(\varepsilon)$:

$$|f(x) - f(p)| = |g(H(x)) - g(q)|.$$

The isomorphism $i\colon Gf_p \to Gg_q$ determines a correspondence between disks D_j of two functions and also a correspondence between circles S^1_i. Choose arbitrary orientation of the edges of one of the graphs and orient the edges of another graph in a such way that isomorphism of the graphs preserves the orientation. Fix also orientations of the spheres. Then the orientation of the edges of the graphs determines the orientation of the circles. Fix further arbitrary orientation preserving homeomorphisms φ_i between the corresponding circles. These homeomorphisms multiplied by identity map of a segment $[-\varepsilon, \varepsilon]$ determine homeomorphisms of cylinders. Also, the products of φ_i by the identity map of $(0, 1]$ define homeomorphisms of cones $\mathrm{Con}(S^1_i)$. We can extend homeomorphisms of cones on sets U'_i using inclined vector field and coordinates that are relevant to them. It follows from the construction of inclined fields that they define homeomorphisms of boundaries of region U_i (on equality of the relevant coordinates).

Thus, we get homeomorphisms of boundaries of D''_i. Extend them inside of D''_i arbitrarily. These homeomorphisms define correspondences of integral trajectories. The correspondence of points of trajectories is given by equality of a difference of values of functions in them with a value at critical points.

It follows from the construction that the obtained map is a homeomorphism sending levels of the function f into levels of the function g. □

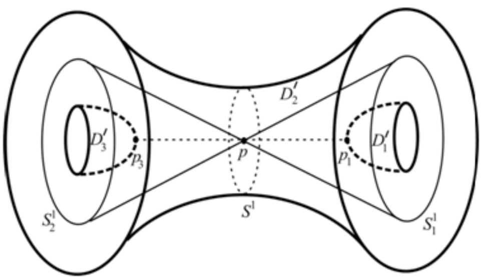

FIGURE 2.2.

Example 2.8. In accordance of the notations described in the proof of Theorem 2.7, for the function $f(x, y, z) = x^2 - y^2 + z^2$ in standard Riemannian metric, the neighborhood $W_p(\epsilon)$ is shown in Figure 2.2, wherein $V_- = D_1' \cup D_3'$, $V_+ = D_2'$, $D_1'' = p_1$, $D_2'' = S^1$, $D_3'' = p_3$.

3. Functions with 4 critical points

Let M be a closed oriented 3-manifold and $f\colon M \to \Re$ be a smooth function with 4 isolated critical points p_1, p_2, p_3, p_4 and correspondent critical values $y_i = f(x_i)$, $i = 1, 2, 3, 4$ such that $y_i < y_j$, if $i < j$. Thus, p_1 is a minimum point and p_4 is a maximum point.

We denote by $U_i = W_{p_i}(\varepsilon)$ a neighborhoods of the points p_i, $i = 2, 3$, defined in Section 2. Let U_1 be a connected component of

$$M \backslash (\mathrm{cl}(U_2) \cup f^{-1}(p_2)),$$

containing a point p_1, and U_4 be a component of

$$M \backslash (\mathrm{cl}(U_3) \cup f^{-1}(p_3)),$$

containing a point p_4.

Consider the surfaces

$$F = \partial(U_1 \cup U_2) \quad \text{and} \quad F' = \partial(U_3 \cup U_4).$$

According to the construction they are homeomorphic to a regular level $f^{-1}(z)$, where $y_2 < z < y_3$. Moreover, part M_0 of the manifold M, which is located between them is homeomorphic to the cylinder $F \times [0, 1]$. Denote by $\{u_i\}$ closed curves of $\mathrm{cl}(U_1) \cap \mathrm{cl}(U_2) \cap M_0$ and by $\{v_i'\}$ of $\mathrm{cl}(U_3) \cap \mathrm{cl}(U_4) \cap M_0$. Let π be the projection of top base of cylinder M_0 to bottom base and let $v_i = \pi(v_i')$.

Definition 3.1. The surface F constructed in such a way together with two sets of closed curves $\{u_i\}$, $\{v_i\}$ on it is called a *diagram* of the function f and denoted by D_f. In this case, the curves $\{u_i\}$, $\{v_i\}$ are called *meridians.*

The diagram of function describes the attaching $U_1 \cup U_2$ to $U_3 \cup U_4$. Such diagram is similar to the Heegaard diagram of a 3-dimensional manifold and for one can define concepts of isomorphism and semiisotopy of the diagrams [3, 8].

Definition 3.2. A diagram $D = \{F, u, v\}$ is said to be *isomorphic* to a diagram $D' = \{F', u', v'\}$ if there is a homeomorphism $h\colon F \to F'$ such that $h(u) = u'$ and $h(v) = v'$. Diagram $D = \{F, u, v\}$ is *semiisotopic* to $D' = \{F, u, v'\}$ if there is an isotopy $g_t\colon F \to F$, $t \in [0, 1]$, such that $g_0 = \mathrm{id}$ and $g_1(v) = v'$.

Definition 3.3. A diagram is called *normalized* if it does not contain two-angles, that are two arcs from different meridian systems, with common ends and without internal points of intersection.

In the construction of the diagram of a function we have an ambiguity in the choice of a structure of a direct product on M_0. Thus, the change of the direct product structure induces an isotopy of curves v_i, leaving curves u_i invariant. On the contrary, each isotopy induces a change from a fixed direct product structure to a new such structure. Thus, the obtained diagrams are semiisotopic. Using semiisotopy (choices of a direct product structure on M_0) we cancel all two-angles in the diagram and obtain the normalized diagram.

Two curves u_i and v_j are called *parallel* if they are isotopic in the complement to other curves, i.e. if they form the boundary of a connected component homeomorphic to $S^1 \times [0,1]$, obtained by splitting F by curves u_i and v_j.

Theorem 3.4. *Two normalized diagrams are semiisotopic if and only if one can pass from the first diagram to the second one by isotopies consisting of permutations of parallel curves.*

Proof. If diagrams do not contain parallel curves, the proof coincides with the proof of the similar statement for Heegaard diagrams [3, Theorem 5.3]. If the parallel curves intersect in two points, then they form two two-angles. Depending on the way of reducing the two-angle we can obtain two distinct normalized diagrams. These diagrams differ by permutation of two parallel curves.

The converse statement is obvious: the permutation of two parallel curves can be obtained using the semiisotopy. □

We will now describe the construction of the graph G_1 defining the gluing of U_2 with U_1. Consider dual graphs of partitioning boundaries into regions. The vertices corresponding to the areas belonging to U_1 and not belonging to U_2 we paint in the first color, those that correspond to the intersections of U_1 and U_2 we paint in the second color, and the remaining vertices of the second graph in the third color. Edges that are incident to vertices of the second color are divided into two half-edges. Then the graph G_1 is obtained from the disjoint union of these two dual graphs after gluing the corresponding vertices of the second color and the half-edges incident to them in accordance with gluing the boundaries of U_1 and U_2.

Each non-colored vertex of degree 3 (we denote by V_0 the set of such vertices) corresponds to a meridian from the first system of meridians. Vertices colored in the first and third colors (we denote the sets of such vertices V_1

and V_3, accordingly), correspond to components into which the first system of meridians divides a surface. Thus, we have a bijection:

$$\psi_1 \colon \{V_0, V_1 \cup V_3\} \to \{\{u_i\}, \pi_0(F \setminus \cup u_i)\}.$$

Similar correspondences ψ_2 arise for the graph G_2 and the second system of meridians.

Definition 3.5. *A scheme* of function f is a quintuple $\{D_f, G_1, \psi_1, G_2, \psi_2\}$, consisting of the diagram D_f of the function, two graphs G_1, G_2, and maps of correspondence ψ_1, ψ_2. Two schemes are called *equivalent* if there are isomorphisms of the diagrams and the graphs agreeing with the maps of correspondence.

Theorem 3.6. *Two functions f and g with 4 critical points on 3-manifolds M, N correspondingly are topologically conjugate if and only if the scheme of one of them is equivalent to the scheme obtained from another one by a semiisotopy of its diagram.*

Proof. Necessity follows from the construction and previous arguments.

Sufficiency. Without loss of generality we may assume that the functions have critical values -2, -1, 2, 3. Similarly to Theorem 3.4 we construct a homeomorphism from $f^{-1}([-2,0])$ to $g^{-1}([-2,0])$ and a homeomorphism from $f^{-1}([1,3])$ to $g^{-1}([1,3])$. Thus, on the sets $f^{-1}([0,1])$ and $g^{-1}([0,1])$ the structures of a direct product are fixed which were used in construction of the diagrams. Then the semiisotopy of the diagrams can be considered as a level-by-level homeomorphism between $f^{-1}([0,1])$ and $g^{-1}([0,1])$. Due to assumptions of the theorem it coincides on the boundaries with the constructed above homeomorphisms from $f^{-1}([-2,0])$ and $f^{-1}([1,3])$, and thus it is an extension of these homeomorphisms up to the required homeomorphism of the manifolds. □

4. Function with an arbitrary finite number of critical points

Let $p_1, \ldots, p_k$ be critical points such that $f(p_1) \leqslant f(p_2) \leqslant \ldots \leqslant f(p_k)$. Fix a Riemannian metric on the manifold and small enough mutually disjoint closed neighborhoods $W_1, \ldots, W_k$ of critical points having the same structure as $W_p(\varepsilon)$ in Lemma 2.3.

Our aim is to construct analogue of a handle decomposition that is the decomposition of the manifold into a union $M = H_1 \cup \ldots \cup H_k$, where $W_i \subset H_i$, $i = 1, \ldots, k$. These neighborhoods $H_1, \ldots, H_k$ will be called *generalized handles* and we will define by an induction.

Let $H_1 := W_1$. Let also $S(W_i)$ be the set of those points, whose positive orbit with respect to the gradient field ∇f intersects W_i. Then we define

$H_i := \mathrm{cl}(S(W_i)\setminus\bigcup_{j<i} H_j)$. Thus, M can be obtained by a sequential gluing of the generalized handles.

Denote by n_i the number of critical values of f in H_i.

Consider the following set $S = \partial H_1 \cup \ldots \cup \partial H_k$. It has a natural structure of the stratified set, and each strata A of dimension 2 belongs to the intersection of two different generalized handles H_j, H_k. We equip that 2-strata A with a pair of numbers (n_j, n_k), equal to the numbers of the critical values of f in the adjacent handles H_j, H_k. By the *diagram* of a function we mean the stratified set S together with those pairs of numbers associated to each 2-strata. As above, the diagrams are called *homeomorphic*, if there is a homeomorphism of the stratified sets preserving the 2-strata equipments.

Theorem 4.1. *Two functions with isolated critical points on* 3*-manifolds are topologically conjugate if and only if it is possible to construct their diagrams that are homeomorphic.*

Proof. Necessity. The restriction of a conjugated homeomorphism on the first stratified set induces a homeomorphism between the first diagram and the diagram constructed on the image this map.

Sufficiency. As above, without loss of generality one can assume that the functions have the same sets of critical values.

The boundary of each generalized handle can be divided into three parts:

- lower base consisting of boundary intersections with handles, which have the smaller numbers;
- upper base consisting of points, in which gradient field is transversal to boundary of the handle and not included in the lower base;
- side walls consisting of points not belong to the bases.

The given homeomorphism of the stratified sets can easily be improved so that it will map side walls on side walls, and preserve their partition into the levels of the function. Then Theorem 2.7 allows us to extend this homeomorphism up to a required conjugated homeomorphism. □

Let us consider the problem when not homeomorphic diagrams correspond to topologically conjugate functions. Similarly to functions with 4 critical points a choice of another Riemannian metric implies that the generalized handles have isotopic attaching maps. The correspondent diagrams will be called *semiisotopic*.

Corollary 4.2. *Two functions with isolated critical points on* 3*-manifolds are topologically conjugate if and only if their diagrams are semiisotopic.*

The proof is similar to the proof of Theorem 3.6.

References

[1] V. I. Arnold. Topological classification of Morse functions and generalisations of Hilbert's 16-th problem. *Math. Phys. Anal. Geom.*, 10(3):227–236, 2007. doi:10.1007/s11040-007-9029-0.

[2] A. V. Bolsinov and A. T. Fomenko. *Integrable Hamiltonian systems*. Chapman & Hall/CRC, Boca Raton, FL, 2004. Geometry, topology, classification, Translated from the 1999 Russian original. doi:10.1201/9780203643426.

[3] A. T. Fomenko and S. V. Matveeev. *Algorithmic and Computer Methods for Three-Manifolds*. MAIA. Springer Netherlands, 1997. 724 p. doi:10.1007/978-94-017-0699-5.

[4] B. I. Hladysh and A. O. Prishlyak. Topology of functions with isolated critical points on the boundary of a 2-dimensional manifold. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 13:Paper No. 050, 17, 2017. doi:10.3842/SIGMA.2017.050.

[5] B. I. Hladysh and A. O. Prishlyak. Simple Morse functions on an oriented surface with boundary. *J. Math. Phys. Anal. Geom.*, 15(3):354–368, 2019. doi:10.15407/mag15.03.354.

[6] B. I. Hladysh and A. O. Pryshlyak. Functions with nondegenerate critical points on the boundary of the surface. *Ukrainian Mathematical Journal*, 68(1):29–40, jun 2016. doi:10.1007/s11253-016-1206-5.

[7] A. S. Kronrod. On functions of two variables. *Uspehi Matem. Nauk (N.S.)*, 5(1(35)):24–134, 1950.

[8] A. O. Prishlyak. On topologically equivalent Morse functions on 3-manifold. *Methods Funct. Anal. Topology*, 5(3):49–53, 1999. URL: http://mfat.imath.kiev.ua/article/?id=103.

[9] A. O. Prishlyak. Conjugacy of Morse functions on surfaces with values on a straight line and circle. *Ukrainian Mathematical Journal*, 52(10):1623–1627, 2000. doi:10.1023/A:1010461319703.

[10] A. O. Prishlyak. Conjugacy of Morse functions on four-dimensional manifolds. *Uspekhi Mat. Nauk*, 56(1(337)):173–174, 2001. doi:10.1070/rm2001v056n01ABEH000370.

[11] A. O. Prishlyak. Topological equivalence of Morse–Smale vector fields with beh2 on three-dimensional manifolds. *Ukrainian Mathematical Journal*, 54(4):603–612, 2002.

[12] A. O. Prishlyak. Topological equivalence of smooth functions with isolated critical points on a closed surface. *Topology and its Applications*, 119(3):257–267, 2002. doi:10.1016/S0166-8641(01)00077-3.

[13] A. O. Prishlyak. A complete topological invariant of Morse-Smale flows and handle decompositions of 3-manifolds. *Fundam. Prikl. Mat.*, 11(4):185–196, 2005. doi:10.1007/s10958-007-0287-y.

[14] A. O. Prishlyak and M. B. Loseva. Optimal Morse–Smale flows with singularities on the boundary of a surface. *J. Math. Sci., New York*, 243:279–286, 2019. doi:10.1007/s10958-019-04539-9.

[15] A. O. Prishlyak and M. B. Loseva. Topology of optimal flows with collective dynamics on closed orientable surfaces. *Proc. Int. Geom. Cent.*, 13(2):50–67, 2020. doi:10.15673/tmgc.v13i2.1731.

[16] G. Reeb. Sur les points singuliers d'une forme de Pfaff complètement intégrable ou d'une fonction numérique. *C. R. Acad. Sci. Paris*, 222:847–849, 1946.

[17] V. V. Sharko. *Functions on manifolds*, volume 131 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Algebraic and

topological aspects, Translated from the Russian by V. V. Minachin [V. V. Minakhin]. doi:10.1090/mmono/131.

[18] F. Takens. The minimal number of critical points of a function on a compact manifold and the Lusternik-Schnirelman category. *Invent. Math.*, 6:197–244, 1968. doi:10.1007/BF01404825.

Received: May 11, 2023, accepted: October 14, 2023.

Bohdana Hladysh
Akademia Nauk Stosowanych w Koninie, Konin, Poland
Email: bohdanahladysh@gmail.com
ORCID: 0000-0001-8935-1453

Maria Loseva
Taras Shevchenko National University of Kyiv, Faculty of computer science and cybernetics, 4th Academician Glushkov avenue, Kyiv, 03127, Ukraine
Email: mv.loseva@gmail.com
ORCID: 0000-0002-2282-206X

Alexandr Prishlyak
Taras Shevchenko National University of Kyiv, Faculty of Mechanics and Mathematics, 4th Academician Glushkov avenue, Kyiv, 03127, Ukraine
Email: prishlyak@knu.ua
ORCID: 0000-0002-7164-807X