Proceedings of the
International Geometry Center
Vol. 16, no. 2 (2023) pp. 161–172

Cohomology algebra of mapping spaces between quaternion Grassmannians

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Abstract. Let $G_{k,n}(\mathbb{H})$ for $2 \leqslant k < n$ denote the quaternion Grassmann manifold of k-dimensional vector subspaces of $\mathbb{H}^n$. In this paper we compute, in terms of the Sullivan models, the rational cohomology algebra of the component of the inclusion $\iota\colon G_{k,n}(\mathbb{H}) \hookrightarrow G_{k,n+r}(\mathbb{H})$ in the space of mappings from $G_{k,n}(\mathbb{H})$ to $G_{k,n+r}(\mathbb{H})$ for $r \geqslant 1$ and, more generally, we show that the cohomology of $Map(G_{k,n}(\mathbb{H}), G_{k,n+r}(\mathbb{H}); \iota)$ contains a truncated algebra $\mathbb{Q}[x]/x_4^{r+n+k^2-nk}$ for $n \geqslant 4$.

Анотація. Нехай $G_{k,n}(\mathbb{H})$, $2 \leqslant k < n$, – кватерніонний грасмановий многовид k-вимірних векторних підпросторів в $\mathbb{H}^n$. В даній роботі, в термінах моделей Суллівана, обчислено алгебру раціональних когомологій компонент включення $\iota\colon G_{k,n}(\mathbb{H}) \hookrightarrow G_{k,n+r}(\mathbb{H})$ в простір відображень з $G_{k,n}(\mathbb{H})$ в $G_{k,n+r}(\mathbb{H})$ для $r \geqslant 1$. Також показано, що когомології $Map(G_{k,n}(\mathbb{H}), G_{k,n+r}(\mathbb{H}); \iota)$ містять усічену алгебру $\mathbb{Q}[x]/x_4^{r+n+k^2-nk}$ для $n \geqslant 4$.

1. INTRODUCTION

Throughout this paper, we rely on the theory of minimal Sullivan models in rational homotopy theory for which [5] is our standard reference. The quaternion Grassmannian $G_{k,n}(\mathbb{H})$ is the set of k-dimensional vector subspaces through the origin in $\mathbb{H}^n$. Moreover, it is a homogeneous space as

$$G_{k,n}(\mathbb{H}) \cong Sp(n)/(Sp(k) \times Sp(n-k)) \qquad \text{for } 1 \leqslant k < n$$

for where $Sp(n)$ is the symplectic group (see [6, Page 43]), and of dimension $2m$, where $m = 2k(n-k)$. In particular, $G_{1,n}(\mathbb{H}) \cong \mathbb{H}P^{4n-1}$, and it is called the quaternion projective space. There is a canonical inclusion

$$\iota\colon G_{k,n}(\mathbb{H}) \hookrightarrow G_{k,n+r}(\mathbb{H})$$

2020 Mathematics Subject Classification: Primary 55P62; Secondary 54C35

Keywords: Grassmann manifolds, mapping space, Sullivan model, L_∞-algebra
DOI: http://dx.doi.org/10.15673/pigc.v16i2.2453

which is induced by $\iota\colon \mathbb{H}^n \to \mathbb{H}^{n+r}$ defined by

$$\iota(x_1, \dots, x_n) = (x_1, \dots, x_n, 0, \dots, 0).$$

It is known in literature that function spaces have been objects of central interest of study to homotopy theorists, where an important problem is to understand the homotopy type of a function space $Map(X, Y)$ by focusing on a path component, denoted here by $Map(X, Y; f)$, for some choice of map $f\colon X \to Y$ as generally, $Map(X, Y)$ is a disconnected space. The classification of the homotopy types of components is not an easy task. However, rational homotopy theory offers a remarkably computational advantage and has been used intensively in recent years to study function spaces. The article of [21], started the rational homotopy type of function spaces in the case where the codomain is an Eilenberg-Maclane space. The first description of a Sullivan model of function spaces is due to [10]. Further, a model of function spaces between complex projective spaces was studied by Moller and Raussen using the Postnikov tower [15]. However, there was no explicit and complete description of the homotopy type of the component of the inclusion for sporadic cases until the recent articles by [7, 17], which describes an explicit and complete description of the rational cohomology and homotopy type of function spaces between complex Grassmannians.

On one hand, several authors (see [1, 4, 8, 11]) studied the classification of self-maps of a complex Grassmann manifold in terms of their induced endomorphisms of the cohomology algebra. Further details on a similar study of maps between two distinct (real) Grassmann manifolds can be found in [12]. The paper [18], studied the existence (or non-existence) of maps of non-zero degree between two different complex (resp. quaternionic) Grassmann manifolds of the same dimension, while [19] solved the same problem for oriented real Grassmann manifolds. Now, in [7, 17], it was shown under some assumptions on r that the cohomology algebra of

$$Map(G_{k,n}(\mathbb{C}), G_{k,n+r}(\mathbb{C}); \iota)$$

contains either a polynomial algebra or a truncated algebra over a generator of degree 2, where

$$\iota\colon G_{k,n}(\mathbb{C}) \hookrightarrow G_{k,n+r}(\mathbb{C})$$

is the canonical inclusion. Our goal in this paper is to show under some assumptions on r that the cohomology algebra of $Map(G_{k,n}(\mathbb{H}), G_{k,n+r}(\mathbb{H}); \iota)$ contains either a polynomial algebra or a truncated algebra over a generator of degree 4. In the process, we shall investigate the rational cohomology algebra of the component of the inclusion $\iota\colon G_{k,n}(\mathbb{R}) \hookrightarrow G_{k,n+r}(\mathbb{R})$ in the space $Map(G_{k,n}(\mathbb{R}), G_{k,n+r}(\mathbb{R}); \iota)$, for $r \geqslant 1$ and $n \geqslant 4$.

2. PRELIMINARIES

Here we fix terminology and recall some standard facts on differential graded algebras. All vector spaces and algebras are taken over a field $\mathbb{Q}$ of rational numbers.

Definition 2.1. A *commutative graded differential algebra* (cdga) is a graded algebra (A, d) such that $xy = (-1)^{|x||y|}yx$ and

$$d(xy) = (dx)y + (-1)^{|pq|}x(dy) \text{ for all } x \in A^p, y \in A^q.$$

It is said to be *connected* if $H^0(A) \cong \mathbb{Q}$. If $V = \bigoplus_{i \geqslant 0} V^i$ with

$$V^{\text{even}} := \bigoplus_{i \geqslant 0} V^{2i}$$

and

$$V^{\text{odd}} := \bigoplus_{i \geqslant 1} V^{2i-1},$$

then $\wedge V$ denotes the *free commutative graded algebra* defined by the tensor product

$$\wedge V = S(V^{\text{even}}) \otimes E(V^{\text{odd}}),$$

where $S(V^{\text{even}})$ is the *symmetric algebra* on V^{even} and $E(V^{\text{odd}})$ is the *exterior algebra* on V^{odd}.

Definition 2.2. A *Sullivan algebra* is a commutative differential graded algebra $(\wedge V, d)$, where

$$V = \bigcup_{k \geqslant 0} V(k) \quad \text{and} \quad V(0) \subset V(1) \subset \cdots$$

such that

$$dV(0) = 0 \quad \text{and} \quad dV(k) \subset \wedge V(k-1).$$

It is called *minimal* if $dV \subset \wedge^{\geqslant 2} V$.

If (A, d) is a cdga of which the cohomology is connected and finite dimensional in each degree, then there always exists a quasi-isomorphism from a Sullivan algebra $(\wedge V, d)$ to (A, d) [5]. To each simply connected space, Sullivan associates a cdga $A_{PL}(X)$ of rational polynomial differential forms on X that uniquely determines the rational homotopy type of X [20]. A minimal Sullivan model of X is a minimal Sullivan model of $A_{PL}(X)$. More precisely,

$$H^*(\wedge V, d) \cong H^*(X; \mathbb{Q})$$

as graded algebras and

$$V \cong \pi_*(X) \otimes \mathbb{Q}$$

as graded vector spaces. Let (A, d) be a cdga.

A *derivation* θ of degree k is a linear mapping $\theta\colon A^n \to A^{n-k}$ such that

$$\theta(ab) = \theta(a)b + (-1)^{k|a|}a\theta(b).$$

Denote by $Der_k A$ the vector space of all derivations of degree k, and

$$DerA = \bigoplus_k Der_k A.$$

The commutator bracket induces a graded Lie algebra structure on $DerA$. Moreover, $(DerA, \delta)$ is a differential graded Lie algebra (see e.g. [20]), with the differential δ defined in the usual way by

$$\delta\theta = d \circ \theta + (-1)^{k+1}\theta \circ d.$$

Let $(\wedge V, d)$ be a Sullivan algebra where V is spanned by $\{v_1 \ldots, v_k\}$. Then, $Der \wedge V$ is spanned by $\theta_1, \ldots, \theta_k$, where θ_i is the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. The derivation θ_i will be denoted by $(v_i, 1)$. Let $\phi\colon (A, d) \to (B, d)$ be a morphism of cdga's.

A *ϕ-derivation* of degree k is a linear mapping $\theta\colon A^n \to B^{n-k}$ for which

$$\theta(ab) = \theta(a)\phi(b) + (-1)^{k|a|}\phi(a)\theta(b).$$

We consider only derivations of positive degree. Denote by $Der_n(A, B; \phi)$ the vector space of ϕ-derivations of degree n for $n > 0$, and by

$$Der(A, B; \phi) = \bigoplus_n Der_n(A, B; \phi)$$

the graded vector space of all ϕ-derivations. The differential graded vector space of ϕ-derivations is denoted by $(Der(A, B; \phi), \partial)$, where the differential ∂ is defined by

$$\partial\theta = d_B \circ \theta + (-1)^{k+1}\theta \circ d_A.$$

In the case $A = B$ and $\phi = 1_B$, the vector space $(Der(B, B; 1), \partial)$ is just a usual differential graded Lie algebra of derivations on the cdga B (see [14]). We note that, there is an isomorphism of graded vector spaces

$$Der(A, B; \phi) \cong Hom(V, B).$$

If $\{v_i\}$ is a basis of V, then the vector space $Der(A, B; \phi)$ is spanned by the unique ϕ-derivation θ denoted by (v_i, b_i) and $(v_i, 1)$ such that

$$\begin{cases} \theta_i(v_i) = b_i, \\ \theta_i(v_j) = 0, \quad i \neq j,\ b_i \in B. \end{cases}$$

3. L_∞-MODELS OF MAPPING SPACES

L_∞ algebras were introduced by Lada in [13] and L_∞ models of function spaces studied by Félix et al. in [2,3].

Definition 3.1. Let S_k be the symmetric group. A *permutation* $\sigma \in S_k$ is an $(i, k-i)$ shuffle if $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(k)$, where $i = 1, \ldots, m$. The *Koszul sign* $\epsilon(\sigma)$ is determined by

$$x_1 \wedge \cdots \wedge x_k = \epsilon(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(k)},$$

where the subscripts indicate the degrees of the graded objects $x_1, \ldots, x_k$.

Definition 3.2. [2] An L_∞ *algebra or a strongly homotopy Lie algebra* is a graded vector space $L = \bigoplus_i L_i$ endowed with a collection of linear maps

$$\ell_k := [, \ldots,] : L^{\otimes k} \to L$$

of degree $k-2$ for $k \geqslant 1$ called brackets, such that

(i) ℓ_k are skew-symmetric, that is, for any k-permutation σ

$$[x_{\sigma(1)}, \ldots, x_{\sigma(k)}] = \operatorname{sgn}(\sigma)\epsilon(\sigma)[x_{\sigma(1)}, \ldots, x_{\sigma(k)}],$$

where $\operatorname{sgn}(\sigma)$ is the sign of σ;

(ii) the Jacobi identities are generalised as follows;

$$\sum_{i+j=k+1} \sum_{\sigma} \operatorname{sgn}(\sigma)\epsilon(\sigma)(-1)^{i(j-1)} \ell_j(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(k)}) = 0,$$

where $\sigma \in S(i, k-i)$.

In particular, if $\ell_k = 0$ for $k \geqslant 3$, we recover the notion of a differential graded Lie algebra (L, d), where

$$[x, y] := \ell_2(x, y) \quad \text{and} \quad dx = \ell_1(x).$$

There is a bijection between L_∞ structures on L and codifferentials

$$d_k \colon \wedge^p (sL) \to \wedge^{p-k+1}(sL)$$

of degree -1 on the coalgebra $\wedge sL$ such that $d^2 = 0$, where

$$d = d_1 + \cdots + d_k + \cdots$$

and sL is the suspension of the graded vector space L defined by

$$(sL)^k = L^{k+1}$$

for all k [13]. We follow [3] for this definition. Define $\widetilde{Der}(A, B; \phi)$ as follows:

$$\widetilde{Der_i}(A, B; \phi) = \begin{cases} Der_i(A, B; \phi) & i > 1, \\ \{\theta \in Der_1(A, B; \phi) : \partial\theta = 0\}, & i = 1. \end{cases}$$

Let $(A, d) = (\wedge V, d)$ be a Sullivan algebra and

$$\theta_1, \dots, \theta_k \in \widetilde{Der}(\wedge V, B; \phi)$$

be ϕ-derivations of respective degrees $n_1, \dots, n_k$, we define their bracket $[\theta_1, \dots, \theta_k] \in \widetilde{Der}(\wedge V, B; \phi)$ of length k by

$$[\theta_1, \dots, \theta_k](v) = (-1)^{\eta} \sum \sum_{i_1, \dots, i_k} \epsilon \phi(v_1 \dots \widehat{v_{i_1}} \dots \widehat{v_{i_k}} \dots v_j)\, \theta_1(v_{i_1}) \dots \theta_k(v_{i_k}),$$

where $dv = \sum v_1 \dots v_k$, $\eta = n_1 + \dots + n_{k-1}$, and ϵ is the suitable sign given by the Koszul convention. These operations may be desuspended to define linear maps ℓ_k for $k \geqslant 1$ each of degree $k-2$ on $s^{-1}\widetilde{Der}(\wedge V, B; \phi)$ by

$$\ell_1(s^{-1}\theta) = -s^{-1}\partial' \theta,$$
$$\ell_k(s^{-1}\theta_1, \dots, s^{-1}\theta_k) = (-1)^{\beta} s^{-1}[\theta_1, \dots, \theta_k],$$

where

$$\beta = \frac{k^2 - k}{2} + \sum_{i=1}^{k-1} (k - i)|\theta_i|,$$

see [3]. It is shown in [3], that

$$(s^{-1}Der(\wedge V, B; \phi), \ell_k)$$

is an L_∞ algebra.

4. MAPPING SPACES BETWEEN QUATERNION GRASSMANNIANS

Consider the quaternion Grassmannian

$$G_{k,n}(\mathbb{H}) \cong Sp(n)/(Sp(k) \times Sp(n-k)) \text{ for } 1 \leqslant k < n$$

The method to compute a Sullivan model of the homogeneous space $G_{k,n}(\mathbb{H})$ is given in details in [9,16]. Thus, a Sullivan model of $G_{k,n}(\mathbb{H})$ for $1 \leqslant k < n$ is given by (see [16])

$$\big(\wedge(b_4, b_8, \dots, b_{4k}, x_4, x_8, \dots, x_{4(n-k)}, y_3, y_7, \dots, y_{4n-1}), d\big)$$

with

$$db_i = 0 = dx_j, \qquad dy_{4p-1} = \sum_{p_1+p_2=p} b_{4p_1} \cdot x_{4p_2}, \quad 1 \leqslant p \leqslant n. \tag{4.1}$$

Lemma 4.1. *If $2 \leqslant k < n$, then the minimal Sullivan model of $G_{k,n}(\mathbb{H})$ is given by*

$$\big(\wedge(b_4, \dots, b_{4k}, y_{4(n-k)+3}, \dots, y_{4n-1}), d\big),$$

where $db_i = 0$ and $dy_{4(n-k)+3} \in \wedge(b_4, \dots, b_{4k})$.

Proof. Consider the Sullivan model from equation (4.1)

$$\big(\wedge(b_4, b_8, \ldots, b_{4k}, x_4, x_8, \ldots, x_{4(n-k)}, y_3, y_7, \ldots, y_{4n-1}), d\big)$$

which is not minimal. Make change of variables and consider the acyclic ideal

$$I = \langle t_4, t_8, \ldots, t_{4(n-k)}, y_{4(n-k)-1}\rangle,$$

where

$$dy_3 = t_4, \quad dy_7 = t_8, \quad \ldots, \quad dy_{4(n-k)-1} = t_{4(n-k)}.$$

Taking the quotient with the acyclic ideal we obtain the minimal model

$$\big(\wedge(b_4, \ldots, b_{4k}, y_{4(n-k)+3}, \ldots, y_{4n-1}), d\big),$$

where $db_i = 0$ and $dy_{4(n-k)+3} \in \wedge(b_4, \ldots, b_{4k})$. □

In the same way, by Lemma 4.1, the minimal Sullivan model of $G_{k,n+r}(\mathbb{H})$ for $2 \leqslant k < n + r$ and $r \geqslant 1$ is given by

$$\big(\wedge(a_4, \ldots, a_{4k}, z_{4(n+r-k)+3}, \ldots, z_{4(n+r)-1}), d\big),$$

where $da_i = 0$ and $dz_{4(n+r-k)+3} \in \wedge(a_4, \ldots, a_{4k})$. We give the following results.

Theorem 4.2. *If $r > n - 1$, and $n \geqslant 4$, then $Map(G_{2,n}(\mathbb{H}), G_{2,n+r}(\mathbb{H}); \iota)$ contains a truncated algebra $\mathbb{Q}[x]/x_4^{r-n+4}$.*

Proof. The minimal model of $G_{2,n+r}(\mathbb{H})$ is given by

$$\wedge V = \big(\wedge(a_4, a_8, z_{4n+4r-5}, z_{4n+4r-1}), d\big),$$

where

$$da_i = 0 \qquad \text{and} \qquad dz_{4n+4r-5} \in \wedge(a_4, a_8).$$

The model of the inclusion $\iota\colon G_{2,n}(\mathbb{H}) \hookrightarrow G_{2,n}(\mathbb{H})$ is as follows

$$\phi\colon \big(\wedge(a_4, a_8, z_{4n+4r-5}, z_{4n+4r-1}), d\big) \to \wedge(b_4, b_8)/\langle dy_{4n-5}, dy_{4n-1}\rangle = B,$$

where

$$\phi(a_4) = b_4, \qquad \phi(a_8) = b_8, \qquad \phi(z_{4n+4r-5}) = \phi(z_{4n+4r-1}) = 0.$$

Moreover,

$$\begin{aligned}
\alpha_{4n+4r-1} &= (z_{4n+4r-1}, 1), & \beta_{4n+4r-5} &= (z_{4n+4r-5}, 1),\\
\alpha_{4n+4r-5} &= (z_{4n+4r-1}, b_4), & \beta_{4n+4r-9} &= (z_{4n+4r-5}, b_4),\\
\alpha_{4n+4r-9} &= (z_{4n+4r-1}, b_8), & \beta_{4n+4r-13} &= (z_{4n+4r-5}, b_8),\\
&\cdots & &\cdots\\
\alpha_{4n+4r-17} &= (z_{4n+4r-1}, b_4^{2n-4}), & \beta_{4n+4r-21} &= (z_{4n+4r-5}, b_4^{2n-4}),
\end{aligned}$$

$$\gamma_8 = (a_8, 1), \qquad \gamma_4 = (a_8, b_4), \qquad \theta_4 = (a_4, 1).$$

As

$$p = (\partial\gamma_8)(z_{4n+4r-5}) \quad \text{and} \quad q = (\partial\gamma_8)(z_{4n+4r-1})$$

are polynomials of degree $\geqslant 4n+4r-8$, if $r > n-1$, then

$$4n+4r-8 > 8n-16.$$

Hence, the polynomials p and q are zero in

$$\big(\wedge(a_4, a_8, z_{4n+4r-5}, z_{4n+4r-1}), d\big).$$

Thus, $\partial\gamma_8 = 0$, and $\partial\gamma_4 = \partial\theta_4 = 0$.

Note that

$$f = \ell_k(\gamma_4, \dots, \gamma_4)(z_{4n+4r-5})$$

and

$$g = \ell_k(\gamma_4, \dots, \gamma_4)(z_{4n+4r-1})$$

are polynomial of degree at least $4n+4r-4-4k$. If $k < r-n+3$, then f and g are polynomials of degree $> 8n-16$. Hence,

$$\ell_k(\gamma_4, \dots, \gamma_4) = 0 \quad \text{for} \quad k = 2, \dots, r-n+3.$$

In $C^\infty(s^{-1}L)$, set $\gamma_4^* = \rho_4$. Then, $[\rho_4^j] \neq 0$ for $j = 1, \dots, r-n+3$. □

Theorem 4.3. *Consider the inclusion*

$$\iota\colon G_{2,n}(\mathbb{H}) \hookrightarrow G_{2,n+r}(\mathbb{H}).$$

If $r > 3n-6$, and $n \geqslant 4$, then Map$(G_{2,n}(\mathbb{H}), G_{2,n+r}(\mathbb{H}); \iota)$ contains a polynomial algebra over a generator of degree 4.

Proof. Let

$$\phi\colon \big(\wedge(a_4, a_8, z_{4n+4r-5}, z_{4n+4r-1}), d\big) \to \wedge(b_2, b_4)/\langle dy_{4n-5}, dy_{4n-1}\rangle,$$

be a model of the inclusion. If $r > 3n-6$, and $n \geqslant 4$, then any odd derivation is of the degree at least

$$4n+4r-5-(8n-16) = 4r-4n+11 \geqslant 11.$$

Hence,

$$L_9 = L_7 = L_5 = \cdots = L_1 = 0.$$

As $L_3 = 0$, then the derivation $\gamma_4 = (a_8, b_4)$ is a cycle. It is enough to show that $\ell_m(\gamma_4, \dots, \gamma_4) = 0$ for $m \geqslant 4$. We show the case when $n+r$ is even, the other case follows in the same way. If $m > \frac{n+r}{4}$, then $\ell_m(\gamma_4, \dots, \gamma_4) = 0$. Assume $4 \leqslant m \leqslant \frac{n+r}{4}$, and let

$$p_1 = \ell_m(\gamma_4, \dots, \gamma_4)(z_{4n+4r-1}) = \sum_{j \geqslant m} \alpha_j b_8^{j-m} \cdot b_4^m b_4^{n+r-2j},$$

$$p_2 = \ell_m(\gamma_4, \dots, \gamma_4)(z_{4n+4r-5}) = \sum_{j \geqslant m} \beta_j b_8^{j-m} \cdot b_4^m \cdots b_4^{n+r-1-2j}.$$

The polynomials p_1 and p_2 are of total degree at least $4n + 4r - 2 - 4m$. But $r > 3n - 6$, then

$$\begin{aligned} 2n + 2r - 4 > 8n - 16 &\Rightarrow 4n + 4r - 4 - 2(n + r) > 8n - 16 \\ &\Rightarrow 4n + 4r - 4m - 4 > 8n - 16. \end{aligned}$$

Therefore, $\ell_m(\gamma_4, \dots, \gamma_4) = 0$ for $m \geqslant 4$. Hence, $H^*(C^\infty(s^{-1}L))$ contains a polynomial algebra over a generator of degree 4. □

5. The general case

Here we extend the above results.

Theorem 5.1. *If $r > nk - k^2 - n + 2k - 1$, then the rational cohomology algebra of $Map(G_{k,n}(\mathbb{H}), G_{k,n+r}(\mathbb{H}); \iota)$ contains a truncated algebra $\mathbb{Q}[x]/x_4^{r+n+k^2-nk}$ for $k \geqslant 2$ and $n \geqslant 4$.*

Proof. Let

$$(\wedge V, d) = \big(\wedge(a_4, \dots, a_{4k}, z_{4(n+r-k)+3}, \dots, z_{4(n+r)-1}), d\big),$$

where $da_i = 0$ and $dz_{4(n+r-k)+3} \in \wedge(a_4, \dots, a_{4k})$ be the minimal Sullivan model of $G_{k,n+r}(\mathbb{H})$. Moreover,

$$B = H^*(G_{k,n}(\mathbb{H}); \mathbb{Q}) = \wedge(b_4, \dots, b_{2k})/\langle dy_{4(n-k)+3}, \dots, dy_{4n-1}\rangle.$$

A Sullivan model of $\iota\colon G_{k,n}(\mathbb{H}) \hookrightarrow G_{k,n+r}(\mathbb{H})$ is given by $\phi\colon (\wedge V, d) \to B$, where

$$\begin{gathered} \phi(a_4) = b_4, \dots, \phi(a_{4k}) = b_{4k}, \\ \phi(z_{4(n+r-k)+3}) = \cdots = \phi(z_{4(n+r)-1}) = 0. \end{gathered}$$

Let $L = Der(\wedge V, B; \phi)$. Note that the lowest odd degree derivation is of degree $4k - 1$, as $k \geqslant 2$, then $L_3 = 0$. Define $\gamma_4 = (a_{4k}, b_{4k-4})$. Then γ_4 is a cycle. Further,

$$\begin{gathered} q_1 = \ell_m(\gamma_4, \dots, \gamma_4)(z_{4(n+r-k)+3}), \\ \dots \\ q_k = \ell_m(\gamma_4, \dots, \gamma_4)(z_{4(n+r)-1}), \end{gathered}$$

are polynomials of degree at least

$$4n + 4r - 4k + 4 - 4m.$$

As the manifold $G_{k,n}(\mathbb{H})$ is of dimension $4k(n - k)$, if

$$m < r + n + k^2 + 1 - nk - k,$$

then $q_1, \dots, q_k$ are polynomials of degree more than $4k(n - k)$. Therefore,

$$\ell_m(\gamma_4, \dots, \gamma_4) = 0 \ \text{ for } \ m = 4, \dots, (r + n + k^2 + 1 - nk - k).$$

In $C^\infty(s^{-1}L)$, let $\gamma_4^* = \omega_4$. Then

$$[\omega_4^i] \neq 0 \quad \text{for} \quad i = 1, \dots, (r + n + k^2 + 1 - nk - k). \qquad \square$$

Theorem 5.2. *If $r > (n-k)(2k-1) + k - 2$, and $n \geqslant 4$, then the cohomology algebra of Map$(G_{k,n}(\mathbb{H}), G_{k,n+r}(\mathbb{H}); \iota)$ for $k \geqslant 2$, contains a polynomial algebra over a generator of degree 4.*

Proof. Let $\phi\colon (\wedge V, d) \to B$ be a Sullivan model of the inclusion

$$\iota\colon G_{k,n}(\mathbb{H}) \hookrightarrow G_{k,n+r}(\mathbb{H})$$

as given in Theorem 5.1. Assume $n \geqslant 4$, in particular, if $r > 2n^2 - 5n - 6$, then any odd derivation of degree

$$\begin{aligned} \geqslant 4n + 4r - 4k + 3 - (4kn - 4k^2) &= 4r + 4n - 4kn + 4k^2 - 4k + 3 \\ &\geqslant 4k^2 - 4k + 3. \end{aligned}$$

Hence,

$$L_1 = L_3 = \dots = L_f = 0,$$

where $f = 4k^2 - 4k + 3$. Define $\gamma_4 = (a_{4k}, b_{4k-4})$ as in Theorem 5.1. Then it suffices to show that $\ell_m(\gamma_4, \dots, \gamma_4) = 0$ for $m \geqslant 4$. Consider $n + r$ even, the other case is dealt in the same way. If $m > \frac{n+r}{4}$, then $\ell_m(\gamma_4, \dots, \gamma_4) = 0$. Assume $4 \leqslant m \leqslant \frac{n+r}{4}$, and let

$$\begin{aligned} p_1 &= \ell_m(\gamma_4, \dots, \gamma_4)(z_{4n+4r-4k+3}), \\ &\cdots \\ p_k &= \ell_m(\gamma_4, \dots, \gamma_4)(z_{4n+4r-1}). \end{aligned}$$

The polynomials $p_1, \dots, p_k$ are of total degree at least $4n + 4r - 4k + 4 - 4m$. But $r > (n-k)(2k-1) + k - 2$, then

$$\begin{aligned} 2n + 2r - 4k + 4 > 8n - 16 &\Rightarrow 4n + 4r - 4k + 4 - 2(n+r) > 8n - 16 \\ &\Rightarrow 4n + 4r - 4m - 4k + 4 > 8n - 16. \end{aligned}$$

Therefore, $\ell_m(\gamma_4, \dots, \gamma_4) = 0$ for $m \geqslant 4$. Hence, $H^*(C^\infty(s^{-1}L))$ contains a polynomial algebra over a generator of degree 4. $\square$

Remark 5.3. We recall here the minimal Sullivan model of $G_{k,n}(\mathbb{H})$ for $2 \leqslant k < n$, and the Sullivan model of $G_{k,n}(\mathbb{R})$ for $1 \leqslant k < n$. We refer to [6, Corollary 1.86 & 1.90] and [16, Theorem 2] for details. The minimal Sullivan model of $G_{k,n}(\mathbb{H})$ is given by

$$(A, d) = (\wedge(b_4, \dots, b_{4k}, y_{4(n-k)+3}, \dots, y_{4n-1}), d),$$

where $db_i = 0$ and $dy_{4(n-k)+3} \in \wedge(b_4, \dots, b_{4k})$, while for $G_{k,n}(\mathbb{R})$, where n is odd, is given by

$$(B, d) = (\wedge(x_4, \dots, x_{4(k-1)}, u_{2k}, x'_4, \dots, x'_{4k}, y_3, y_7, \dots, y_{4n-1}), d),$$

where

$$dx_i = 0 = dx_j \qquad \text{and} \qquad dy_{4p-1} = \sum_{r+q=p} x_{4r} \cdot x'_{4q},\ \ 1 \leqslant p < n.$$

Consider the map $\phi\colon (A, d) \to (B, d)$, and the ideal

$$I = \Big\langle \sum_{r+q=p} x_{4r} \cdot x'_{4q} \Big\rangle.$$

Then,

$$H^*(B, d) = \wedge(x'_4, \dots, x'_{4k})/I.$$

Hence, there is a quasi-isomorphism

$$\bar{\phi}\colon \big(\wedge(b_4, \dots, b_{4k}, y_{4(n-k)+3}, \dots, y_{4n-1}), d\big) \to H^*(B, d),$$

where

$$\bar{\phi}(b_4) = x'_4, \quad \dots, \quad \bar{\phi}(b_{4k}) = x'_{4k},$$
$$\bar{\phi}(y_{4(n-k)+3}) = 0, \quad \dots, \quad \bar{\phi}(y_{4n-1}) = 0.$$

The case for n even is treated in the same way. The quasi-isomorphism $\bar{\phi}$ implies that $G_{k,n}(\mathbb{H})$ and $G_{k,n}(\mathbb{R})$ have the same dimension. Thus, computing the rational cohomology algebra of the component of the inclusion $\iota\colon G_{k,n}(\mathbb{R}) \hookrightarrow G_{k,n+r}(\mathbb{R})$ in the space of mappings from $G_{k,n}(\mathbb{R})$ to $G_{k,n+r}(\mathbb{R})$ one recovers the same results for the inclusion between the quaternion Grassmannians.

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Received: February 15, 2023, accepted: May 7, 2023.

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