

# Pseudocompact and precompact topological subsemigroups of topological groups

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**Abstract.** It is known that every pseudocompact topological group is precompact, we extend this result to a class of subsemigroup of topological groups. Then we use this results to prove that cancellative locally compact countably compact topological semigroups with open shifts are topological groups and to give a sufficient condition under which a locally compact monothetic topological semigroup is a compact topological group.

**Аноація.** Відомо, що кожна псевдокомпактна топологічна група є передкомпактною. В даній роботі цей результат поширено на клас підгруп топологічних груп. В якості застосування показано, що локально компактні зліченно компактні топологічні напівгрупи із скороченнями та відкритими зсувами є топологічними групами, а також отримано достатню умову, за якої локально компактна монотетична топологічна напівгрупа є компактною топологічною групою.

## 1. INTRODUCTION

The literature exhibits a variety of sufficient conditions for a cancellative topological semigroup  $S$  to be a topological group. The first of them is given in [12], where it was proven that every cancellative Hausdorff compact topological semigroup is a topological group. A similar result can be found in [5], where it is used sequential compactness instead of compactness. In [16] it is proven that every cancellative countably compact sequential topological semigroup is a topological group. The problem of when a cancellative topological semigroup is a topological group was recently addressed in [2], where the authors proved that a polybounded cancellative  $T_1$

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topological semigroup is a topological group. In [9], it was recently proven the following fact.

**Theorem 1.1.** *Every commutative cancellative locally compact pseudocompact topological semigroup with open shifts is a compact topological group.*

In this paper we expanded this list of results in Section 3, for that we use pseudocompactness, precompactness and local compactness mainly. The fact that every commutative cancellative semigroup can be embedded in a group (see [8]) plays a crucial role in the proof of Theorem 1.1, in Theorem 3.11 we removed the commutativity but we use countable compactness instead of pseudocompactness.

## 2. PRELIMINARIES

Let  $X$  be a topological space. We say that  $X$  is *pseudocompact* if each continuous function  $f : X \rightarrow \mathbb{R}$  is bounded.  $X$  is called *feebly compact* if every locally finite family of open sets in  $X$  is finite. By [11, Theorem 1.1.3], feebly compact is equivalent to pseudocompact for Tychonoff spaces. If  $X$  is a Tychonoff space, we denote by  $\beta X$  the Čech-Stone compactification of  $X$ . If  $X$  is a Tychonoff topological semigroup such that  $X \times X$  is pseudocompact, [4, Theorem 2.2] tells us that the semigroup operation of  $X$  extends to a continuous semigroup operation on  $\beta X$ . If  $X$  is pseudocompact, locally compact and  $T_2$ , then  $X \times X$  is pseudocompact.

A subset  $A$  of  $X$  is said to be a *retract* of  $X$  if there exists a continuous mapping  $f : X \rightarrow A$  such that  $f(a) = a$  for every  $a \in A$ . If  $A$  is a clopen in  $X$ , then  $A$  is a retract of  $X$ . If  $A$  is a retract of  $X$  and  $f : A \rightarrow Y$  is a continuous map, then  $f$  extends a continuous map on all of  $X$ . Therefore, each retract of a pseudocompact space is pseudocompact too.

A family  $\mathcal{F}$  of subsets of  $X$  has *finite intersection property* (FIP) if for every finite subset  $\mathcal{A}$  of  $\mathcal{F}$ , we have that  $\bigcap \mathcal{A} \neq \emptyset$ . A space  $X$  is countably compact if each countable family of closed sets in  $X$ , with the FIP, has non-empty intersection. Countable compactness implies pseudocompactness for any, non-necessary Tychonoff, space.

Let  $S$  be a semigroup, an element  $x \in G$  is called *idempotent* if  $x^2 = x$ . If  $S$  has neutral element, we say that  $S$  is a *monoid* and its neutral element will be denoted by  $1_S$ . If  $S$  is a topological monoid, we will denote by  $\mathfrak{V}_{1_S}$  the set of neighbourhoods of  $1_S$ . If  $G$  is a topological group,  $\mathfrak{V}_{1_G}^*$  will be denote the set of symmetric neighbourhoods of  $1_G$ . The mapping from  $G$  to  $G$  defined by  $x \mapsto x^{-1}$ , for every  $x \in X$ , will be denoted by  $\text{In}_G$ .

A subset  $A$  of topological group  $G$  is *precompact* if for a given  $U \in \mathfrak{V}_{1_G}$ , there exists a finite subset  $F$  of  $G$  such that  $A \subseteq (FU) \cap (UF)$ . If  $A$  and  $B$  are precompact subsets of  $G$ , then  $AB$ ,  $\overline{A}$  and  $B^{-1}$  are precompact sets.

Moreover, if  $C \subseteq B$ , then  $C$  is precompact too. Obviously, each compact topological group is precompact.

If  $G$  is a topological group,  $V \in \mathfrak{V}_{1_G}$  and  $A \subseteq G$ , we say that  $A$  is  $V$ -disjoint if  $a \notin bV$  for all distinct  $a, b \in A$ . The Zorn's Lemma allows us to have a maximal  $A \subseteq G$ ,  $V$ -disjoint.

A topological semigroup  $S$  has open shifts, if for each  $a \in S$  the left shift  $S \rightarrow S, x \mapsto ax$  and the right shift  $S \rightarrow S, x \mapsto xa$  are open.

### 3. MAIN RESULTS

Every topological semigroup is assumed to be nonempty and  $T_2$ . If  $X$  is a topological space and  $A \subseteq X$ , then  $\text{cl}_X(A)$ , or simply  $\overline{A}$ , if there is no possibility of confusion, denotes the closure  $A$  in  $X$ .

**Lemma 3.1** ([1, Lemma 3.7.26]). *Let  $A$  be a precompact subset of a topological group  $G$ . If  $\{x_n : n \in \mathbb{N}\} \subseteq A$ , then*

$$1_G \in \text{cl}_G(\{x_i x_j^{-1} : i < j, i, j \in \mathbb{N}\}).$$

**Lemma 3.2.** *If  $S$  is a precompact subsemigroup of a topological group  $G$ , then  $\overline{S}$  is a submonoid of  $G$ .*

*Proof.* Note that  $\overline{S}$  is a subsemigroup of  $G$ . Let  $x \in \overline{S}$ . Then

$$\{x^n : n \in \mathbb{N}\} \subseteq \overline{S},$$

and by Lemma 3.1,  $1_G \in \overline{\{x^{-n} : n \in \mathbb{N}\}}$ . But  $\text{In}_G$  is a homeomorphism, therefore

$$\begin{aligned} 1_G = \text{In}_G(1_G) &\in \text{In}_G(\overline{\{x^{-n} : n \in \mathbb{N}\}}) = \overline{\text{In}_G(\{x^{-n} : n \in \mathbb{N}\})} \\ &= \overline{\{x^n : n \in \mathbb{N}\}} \subseteq \overline{S}, \end{aligned}$$

whence  $\overline{S}$  is a submonoid of  $G$ . □

Now [13, Corollary 3.4] tells us that the closure of any subgroup of a precompact topological group is a group, since subgroups of precompact topological groups are precompact too, then the following theorem generalizes this result.

**Theorem 3.3.** *If  $S$  is a precompact subsemigroup of a topological group  $G$ , then  $\overline{S}$  is a subgroup of  $G$*

*Proof.* Since  $\overline{S}$  is a submonoid of  $G$ , to prove that  $\overline{S}$  is a subgroup of  $G$  it is enough to show that  $x^{-1} \in \overline{S}$  for any  $x \in \overline{S}$ . Indeed, let  $x \in \overline{S}$ , since that  $x\overline{S}$  is a precompact subsemigroup of  $G$  and  $x\overline{S} = \overline{xS}$ , then by Lemma 3.2

we have that  $x\bar{S}$  is a monoid, then  $1_G \in x\bar{S}$ , hence there exists  $y \in \bar{S}$ , such that  $xy = 1_G$ , then  $x^{-1} = y \in \bar{S}$ .  $\square$

If  $S$  is an open precompact subsemigroup of a topological group  $G$ , then Lemma 3.3 implies that  $S$  is an open subsemigroup of the topological group  $\bar{S}$ . By [9, Corollary 4] we have that  $S$  is a closed subgroup of  $G$ . This together with Theorem 3.3, give us the following corollary.

**Corollary 3.4.** *Let  $S$  be a precompact subsemigroup of a topological group  $G$ . If  $S$  is either open or closed in  $G$ , then  $S$  is a closed subgroup of  $G$ .*

If  $G = \mathbb{R}/\mathbb{Z}$ , then  $S = (\mathbb{Z} + \sqrt{2}\mathbb{N})/\mathbb{Z}$  is a precompact subsemigroup of  $G$  that fails to be a subgroup. This proves that closedness can not be removed in Theorem 3.3.

From [11, 2.3.1] it is known that every pseudocompact topological group is precompact, the next two theorems extend this result to a class of subsemigroups of topological groups.

**Theorem 3.5.** *Each open pseudocompact submonoid of a topological group is precompact.*

*Proof.* Let  $S$  be an open pseudocompact submonoid of a topological group  $G$ . Put  $H = S \cap S^{-1}$ . Then  $H$  is an open subgroup of  $G$  and therefore  $H$  is closed. So that  $H$  is open and closed in  $S$ . Then it is a retract of  $S$ , thus we have that  $H$  is a subgroup pseudocompact and hence  $H$  is also precompact. Hence

$$S = \bigcup_{x \in S} xH$$

is a topological sum of the family of pseudocompact topological spaces  $\{xH : x \in S\}$ . The pseudocompactness of  $S$  and [7, Theorem 3.10.25] imply that there are  $x_1, x_2, \dots, x_n \in S$  such that  $S = \bigcup_{i=1}^n x_iH$ , then  $S$  is precompact, being a finite union of precompact sets.  $\square$

We adapted the argument used in the proof of [11, 2.3.1] to prove the following result.

**Theorem 3.6.** *Each open pseudocompact subsemigroup of an abelian topological group is precompact.*

*Proof.* Let  $S$  be an open subsemigroup of an abelian topological group  $G$  and  $x_0 \in S$ . Let  $U \in \mathfrak{V}_{1_G}^*$  such that  $U \subseteq x_0^{-1}S$ . Then there exists  $V \in \mathfrak{V}_{1_G}^*$  such that  $V^2 \subseteq U$ . We can also choose  $W \in \mathfrak{V}_{1_G}^*$  such that  $W^4 \subseteq V$ . By Zorn's Lemma there exists a maximal  $V$ -disjoint subset  $A$  of  $G$ . By the

proof of [11, Lemma 2.3.1], it is known that  $G = AV = AU$  and also that  $\{aW : a \in A\}$  is discrete. If  $x \in S$ , then  $x = au$ , where  $a \in A$  and  $u \in U$ , thus

$$\begin{aligned} a &= xu^{-1}x_0x_0^{-1} \in xx_0^{-1}x_0U^{-1} = xx_0^{-1}x_0U \subseteq \\ &\subseteq xx_0^{-1}S = x_0^{-1}xS \subseteq x_0^{-1}S. \end{aligned}$$

Therefore  $a \in A \cap (x_0^{-1}S)$ , and thus  $S \subseteq \bigcup\{aU : a \in A \cap (x_0^{-1}S)\}$ . Now, since

$$\{aW : a \in A \cap (x_0^{-1}S)\} \subseteq \{aW : a \in A\}$$

we have that

$$\{aW : a \in A \cap (x_0^{-1}S)\}$$

is discrete. If  $a \in A \cap (x_0^{-1}S)$  and  $w \in W$ , then  $a = x_0^{-1}s$  for some  $s \in S$  and therefore

$$\begin{aligned} aw &= x_0^{-1}sw \in x_0^{-1}sW \subseteq x_0^{-1}sW^4 \subseteq x_0^{-1}sV \subseteq x_0^{-1}sV^2 \\ &\subseteq x_0^{-1}sU \subseteq x_0^{-2}sS \subseteq x_0^{-2}S, \end{aligned}$$

so that  $aW \subseteq x_0^{-2}S$ .

This proves that  $\{aW : a \in A \cap (x_0^{-1}S)\}$  is a discrete family of open sets in  $x_0^{-2}S$ , but  $x_0^{-2}S$  is pseudocompact, therefore  $A \cap (x_0^{-1}S)$  is a finite set. We have proven that for each  $U \in \mathfrak{A}_{1G}^*$ , such that  $U \subseteq x_0^{-1}S$ , there exists a finite subset  $A$  of  $G$  such  $S \subseteq AU$ . Let us see that  $S$  is precompact, indeed let  $U \in \mathfrak{A}_{1G}$ , then  $U \cap (x_0^{-1}S) \in \mathfrak{A}_{1G}$ . We can find  $V \in \mathfrak{A}_{1G}^*$  such that

$$V \subseteq U \cap (x_0^{-1}S) \subseteq x_0^{-1}S$$

by what we have proven, there exists a finite subset  $A$  of  $G$  such that  $S \subseteq AV \subseteq AU$ , this proves that  $S$  is precompact.  $\square$

**Corollary 3.7.** *Let  $S$  be a subsemigroup of an abelian topological group  $G$  such that  $S \times S$  is pseudocompact. Then  $\overline{S}$  is a precompact subgroup of  $G$ .*

*Proof.* Since  $S \times S$  is pseudocompact and  $G$  is abelian, we have that

$$K = SS^{-1}$$

is a pseudocompact topological group, therefore it is precompact due to [11, Lemma 2.3.1]. Notice that  $S$  is precompact, because it is a subset of  $K$ , hence Theorem 3.3 implies that  $\overline{S}$  is a precompact subgroup of  $G$ .  $\square$

**Theorem 3.8.** *Let  $S$  be a pseudocompact subsemigroup of a topological group  $G$ . Then  $S$  is a subgroup of  $G$  provided that one of the following conditions holds:*

- (i)  $G$  is abelian and  $S$  is open;

- (ii)  $S$  is an open monoid;
- (iii)  $S \times S$  is pseudocompact,  $G$  is abelian and  $S$  is closed.

*Proof.* Let us prove (i). From Lemma 3.2 together with Theorem 3.6 we have that  $\bar{S}$  is a subgroup of  $G$ . Therefore  $S$  is an open subsemigroup of the precompact topological group  $\bar{S}$ . Then [9, Corollary 1] implies that  $S$  is a subgroup of  $G$ . We can apply a similar argument to prove (ii). Finally, (iii) follows immediately from Corollary 3.7.  $\square$

A topological semigroup  $S$  is said to be *monothetic* if there is  $x \in S$  such that

$$S = \overline{\{x^n : n \in \mathbb{N}\}}.$$

In this case,  $x$  is called a *generator* of  $S$ .

From [15, Theorem 3.2 b) iii)] it is known that every compact monothetic semigroup with two distinct generators is a topological group, now we can extend this result to locally compact pseudocompact monothetic semigroup.

**Corollary 3.9.** *Each locally compact pseudocompact monothetic semigroup with two distinct generators is a compact topological group.*

*Proof.* Let  $S$  be a locally compact pseudocompact monothetic semigroup with two distinct generators  $a$  and  $b$ . It is not hard to prove that  $a$  and  $b$  are generators of  $\beta S$ . By [15, Theorem 3.2 b) iii)],  $\beta S$  is a topological group. The local compactness of  $S$  implies that  $S$  is an open pseudocompact of the abelian topological group  $\beta S$ , therefore we have that  $S$  is closed topological subgroup of  $\beta S$ , thus  $S$  is compact.  $\square$

By [15, Theorem 3.2 b) ii)] every compact monothetic connected topological semigroup is a group, this result can be easily extended to locally compact pseudocompact monothetic connected topological semigroup  $S$ , because if  $S$  is connected,  $\beta S$  is a connected compact topological group and therefore  $S$  is an open pseudocompact subsemigroup of an abelian topological group and we can apply Theorem 3.6 and say that  $S$  is a compact topological group.

Corollary 3.9 is framed within the context of Koch's problem, which asks whether every locally compact monothetic monoid is compact, we give a positive answer in terms of the pseudocompactness. Even when negative answers are already known for this problem, [3] contains positive answers in special cases.

By [6, Theorem 1.8] we have that each compact topological semigroup contains an idempotent, in the following lemma, we extend this result to countably compact locally compact topological semigroups.

**Lemma 3.10.** *A locally compact countably compact topological semigroup contains an idempotent.*

*Proof.* Let  $S$  be a locally compact countably compact topological semigroup and let  $x \in S$ . Since  $\beta S$  is a compact topological semigroup, from [10, Lemma 9.17] we have that

$$\bigcap_{n=1}^{\infty} \text{cl}_{\beta S} (\{x^k : k \geq n\})$$

is a compact subgroup of  $\beta S$ . Note that  $\{\text{cl}_S (\{x^k : k \geq n\})\}_{n \in \mathbb{Z}^+}$  is a countable family of closed subgroups in  $S$  with the FIP, therefore

$$\bigcap_{n=1}^{\infty} \text{cl}_S (\{x^k : k \geq n\}) \neq \emptyset,$$

but

$$\begin{aligned} \bigcap_{n=1}^{\infty} \text{cl}_S (\{x^k : k \geq n\}) &= \bigcap_{n=1}^{\infty} (\text{cl}_{\beta S} (\{x^k : k \geq n\} \cap S)) \\ &= \left( \bigcap_{n=1}^{\infty} \text{cl}_{\beta S} (\{x^k : k \geq n\}) \right) \cap S \end{aligned}$$

since  $S$  is open in  $\beta S$ , we have that  $\bigcap_{n=1}^{\infty} \text{cl}_S (\{x^k : k \geq n\})$  is an open subsemigroup of the compact group  $\bigcap_{n=1}^{\infty} \text{cl}_{\beta S} (\{x^k : k \geq n\})$ , hence

$$\bigcap_{n=1}^{\infty} \text{cl}_S (\{x^k : k \geq n\})$$

is a subgroup of  $S$ . The element neutral of each subgroup of  $S$  is an idempotent in  $S$ , this proves that  $S$  contains an idempotent.  $\square$

Finally we present a modification of Theorem 1.1.

**Theorem 3.11.** *Each locally compact countably compact cancellative topological semigroup with open shifts is a compact topological group.*

*Proof.* Let  $S$  be locally compact countably compact cancellative topological semigroup with open shifts. In order to show that  $S$  is a group, we will prove that  $xS = S = Sx$  for every  $x \in S$ . Indeed, let  $x \in S$ , from Lemma 3.10,  $S$  has an idempotent which is the neutral element  $1_S$  because  $S$  is cancellative. Now, the openness and continuity of the shifts imply that  $xS$  is a locally compact countably compact topological semigroup, we can

apply Lemma 3.10 again to say that  $xS$  has an idempotent which coincides with  $1_S$ . Thus there exists  $y \in S$  such that  $xy = 1_S$ , then

$$S = 1_S S = (xy)S = x(yS) \subseteq xS,$$

this implies that  $S = xS$ . Similarly it is proven that  $Sx = S$ , so that  $S$  is a compact group.  $\square$

In [16] it is proven that cancellative countably compact sequential topological semigroup is a group, in Theorem 3.11 we give a similar result, but we have change sequentialness by local compactness plus openness of the shifts.

The Wallace's problem asks whether every  $T_2$  cancellative countably compact topological semigroup is a topological group. A negative answer to this problems is given in [14]. Theorem 3.11 gives us a positive answer in terms of local compactness.

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