Proceedings of the
International Geometry Center
Vol. 16, no. 2 (2023) 173–182

On foliations of bounded mean curvature on closed three-dimensional Riemannian manifolds

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Abstract. The notion of a systole of a foliation $sys(\mathcal{F})$ on an arbitrary foliated closed Riemannian manifold $(M, \mathcal{F})$ is introduced. A lower bound on $sys(\mathcal{F})$ for the foliation $\mathcal{F}$ on closed 3-dimensional Riemannian manifold M, the modulus of mean curvature of the leaves of which is bounded above by a fixed constant H_0 is given. As a consequence, we estimate the number of Reeb components of such a foliation.

Анотація. В роботі введене поняття систоли шарування $sys(\mathcal{F})$ на довільному замкненому рімановому многовиді $(M, \mathcal{F})$ та отримані нижні оцінки на $sys(\mathcal{F})$ для 3-многовидів з шаруваннями, у який середня кривина шарів обмежена зверху деякою константою H_0. В якості наслідку отримано оцінки на число компонент Ріба таких шарувань.

1. Introduction

Let (M, g) be a closed oriented three-dimensional Riemannian manifold and $\mathcal{F}$ be a transversely oriented smooth foliation of codimension one on M. A foliation is always assumed to be smooth, i.e. belonging to the class C^∞. Recall that a foliation $\mathcal{F}$ is *taut* if there exists a closed transversal passing through each leaf of $\mathcal{F}$. D. Sullivan [10] gave a description of taut foliations. Namely, he proved that a foliation $\mathcal{F}$ is taut if and only if $\mathcal{F}$ is *minimal*, i.e. its leaves are minimal submanifolds of M with respect to some Riemannian metric on M. The latter is also equivalent to the following statement: $\mathcal{F}$ does not contain generalized Reeb components (see below).

The following theorem, which we recently proved, states that if one requires that the mean curvature of the leaves is not too large in absolute value, then the foliation remains in the class of taut foliations.

2020 Mathematics Subject Classification:
Keywords: Reeb component, foliation, Riemannian manifold, systole, curvature
DOI: http://dx.doi.org/10.15673/pigc.v16i2.2510

Theorem 1.1. [2] *Let $V_0 > 0$, $i_0 > 0$, $K_0 \geqslant 0$ be fixed constants, and M be a closed oriented three-dimensional Riemannian. Denote by* $\mathrm{Vol}(M)$, *K and* $\mathrm{inj}(M)$ *respectively the volume, sectional curvature and injectivity radius of M. Suppose that the following conditions hold:*

(i) $\mathrm{Vol}(M) \leqslant V_0$;

(ii) $K \leqslant K_0$;

(iii) $\min\{\mathrm{inj}(M), \frac{\pi}{2\sqrt{K_0}}\} \geqslant i_0$.

Denote

$$H_0 = \begin{cases} \min\left\{\frac{2\sqrt{3}i_0^2}{V_0}, \sqrt[3]{\frac{2\sqrt{3}}{V_0}}\right\}, & \text{if } K_0 = 0, \\ \min\left\{\frac{2\sqrt{3}i_0^2}{V_0}, x_0\right\}, & \text{if } K_0 > 0, \end{cases}$$

where x_0 is the root of the equation

$$\frac{1}{K_0}\left(\operatorname{arccot}\frac{x}{\sqrt{K_0}}\right)^2 - \frac{V_0}{2\sqrt{3}}x = 0.$$

Then each smooth transversely oriented foliation $\mathcal{F}$ of codimension one on M, such that the absolute values of the mean curvature of its leaves satisfy the inequality $|H| < H_0$, is taut. In particular, it is minimal for some Riemannian metric on M.

If we increase the parameter H_0, which limits the mean curvature modulus of the leaves from above, then the foliation may already contain generalized Reeb components, and the topology of the foliations may change significantly. However, if we fix H_0, then the class of foliations whose modulus of mean curvature of the leaves does not exceed H_0 will have some geometrical and topological restrictions.

Using the lower estimate on foliation systole $sys(\mathcal{F})$ given in Theorem 3.1 below, we obtain an upper estimate for the number of Reeb components of a foliation $\mathcal{F}$ whose modulus of mean curvature of the leaves does not exceed H_0. Namely, we prove the following theorem.

Theorem 3.1. (Main theorem) *Let M be a closed oriented three-dimensional Riemannian manifold satisfying the conditions* (i)-(iii) *of Theorem 1.1. Let $\mathcal{F}$ be a codimension one transversally oriented foliation on M, whose leaves have a modulus of mean curvature bounded above by the fixed constant H_0. Then the number of Reeb components of the foliation $\mathcal{F}$ does not exceed*

$$\frac{4H_0V_0}{\sqrt{3}C_0^2},$$

where

$$C_0 := \begin{cases} 2\min\left\{i_0, \frac{1}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}}\right\}, & \text{if } K_0 > 0, \\ 2\min\left\{i_0, \frac{1}{H_0}\right\}, & \text{if } K_0 = 0. \end{cases}$$

2. Preliminaries

2.1. **Generalized Reeb components.** A subset of the manifold M with a given foliation $\mathcal{F}$ on it is called *a saturated set* if it is a union of leaves of the foliation $\mathcal{F}$. A saturated set A of a three-dimensional compact orientable manifold M with a given transversely orientable foliation $\mathcal{F}$ of codimension one is called a *generalized Reeb component* if A is a connected three-dimensional manifold with a boundary and any transversal to $\mathcal{F}$ vector field restricted to the boundary of A is directed either everywhere inwards or everywhere outwards of the Reeb component A. It is clear that ∂A consists of a finite set of compact leaves of the foliation $\mathcal{F}$. In particular, the Reeb component R (see [11]) is a generalized Reeb component. It is not difficult to show that in fact ∂A is a family of tori (see [7]).

The next result is due to G. Reeb.

Theorem 2.1.1 (Reeb)**.** *Let (M, g) be a closed oriented three-dimensional Riemannian manifold and $\mathcal{F}$ be a smooth transversely oriented foliation of codimension one on M. Then*

$$d\chi = 2H\mu,$$

where χ is the volume form of the foliation $\mathcal{F}$, and μ is the volume form on M.

Proof. See, for example, [2, Lemma 1]. □

Proposition 2.1.2. *Let (M, g) be a closed oriented three-dimensional Riemannian manifold with a given transversely oriented smooth foliation $\mathcal{F}$ of codimension one. Suppose that $\mathcal{F}$ contains a generalized Reeb component A and the modulus of the mean curvature of the foliation $\mathcal{F}$ is bounded above by $|H| \leqslant H_0$. Then*

$$\operatorname{Area}(\partial A) \leqslant 2H_0 \operatorname{Vol}(A).$$

Proof. Note that

$$0 < \operatorname{Area}(\partial A) = \Big|\int\limits_{\partial A} \chi\Big| = \Big|\int\limits_{A} d\chi\Big| \overset{(Reeb)}{=}$$

$$= \Big|2\int_A H\mu\Big| \leqslant 2\int_A H_0\mu = 2H_0 \operatorname{Vol}(A).$$

Hence $\mathrm{Area}(\partial A) \leqslant 2H_0 \mathrm{Vol}(A)$. □

Corollary 2.1.3. *The generalized Reeb component A is an obstruction to the foliation $\mathcal{F}$ being taut.*

Remark 2.1.4. The converse is also true: if the foliation is not taut, then it contains a generalized Reeb component (see [7]).

2.2. **Novikov's theorem and the vanishing cycle.** We define the following concepts in accordance with [8, Chapter VII]:

Definition 2.2.1. A map $f\colon N \to M$ is *integral* for $\mathcal{F}$ if $f(N)$ is contained in some leaf L of $\mathcal{F}$. In this case L is referred as the *support* of f.

Definition 2.2.2. An integral loop $\alpha : S^1 \to M$ is a *vanishing cycle* if there exists a homotopy $A\colon S^1 \times I \to M$ through integral loops $A_t := A|_{S^1 \times t}$ for $\mathcal{F}$ such that $A_0 = \alpha$ and A_t is null-homotopic in its support for $0 < t \leqslant 1$. A vanishing cycle α is *non-trivial* if α is not null-homotopic in its support.

The following well-known Novikov's theorem gives a topological obstruction to the existence of taut foliations.

Theorem 2.2.3. [9]. *Let $(M, \mathcal{F})$ be a compact, orientable and transversely orientable foliated 3-manifold.*

(i) *Then the following conditions are equivalent:*
   (a) *$\mathcal{F}$ has a Reeb component;*
   (b) *there is a leaf L of $\mathcal{F}$ being not π_1-injective, i.e. the inclusion $i\colon L \to M$ induces a homomorphism $i_*\colon \pi_1(L) \to \pi_1(M)$ with nontrivial kernel;*
   (c) *some leaf of $\mathcal{F}$ contains a non-trivial vanishing cycle.*

(ii) *The leaf supporting a non-trivial vanishing cycle is a torus bounding a Reeb component.*

Let us briefly recall the construction underlying the proof of Novikov's theorem. Suppose that a simple closed integral regular curve $\alpha\colon S^1 \to M$ belongs to the leaf $L \in \mathcal{F}$ and represents a non-trivial element of

$$\ker(i_*\colon \pi_1 L \to \pi_1 M).$$

We can find an immersion $p\colon D \to M$ of a two-dimensional disk D such that $p(\partial D) = \alpha$. This immersion can be brought to a general position by a small perturbation. It means that the induced foliation $\mathcal{F}' := p^{-1}(\mathcal{F} \cap p(D))$ on D has only Morse singularities. Moreover, by a small perturbation on can assume that each leaf of $\mathcal{F}'$ contains at most one singular point (see [5, Lemma

9.2.1]). Also, the resulting foliation outside singular points of D can be oriented: this follows from the orientability of the disk D, the foliation $\mathcal{F}$, and the manifold M. Therefore, there exists a smooth vector field X tangent to $\mathcal{F}'$ with zeros corresponding to the singular points of $\mathcal{F}'$. Recall that a separatrix coming out of a singular point and returning to it, together with the singular point (a saddle) is called a *separatrix loop*. By the construction, a saddle singular point of $\mathcal{F}'$ belongs to at most two separatrix loops. Let us identify closed orbits and separatrix loops of $\mathcal{F}'$ with the images of corresponding loops $f\colon S^1 \to D$ which bypass them once along the trajectories of the vector field X. The integral loop is called *essential* whenever the loop $p \circ f$ represents nontrivial element of the fundamental group of its support and *inessential* otherwise. Note, that by Reeb stability theorem (see [11]) inessential closed orbits have a "good neighborhood" consisting of inessential closed orbits.

Definition 2.2.4. Let $\mathcal{P} \subset D$ be a subset such that either

- $\mathcal{P}$ is homeomorphic to a disk whose boundary $\partial\mathcal{P}$ is either a closed orbit of $\mathcal{F}'$ or a separatrix loop, or
- $\mathcal{P}$ is a pinched annulus (see Fig.2.1) with a boundary consisting of two separatrix loops with a common saddle point

and $\partial\mathcal{P}$ has a "good collar" in $\mathcal{P}$, i.e. a collar consisting of inessential closed orbits of $\mathcal{F}'$. Clearly, p-image of $\partial\mathcal{P}$ will represent a vanishing cycle. We will refer to $\partial\mathcal{P}$ as a *vanishing cycle* as well.

One of S. P. Novikov's key results in [9] is the proof of the existence of a non-trivial vanishing cycle $\partial\mathcal{P}$ inside of D.

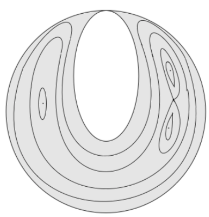

FIGURE 2.1. Pinched annulus $\mathcal{P}$.

2.3. **Some quantitative estimates.** Recall the following comparison theorem for the normal curvatures.

Theorem 2.3.1 ([3, §22.3.2.]). *Let $p \in M$ and $\beta\colon [0, r] \to M$ be a radial geodesic of the ball $B(p, r)$ of radius r centered at p. Let also $\beta(r)$ be a point not conjugate to p along β. Choose the radius r so that there are no* *conjugate points in the space of constant curvature* K_0 *within the radius of length* r*. Then if at each point* $\beta(t)$ *the sectional curvatures* K *of the manifold* M *do not exceed* K_0*, then the normal curvature* k_n^S *of the sphere* $S(p,r)$ *at the point* $\beta(r)$ *with respect to the normal* $-\beta'$ *is not less than the normal curvature* k_n^0 *of the sphere of radius* r *in the space of constant curvature* K_0.

By Theorem 2.3.1 all normal curvatures of the sphere $S(r) \subset M$ of radius r are positive, provided that $r < i_0$ and the normal to the sphere $S(r)$ is directed inside the ball $B(r)$ which it bounds[1]. We will call such a normal *inward.*

A hypersurface $S \subset M$ in a Riemannian manifold M is called *supporting* to a subset $A \subset M^n$ at a point $p \in \partial A \cap S$ with respect to a normal $n_p \perp T_pS$, if S cuts some spherical neighborhood B_p of the point p into two components, and $A \cap B_p$ is contained in that component to which the normal n_p is directed. Say that the sphere $S(r) \subset M$ $(r < i_0)$ is *supporting* to the set $A \subset M$ at the point $q \in A \cap S(r)$ if it is the supporting sphere to A at the point q with respect to the inward normal.

We have the following obvious result.

Lemma 2.3.2. [2] *Assume that the surface* $F \subset M$ *is tangent to the sphere* $S(r_0)$, $(r_0 < i_0)$*, at a point* q *and the sphere* $S(r_0)$ *is the supporting sphere to* F *at the point* q*. Then* $k_n^S(v) \leqslant k_n^F(v)$ *for all* $v \in T_qS(r_0)$*, where* $k_n^S(v)$ *and* $k_n^F(v)$ *denote corresponding normal curvatures of* $S(r_0)$ *and* F *at the point* q *in the direction* v.

As a consequence of Lemma 2.3.2 and Theorem 2.3.1 we obtain the following inequalities at the touching point q:

$$0 < H_r^0 \leqslant H_r(q) \leqslant H(q), \tag{2.1}$$

where H_r^0 and H_r are the mean curvatures of the spheres $S(r)$ bounding the ball of radius $r, r < i_0$, in the space of constant curvature K_0 and the manifold M respectively, and H is the mean curvature of the surface F.

2.4. **Systoles.** Recall that the systole in a Riemannian manifold M with nontrivial fundamental group, denoted by sys, is the length of the smallest non null-homotopic loop in M. Under the condition of closeness M such a loop exists and is necessary a closed geodesic. The proof does not differ from the proof of the existence of a closed geodesic in its homotopy class (see [6, Chapter 12]). These arguments allow to prove the following statement.

[1] Note that the sphere indeed bounds the ball, since by definition $r < inj(M)$.

Proposition 2.4.1. *Let $(M, \mathcal{F})$ be a foliated closed Riemannian manifold. Suppose that $\pi_1(L) \neq 0$ for some leaf $L \in \mathcal{F}$. Then there exists an integral essential loop l_{sys} in M with the smallest length among all integral essential loops in $(M, \mathcal{F})$, which is necessarily a closed geodesic in its support.*

Proof. First note that for any two points $x, y \in L$ of any leaf $L \in \mathcal{F}$ we have the following inequality:

$$\rho_L(x, y) \geqslant \rho(x, y),$$

where ρ_L and ρ are Riemannian distance in L and M respectively. The Riemannian metric on L is supposed to be induced from M. It means that the space $C_{\mathcal{F}}(S^1, M)$ of integral continuous maps $f\colon S^1 \to M$ (see above) is a subspace of the space $C(S^1, M)$ of continuous maps of S^1 to M, where $C(S^1, M)$ is endowed with the topology of uniform convergence on compact sets. As shown in [1], arbitrary connected locally compact space X the subset $C_{\mathcal{F}}(X, M)$ is closed in $C(X, M)$.

Further, note that the length of an essential integral loop is bounded from bellow by some constant $d > 0$. Otherwise, we can find an equicontinuous sequence of essential integral loops $\{l_i\}$ with decreasing lengths and converging to some point $m \in M$. This means that there exists a loop l_i, belonging to a foliated neighborhood U of m, and being contractible to a point inside of supporting plaque in U. This contradicts to the assumption that l_i is essential.

Now consider the sequence $\{l_i\}$ of integral essential piecewise differentiable loops parametrized by arc length and such that $|l_i| \to d$. Clearly, the set $\{l_i\}$ is equicontinuous and there is a subsequence of l_i, which converges uniformly to the continuous integral loop $l_0\colon S^1 \to L_0$, where L_0 is a leaf of $\mathcal{F}$ supporting l_0. In the same way as it was done in [6, Chapter 12] in the proof of the existence of a closed geodesic in its homotopy class onw can approximate l_0 by integral broken geodesic $l_{sys} \in L_0$ to prove that $|l_{sys}| = d$ and l_{sys} is actually a closed geodesic in L_0.

It remains to show that l is an essential. If we assume the contrary, then using the Reeb's local stability theorem (see [4, Proposition 11.4.8]), one can show that loops l_i close to l_{sys} will be inessential which leads to a contradiction. □

Denote by $sys(\mathcal{F})$ the length of the systolic geodesic l_{sys}.

We conclude this section by recalling the following Lowner's theorem, which we use bellow and which gives an upper bound on the length of the shortest closed geodesic in a Riemannian two-dimensional torus.

Theorem 2.4.2 (Lowner)**.** *Let* (T^2, g) *be a two-dimensional Riemannian torus and* sys *denotes the systole of* (T^2, g). *Then*

$$sys^2 \leqslant \frac{2}{\sqrt{3}} \text{Area}(T^2). \tag{2.2}$$

3. Main result

Let us prove the following theorem.

Theorem 3.1. *Let* M *be a closed oriented three-dimensional Riemannian manifold satisfying the conditions* (i)-(iii) *of Theorem* 1.1. *Let also* $\mathcal{F}$ *be a codimension one transversally oriented foliation on* M*, whose leaves have a modulus of mean curvature bounded above by the same constant* H_0. *Then*

$$sys(\mathcal{F}) \geqslant C_0 := \begin{cases} 2\min\{i_0, \frac{1}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}}\}, & \textit{if } K_0 > 0, \\ 2\min\{i_0, \frac{1}{H_0}\}, & \textit{if } K_0 = 0. \end{cases}$$

Proof. Case 1. If $\frac{sys(\mathcal{F})}{2} \geqslant i_0$, then the result follows immediately.

Case 2. Suppose $\frac{sys(\mathcal{F})}{2} < i_0$. Let l_{sys} be an integral closed geodesic which is not null-homotopic in its support and whose length

$$sys = sys(\mathcal{F}) < 2i_0.$$

Then there is an immersion

$$p\colon D \to \text{int } B(r), \; r \in \left(\tfrac{sys}{2}, i_0\right)$$

of a disk D which is in general position with respect to $\mathcal{F}$ and such that $p(\partial D) = l_{sys}$. As shown in the section 2.2 there is a vanishing cycle which belongs to

$$T^2 \cap p(D) \subset \text{int } B(r),$$

where $T^2 \in \mathcal{F}$ is a torus bounding a Reeb component R.

Let $r \in (\frac{sys}{2}, i_0)$ be a regular value of the mapping

$$pr_r|_{(\text{int } B(i_0))\cap T^2} : (\text{int } B(i_0)) \cap T^2 \to \mathbb{R} \tag{3.1}$$

such that $pr_r(r, \phi_1, \phi_2) = r$, where (r, ϕ_1, ϕ_2) is a normal coordinate system in the ball $B(i_0)$.

By [2, Proposition 2] in the case $S(r) \cap T^2 \neq \varnothing$ the sphere $S(r)$ will be a supporting sphere with respect to the inward normal at the tangent point q for some inner leaf of the Reeb component R. Note, that due to Sard's theorem the set of regular values of the mapping (3.1) has a full measure in the interval $(\frac{sys}{2}, i_0)$ and the value r can be taken arbitrarily close to $\frac{sys}{2}$.

In the case of $S(r) \cap T^2 = \varnothing$ we will achieve the tangency of the sphere $S(r)$ and T^2 by decreasing the radius r, and the sphere $S(r)$ will become supporting for the torus T^2.

Then it follows from (2.1) that

$$H_r^0 = \left\{ \begin{array}{ll} \sqrt{K_0}\cot(r\sqrt{K_0}), & \text{if } K_0 > 0, \\ \frac{1}{r}, & \text{if } K_0 = 0 \end{array} \right\} \leqslant H_0.$$

Thus, we conclude that $sys(\mathcal{F})$ should satisfy the following inequality:

$$sys(\mathcal{F}) \geqslant \begin{cases} \frac{2}{\sqrt{K_0}} \operatorname{arccot} \frac{H_0}{\sqrt{K_0}}, & \text{if } K_0 > 0, \\ \frac{2}{H_0}, & \text{if } K_0 = 0. \end{cases}$$

Combining Case 1 and Case 2 we obtain the result. □

As a corollary of Theorem 3.1 we obtain the main result of this paper.

Corollary 3.1.1 (Main theorem)**.** *The number of Reeb components of the foliation $\mathcal{F}$ does not exceed*

$$\frac{4H_0V_0}{\sqrt{3}C_0^2}.$$

Proof. From Theorem 2.4.2 and Proposition 2.1.2 we have that

$$\frac{\sqrt{3}}{2}C_0^2 \leqslant \operatorname{Area}(\partial R) \leqslant 2H_0\operatorname{Vol}(R). \tag{3.2}$$

Now, it follows from (3.2) that

$$\operatorname{Vol}(R) \geqslant \frac{\sqrt{3}C_0^2}{4H_0}.$$

Since the interiors of Reeb components do not intersect, the number of Reeb components does not exceed

$$\frac{4H_0V_0}{\sqrt{3}C_0^2}.$$

□

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Received: May 8, 2023, accepted: June 15, 2023.

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