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On weakly 1-convex sets in the plane

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Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. The present work considers the properties of generally convex sets in the plane known as weakly 1-convex. An open set is called *weakly* 1-*convex* if for any boundary point of the set there exists a straight line passing through this point and not intersecting the given set. A closed set is called *weakly* 1-*convex* if it is approximated from the outside by a family of open weakly 1-convex sets. A point of the complement of a set to the whole plane is called a *1-nonconvexity point* of the set if any straight passing through the point intersects the set. It is proved that if an open, weakly 1-convex set has a non-empty set of 1-nonconvexity points, then the latter set is also open. It is also shown that the non-empty interior of a closed, weakly 1-convex set in the plane is weakly 1-convex.

Анотація. У роботі розглядаються властивості узагальнено опуклих множин на площині, які називаються слабко 1-опуклими. Відкрита множина називається *слабко* 1-*опуклою*, якщо через довільну точку межі множини проходить пряма, яка цю множину не перетинає. Замкнена множина називається *слабко* 1-*опуклою*, якщо вона апроксимується ззовні сім'єю відкритих слабко 1-опуклих множин. Точка доповнення множини до всієї площини називається *точкою* 1- *неопуклості* множини, якщо довільна пряма, яка проходить через цю точку, перетинає задану множину. Доведено, що непорожня множина точок 1-неопуклості відкритої, слабко 1-опуклої множини на площині відкрита. Також доведено, що непорожня внутрішність замкненої, слабко 1-опуклої множини на площині слабко 1-опукла.

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1. Introduction

We consider sets in the real plane $\mathbb{R}^2$. Let us use the following standard notations. For a subset $A \subset \mathbb{R}^2$ let $\overline{A}$ be its closure, $\operatorname{Int} A$ be its interior, and $\partial A = \overline{A} \backslash \operatorname{Int} A$ be its boundary.

Definition 1.1 ([6]). *A subset $E \subset \mathbb{R}^2$ is called **1-convex** if for any point $x \in \mathbb{R}^2 \backslash E$ there exists a straight line γ such that $x \in \gamma$ and $\gamma \cap E = \varnothing$.*

Definition 1.2 ([7]). *An open subset $E \subset \mathbb{R}^2$ is called **weakly** 1**-convex** if for any point $x \in \partial E$ there exists a straight line γ such that $x \in \gamma$ and $\gamma \cap E = \varnothing$.*

Lemma 1.3 ([2]). *Any component of an open weakly 1-convex subset $E \subset \mathbb{R}^2$ is convex.*

Definition 1.4 ([1]). *A set A **is approximated from the outside** by a family of open sets A_k, $k \in \mathbb{N}$, if $\overline{A}_{k+1}$ is contained in A_k, and $A = \bigcap_k A_k$.*

It can be proved that any set approximated from the outside by a family of open sets is closed.

Definition 1.5 ([2,7]). *A closed subset $E \subset \mathbb{R}^2$ is called **weakly** 1**-convex** if it can be approximated from the outside by a family of open weakly 1-convex sets.*

Thus, any weakly 1-convex set A is either open or closed. Among closed weakly 1-convex sets there are also sets with empty interior:

$$A = \overline{A} = \overline{A} \backslash \operatorname{Int} A = \partial A.$$

Lemma 1.6 ([4]). *Let $E \subset \mathbb{R}^2$ be a closed subset with a finite number of components and such that $\operatorname{Int} E \neq \varnothing$. If E is weakly 1-convex, then $\operatorname{Int} E$ is weakly 1-convex.*

A simple example showing that not every closed set with weakly 1-convex interior is also weakly 1-convex is the following. Take any closed convex set in the plane and connect any two of its boundary points with a curve contained in the complement of the set to the whole plane. The union of the curve and the convex set is not a weakly 1-convex set whereas its interior is weakly 1-convex, obviously.

Denote the classes of 1-convex and weakly 1-convex sets in $\mathbb{R}^2$ by $\mathbf{C}_1^2$ and $\mathbf{WC}_1^2$, respectively. Any open set of the class $\mathbf{C}_1^2$ clearly belongs to the class $\mathbf{WC}_1^2$. The converse statement is not true. It turns out that the class $\mathbf{WC}_1^2\backslash\mathbf{C}_1^2$ of open weakly 1-convex but not 1-convex sets is not empty. Moreover, the following proposition holds:

Lemma 1.7 ([2]). *An open set of the class $\mathbf{WC}_1^2\backslash\mathbf{C}_1^2$ consists of at least three connected components.*

The examples of open and closed sets of the class $\mathbf{WC}_1^2\backslash\mathbf{C}_1^2$ with three and more connected components are constructed in [3, Examples 1-4]. It is also proved in [4] that the closed sets of the class $\mathbf{WC}_1^2\backslash\mathbf{C}_1^2$ consist of not less than three connected components.

Definition 1.8 ([5]). *A point $x \in \mathbb{R}^2\backslash E$ is called a* ***1-nonconvexity point of a subset*** *$E \subset \mathbb{R}^2$ if any straight passing through x intersects E.*

The set of all 1-nonconvexity points of a subset $E \subset \mathbb{R}^2$ is denoted by $E^{\triangle}$. Thus, a set E is not 1-convex iff $E^{\triangle} \neq \varnothing$.

Lemma 1.9 ([5]). *Let $E \subset \mathbb{R}^2$ be an open subset of the class $\mathbf{WC}_1^2\backslash\mathbf{C}_1^2$ with a finite number of components. Then $E^{\triangle}$ is open.*

In the present work, Lemmas 1.6 and 1.9 are proved without any restrictions on the number of components of the sets.

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2. Topological properties of open and closed weakly 1-convex sets in the plane

Given two points x, $y \in \mathbb{R}^2$, we will denote by $|x-y|$ the length of the straight segment between those points. Also, for $\varepsilon > 0$, let

$$U(y,\varepsilon) := \{x \in \mathbb{R}^2 : |x-y| < \varepsilon\}$$

be the ε-neighborhood of a point $y \in \mathbb{R}^2$.

Let

$$\gamma_\alpha(y) := \{(t\cos\alpha + y_1, t\sin\alpha + y_2) : -\infty < t < +\infty\}, \qquad \alpha \in [0, 2\pi),$$

be a straight line passing through a point $y = (y_1, y_2) \in \mathbb{R}^2$. Evidently,

$$\gamma_\alpha(y) = \gamma_{\alpha+\pi}(y), \;\; \alpha \in [0,\pi).$$

Lemma 2.1. *Let $E \subset \mathbb{R}^2$ be an open, not 1-convex subset and let E_j, $j \in N \subseteq \mathbb{N}$, be the connected components of E. Then for any point $y \in E^{\triangle}$*

there exists a subset

$$G(y) = \bigcup_{j \in M(y)} E_j, \; M(y) \subset N,$$

such that $M(y)$ is finite and $y \in G(y)^{\triangle}$.

Proof. Regard $\mathbb{R}^2$ as the complex plane $\mathbb{C}$ and let $O \in \mathbb{C}$ be the origin. Let also $S^1 = \{z : |z| = 1\}$ be the unit circle. Consider the homeomorphism $\phi\colon \mathbb{C}\backslash\{O\} \to S^1 \times (0, +\infty)$, defined by the formula

$$\phi(z) = \Big(\frac{z}{|z|}, |z|\Big).$$

Denote by $\sigma'\colon S^1 \times (0, +\infty) \to S^1$ the central projection on the circle, i.e.,

$$\sigma'(z) = \frac{z}{|z|},$$

and let

$$\sigma''(z) = \sigma'(z)e^{i\pi}.$$

Then σ' is continuous and open as the projection on a coordinate and so is $\sigma''(z)$.

Now fix a point $y \in E^{\triangle}$. Without loss of generality, assume that $y = O$. Then the images $\sigma'(E_j)$, $\sigma''(E_j)$ of each component E_j, $j \in N$, are either open arcs of S^1 or the whole S^1. Moreover,

$$\bigcup_{j \in N} \big(\sigma'(E_j) \cup \sigma''(E_j)\big)$$

is a cover of the unit circle S^1. By the Heine-Borel theorem, there exists a finite subcover of S^1 of the form

$$\bigcup_{\substack{i \in M'(y),\\ j \in M''(y)}} \sigma'(E_i) \cup \sigma''(E_j),$$

where $M'(y) \subset N$ and $M''(y) \subset N$.

Then the corresponding set $G(y)$, where $M(y) = M'(y) \cup M''(y)$, is what we need. Indeed, $M(y)$ is finite and $y \in G(y)^{\triangle}$, since there is a one-to-one correspondence between the lines $\gamma_\alpha(O)$, $\alpha \in [0, 2\pi)$, and pairs of antipodal points of S^1. □

Denote by $\mathrm{D}_x A$ the set of all points of the straight lines passing through a point $x \in \mathbb{R}^2\backslash A$ and intersecting a subset $A \subset \mathbb{R}^2$. If A is not empty, then $x \in \mathrm{D}_x A$. If A is open, then the set $\mathrm{D}_x A\backslash\{x\}$ is open.

Lemma 2.2. *Let $E \subset \mathbb{R}^2$ be open and not 1-convex subset and $y \in E^{\triangle}$. Then for any line $\gamma_\alpha(y)$, $\alpha \in [0, 2\pi)$, there exists a point $x_\alpha(y) \in \gamma_\alpha(y) \cap E$*

such that $U\big(x_\alpha(y), d(y)\big) \subset E$, *where* $d(y) > 0$ *depends only on* y *and does not depend on* α.

Proof. Let E_j, $j \in N \subseteq \mathbb{N}$, be the components of E. Fix y and consider a subset $G = \bigcup\limits_{j \in M} E_j \subset E$ satisfying the conditions of Lemma 2.1.

Consider two groups of the components E_j, $j \in M$. The first group consists of the components E_j, $j \in M'$, such that $\mathrm{D}_y E_j \backslash \{y\}$, $j \in M'$, are open angles of values $\leqslant \pi$, and the second one consists of the components E_j, $j \in M''$, such that $\mathrm{D}_y E_j$, $j \in M''$, coincide with $\mathbb{R}^2$. Moreover, since E_j, $j \in M$, are open and connected, we have that $M' \cup M'' = M$.

Consider the first group. Let

$$\mathrm{D}_{i,j} = \mathrm{D}_y E_i \cap \mathrm{D}_y E_j, \; i, j \in M,$$

see Figure 2.1a). As y is a 1-nonconvexity point of G, for any index $i \in M'$ there exist the indices $j(i) \in M$ such that $\mathrm{D}_{i,j(i)} \neq \varnothing$. Since the vertical angles $\mathrm{D}_y E_i \backslash \{y\}$, $i \in M'$, are open, we can reduce them so that the intersections of the reduced angles remain non-empty. Denote the reduced angles by $\widetilde{\mathrm{D}}_y E_i$. Then

$$\overline{\widetilde{\mathrm{D}}_y E_i} \subset \mathrm{D}_y E_i, \; i \in M'.$$

The boundary of each $\widetilde{\mathrm{D}}_y E_i$, $i \in M'$, consists of two straight lines passing through y. Denote them by $\gamma_{\alpha_i^1}(y)$, $\gamma_{\alpha_i^2}(y)$, see Figure 2.1b). Moreover,

$$\gamma_{\alpha_i^1}(y), \gamma_{\alpha_i^2}(y) \subset \mathrm{D}_y E_i,$$

and thus,

$$\gamma_{\alpha_i^1}(y) \cap E_i \neq \varnothing, \qquad \gamma_{\alpha_i^2}(y) \cap E_i \neq \varnothing.$$

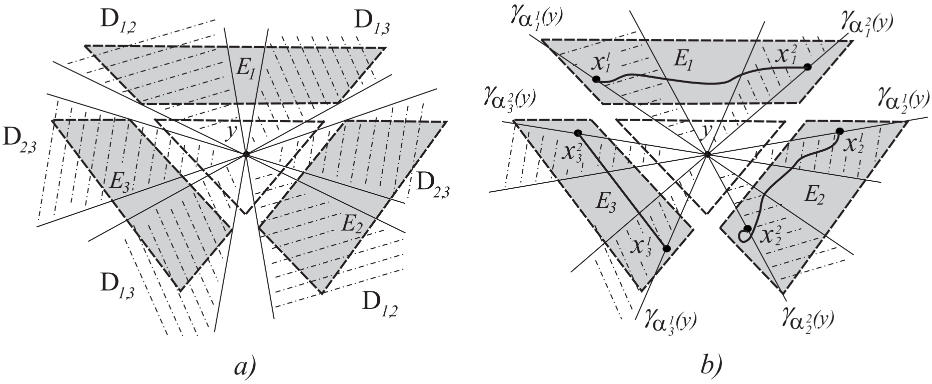

FIGURE 2.1.

The we can take any points

$$x_i^1 \in \gamma_{\alpha_i^1}(y) \cap E_i, \qquad\qquad x_i^2 \in \gamma_{\alpha_i^2}(y) \cap E_i,\ (i \in M'),$$

and choose any curves $\lambda_i \subset E_i$ connecting x_i^1 and x_i^2.

Now consider the second group. Let also x_i^1, x_i^2, $i \in M''$ be points of E_i, $i \in M''$, such that a curve $\lambda_i \subset E_i$ connecting them, satisfies the condition

$$\mathrm{D}_y\lambda_i = \mathrm{D}_y E_i = \mathbb{R}^2.$$

Consider the following function of $x \in E_i$, $i \in M$:

$$d_i(x) = \inf_{x^0 \in \partial E_i} |x - x^0|.$$

It is continuous in its domain. Then its restriction to the compact λ_i reaches its minimum $d_i > 0$ on this compact, i.e.,

$$d_i = \min_{x \in \lambda_i} d_i(x),\ i \in M.$$

Since G has finite number of components, there exists

$$d = \min_{i \in M} d_i > 0.$$

Then $U(x, d) \subset G \subset E$ for any point $x \in \lambda_i$, $i \in M$. Moreover, for any $\alpha \in [0, 2\pi)$ there exists $i \in M$, such that $\gamma_\alpha(y) \cap \lambda_i \neq \varnothing$ by construction. Lemma is proved. □

Theorem 2.3. *Let $E \subset \mathbb{R}^2$ be an open subset of the class $\mathbf{WC}_1^2 \backslash \mathbf{C}_1^2$. Then $E^\triangle$ is open.*

Proof. Let E_j, $j \in N \subseteq \mathbb{N}$, be the components of E. Fix an arbitrary point $y \in E^\triangle$ and prove that it is an inner point of $E^\triangle$. Since E is weakly 1-convex, $y \notin \partial E$. Then there exists a number $\varepsilon_1 > 0$ such that

$$U(y, \varepsilon_1) \subset (\mathbb{R}^2 \backslash \overline{E}).$$

By Lemma 2.2, for the fixed y there exist a constant $d > 0$ and points $x_\alpha \in \gamma_\alpha(y) \cap E$, $\alpha \in [0, 2\pi)$, such that $U(x_\alpha, d) \subset E$.

Let $\varepsilon = \min\{\varepsilon_1, d\}$. Consider the neighborhood $U(y, \varepsilon)$ of the point y. Let $z \in U(y, \varepsilon)$ and $\gamma_\alpha(z)$, $\alpha \in [0, 2\pi)$, be an arbitrary line passing through the point z. Draw the line $\gamma_\alpha(y)$ parallel to $\gamma_\alpha(z)$. Since

$$U(x_\alpha, \varepsilon) \subseteq U(x_\alpha, d) \subset E$$

for the point x_α corresponding to $\gamma_\alpha(y)$, it follows that

$$\gamma_\alpha(z) \cap U(x_\alpha, \varepsilon) \neq \varnothing$$

and, therefore, $\gamma_\alpha(z) \cap E \neq \varnothing$ for any $\alpha \in [0, 2\pi)$, see Figure 2.2a). Since the point z was chosen arbitrarily, all points of $U(y, \varepsilon)$ are 1-nonconvexity points of E. Thus, y is an inner point of $E^\triangle$. □

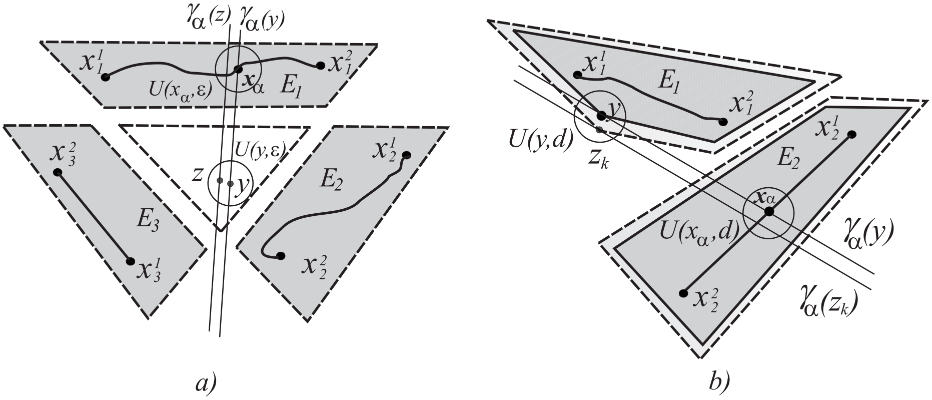

FIGURE 2.2.

Theorem 2.4. *Let $E \subset \mathbb{R}^2$ be a closed subset such that* $\operatorname{Int} E \neq \varnothing$. *If E is weakly* 1*-convex, then* $\operatorname{Int} E$ *is weakly* 1*-convex.*

Proof. Suppose that $\operatorname{Int} E$ is not weakly 1-convex. Then there exists a 1-nonconvexity point $y \in \partial E$ of the set $\operatorname{Int} E$.

By Lemma 2.2, for the point y there exist a constant $d > 0$ and points $x_\alpha \in \gamma_\alpha(y) \cap \operatorname{Int} E$, $\alpha \in [0, 2\pi)$, such that $U(x_\alpha, d) \subset \operatorname{Int} E$.

Consider the neighborhood $U(y, d)$ of the point y, see Figure 2.2b). Since E is weakly 1-convex, there exists a family of open weakly 1-convex sets G_k, $k \geqslant 1$, approximating E from the outside. Then one can find an index k_0 such that $\partial G_k \cap U(y, d) \neq \varnothing$ for all $k \geqslant k_0$. For each $k \geqslant k_0$ choose a point $z_k \in \partial G_k \cap U(y, d)$ and draw an arbitrary line $\gamma_\alpha(z_k)$, $\alpha \in [0, 2\pi)$, through z_k. Consider the line $\gamma_\alpha(y)$ parallel to $\gamma_\alpha(z_k)$. Since

$$U(x_\alpha, d) \subset \operatorname{Int} E \subset E$$

for the point x_α corresponding to $\gamma_\alpha(y)$, it follows that

$$\gamma_\alpha(z_k) \cap U(x_\alpha, d) \neq \varnothing$$

and therefore

$$\gamma_\alpha(z_k) \cap E \neq \varnothing.$$

As $G_k \supset E$, $k \in \mathbb{N}$, we have that

$$\gamma_\alpha(z_k) \cap G_k \neq \varnothing, \ k \geqslant k_0.$$

Since the line $\gamma_\alpha(z_k)$ is arbitrary, the point $z_k \in \partial G_k$ is a 1-nonconvexity point of G_k for all $k \geqslant k_0$, which gives a contradiction. Thus, the assumption is wrong and theorem is proved. □

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