Proceedings of the
International Geometry Center
Vol. 16, no. 1 (2023) pp. 50–58

On the asymptotic behavior at infinity of one mapping class

Ruslan Salimov, Bogdan Klishchuk, Mariia Stefanchuk

Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. We study the asymptotic behavior at infinity of ring Q-homeomorphisms with respect to p-modulus for $p > n$.

Анотація. Ми досліджуємо асимптотичну поведінку на нескінченності кільцевих Q-гомеоморфізмів відносно p-модуля для $p > n$.

1. Introduction

Let us recall some definitions, see [36]. Let Γ be a path family in $\mathbb{R}^n$, $n \geqslant 2$. A Borel function $\rho\colon \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, (abbr. $\rho \in \mathrm{adm}\Gamma$), if

$$\int\limits_{\gamma} \rho(x)ds \;\geqslant\; 1$$

for all locally rectifiable $\gamma \in \Gamma$.

Let $p \in (1, \infty)$. Recall also that the *p–modulus* of Γ is the quantity

$$\mathrm{M}_p(\Gamma) \;=\; \inf_{\rho \in \mathrm{adm}\Gamma} \int\limits_{\mathbb{R}^n} \rho^p(x)dm(x),$$

where $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$.

Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, $x_0 \in D$ and $d_0 = \mathrm{dist}(x_0, \partial D)$. For arbitrary sets E, F and G of $\mathbb{R}^n$ we denote by $\Delta(E, F, G)$ a set of all

Keywords: ring Q-homeomorphisms, p-modulus of a family of curves, quasiconformal mappings, condenser, p-capacity of a condenser

Ключові слова: кільцеві Q-гомеоморфізми, p-модуль сім'ї кривих, квазіконформні відображення, конденсатор, p-ємність конденсатора

DOI: http://dx.doi.org/10.15673/tmgc.v16i1.2394

continuous curves $\gamma\colon [a,b] \to \mathbb{R}^n$ that connect E and F in G, i.e., such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in G$ for $a < t < b$. Set

$$\begin{aligned} \mathbb{A}(x_0, r_1, r_2) =&\{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}, \\ S_i = S(x_0, r_i) =&\{x \in \mathbb{R}^n : |x - x_0| = r_i\},\ i = 1, 2. \end{aligned}$$

Let $Q\colon D \to [0,\infty]$ be a measurable function. Recall that a homeomorphism $f\colon D \to \mathbb{R}^n$ is said to be a *Q-homeomorphism with respect to p-modulus* if

$$\mathrm{M}_p(f\Gamma) \leqslant \int\limits_D Q(x)\rho^p(x)\ dm(x) \tag{1.1}$$

for every family Γ of paths in D and every admissible function ρ for Γ.

This conception is a natural generalization of the geometric definition of a quasiconformal mapping: if $Q(x) \leqslant K < \infty$ a.e., then

- f is quasiconformal for $p = 2$ in $\mathbb{C}$, (see [1, Definition A, p. 21-22]) and for $p = n$ in $\mathbb{R}^n$, $n \geqslant 2$, ([36, 13.1 & 34.6]);
- f has local Lipschitz property, for $n-1 < p < n$, f^{-1} is Lipschitz, for $p > n$, and the bounds for p are sharp (see [5]).

Note that the estimate of the type (1.1) was first established in the classical quasiconformal theory, (see [13, p. 221]). Further, it was obtained in [2, Lemma 2.1], for quasiconformal mappings in space $\mathbb{R}^n$, $n \geqslant 2$. This class of Q-homeomorphisms with respect to the n-modulus was first considered in the papers [16–18], see also the monograph [19].

The main goal of the theory of Q-homeomorphisms is to clarify various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbb{R}^n$ first in the case $Q \in BMO$ (bounded mean oscillation) in the papers [17, 18] and then in the case of $Q \in FMO$ (finite mean oscillation) and other cases in the papers [9, 10, 21, 24].

The following concept generalizes and localizes the concept of a Q-homeomorphism. It is motivated by the ring definition of quasiconformal mappings in the sense of Gehring (see [4]), introduced originally by Ryazanov, Srebro, and Yakubov on the plane, and later extended by Ryazanov and Sevost'yanov in the space $\mathbb{R}^n, n \geqslant 2$, (see [22], [19, Chapters VII and XI]).

Let $Q\colon D \to [0,\infty]$ be Lebesgue measurable function. We say that a homeomorphism $f\colon D \to \mathbb{R}^n$ is a *ring Q-homeomorphism with respect to p-modulus at $x_0 \in D$* if the relation

$$\mathrm{M}_p\big(\Delta(fS_1, fS_2, fD)\big)\ \leqslant\ \int\limits_{\mathbb{A}} Q(x)\eta^p(|x - x_0|)dm(x)$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0$, $d_0 = \mathrm{dist}(x_0, \partial D)$, and for any measurable function $\eta\colon (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r)dr = 1.$$

The theory of ring Q-homeomorphisms for $p = n$ was studied in [19, 22, 23, 34], for $1 < p < n$ in [6–8, 25–27] and for $p > n$ in [11, 28–33]. This theory can be applied to the mappings of finite distortion belonging to the Orlicz–Sobolev classes $W^{1,\varphi}_{\mathrm{loc}}$ under the Calderon condition, and, in particular, to the Sobolev classes $W^{1,p}_{\mathrm{loc}}$ with $p > n - 1$, (see [8, 12]).

In this paper, we obtain an analogue of Martio-Rickman-Vaisal's theorem on the growth at infinity of quasi-regular mappings (see [15]). The case $p = n$ was studied in [34].

Denote by ω_{n-1} the area of the unit sphere $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ and by

$$q_{x_0}(r) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{S(x_0,r)} Q(x) d\mathcal{A}$$

the integral mean over the sphere $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$, here $d\mathcal{A}$ is the element of the surface area.

Now we formulate a criterion which guarantees that a homeomorphism is a ring Q-homeomorphism with respect to p-modulus for $p > 1$ in $\mathbb{R}^n$, $n \geqslant 2$.

Proposition 1.1. [26, Theorem 2.3] *Let D be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and $Q\colon D \to [0, \infty]$ be a Lebesgue measurable function such that $q_{x_0}(r) \neq \infty$ for a.e. $r \in (0, d_0)$, $d_0 = \mathrm{dist}(x_0, \partial D)$. A homeomorphism $f\colon D \to \mathbb{R}^n$ is a ring Q-homeomorphism with respect to p-modulus at a point $x_0 \in D$ if and only if the inequality*

$$\mathrm{M}_p\big(\Delta(fS_1, fS_2, f\mathbb{A})\big) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_1}^{r_2} \frac{dr}{r^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(r)}\right)^{p-1}}$$

holds for any $0 < r_1 < r_2 < d_0$.

Following [14], a pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and C is a nonempty compact set contained in A, is called *condenser*. We say that a condenser $\mathcal{E} = (A, C)$ *lies in a domain* D if $A \subset D$.

Clearly, if $f\colon D \to \mathbb{R}^n$ is a homeomorphism and $\mathcal{E} = (A, C)$ is a condenser in D then $(f(A), f(C))$ is also condenser in $f(D)$. Further, we denote $f(\mathcal{E}) = (f(A), f(C))$.

We will recall also the notion of the p-capacity. Given a condenser

$$\mathcal{E} = (A, C)$$

denote by $\mathcal{C}_0(A)$ the set of all continuous functions $u\colon A \to \mathbb{R}^1$ with compact support. Let also $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ be the family of nonnegative functions $u\colon A \to \mathbb{R}^1$ such that

(1) $u \in \mathcal{C}_0(A)$,

(2) $u(x) \geqslant 1$ for $x \in C$,

(3) u belongs to the class ACL and $|\nabla u| = \left(\sum\limits_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2\right)^{\frac{1}{2}}$.

Then for $p \geqslant 1$ the quantity

$$\mathrm{cap}_p\mathcal{E} = \mathrm{cap}_p(A, C) = \inf_{u\in\mathcal{W}_0(\mathcal{E})} \int\limits_A |\nabla u|^p dm(x)$$

is called the *p-capacity* of the condenser $\mathcal{E}$.

It is known ([35, Theorem 1]) that for $p > 1$

$$\mathrm{cap}_p\mathcal{E} = \mathrm{M}_p\big(\Delta(\partial A, \partial C, A\backslash C)\big). \tag{1.2}$$

Also for $p > n$ we have the another inequality

$$\mathrm{cap}_p(A, C) \geqslant n\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} \left[m^{\frac{p-n}{n(p-1)}}(A) - m^{\frac{p-n}{n(p-1)}}(C)\right]^{1-p}, \tag{1.3}$$

where Ω_n is a volume of the unit ball in $\mathbb{R}^n$ (see, e.g., (8.7) in [20]).

Let us recall the so-called isodiametric inequality or Bieberbach inequality (1915), see [3, Corollary 2.10.33]. Here and in what follows, $\mathrm{diam}(\cdot)$ denotes the Euclidean diameter in $\mathbb{R}^n, n \geqslant 2$.

Proposition 1.2. *Let E be a compact set in $\mathbb{R}^n, n \geqslant 2$. Then*

$$m(E) \leqslant 2^{-n}\Omega_n\,(\mathrm{diam}E)^n\,,$$

where Ω_n is a volume of the unit ball in $\mathbb{R}^n$.

2. Main results

Now we present the main result of our paper concerning the behavior at infinity of ring Q-homeomorphisms with respect to p-modulus for $p > n$.

Theorem 2.1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a ring Q-homeomorphism with respect to p-modulus at a point x_0 with $p > n$, where x_0 is some point in $\mathbb{R}^n$. Suppose that for some numbers $r_0 > 1$ and $\kappa = \kappa(x_0) > 0$ the following condition*

$$q_{x_0}(t) \leqslant \kappa t^{p-n}\,(\ln t)^{\alpha} \tag{2.1}$$

holds for a.e. $t \in [r_0, +\infty)$*. If* $\alpha \in [0, p-1)$*, then*

$$\liminf_{R\to\infty} \frac{\mathrm{diam} f(B(x_0,R))}{(\ln R)^{\frac{p-\alpha-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-\alpha-1}\right)^{\frac{p-1}{p-n}} > 0.$$

If $\alpha = p-1$*, then*

$$\liminf_{R\to\infty} \frac{\mathrm{diam} f(B(x_0,R))}{(\ln\ln R)^{\frac{p-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}} > 0,$$

where $B(x_0, R) = \{x \in \mathbb{R}^n : |x - x_0| \leqslant R\}$*.*

Proof. Consider a condenser $\mathcal{E} = (A, C)$ in $\mathbb{R}^n$, where

$$A = \{x \in \mathbb{R}^n : |x - x_0| < R\},$$
$$C = \{x \in \mathbb{R}^n : |x - x_0| \leqslant r_0\},$$

and $1 < r_0 < R < \infty$. Then $f(\mathcal{E}) = (f(A), f(C))$ is a ringlike condenser in $\mathbb{R}^n$ and by (1.2), we have the equality

$$\mathrm{cap}_p f(\mathcal{E}) = \mathrm{M}_p\Big(\Delta\big(\partial f(A),\ \partial f(C),\ f(A\backslash C)\big)\Big).$$

Then due to (1.3) we have that

$$\mathrm{cap}_p(f(A), f(C)) \geqslant n\Omega_n^{\frac{p}{n}} \left(\tfrac{p-n}{p-1}\right)^{p-1} \left[m^{\frac{p-n}{n(p-1)}}(f(A)) - m^{\frac{p-n}{n(p-1)}}(f(C))\right]^{1-p},$$

whence

$$\mathrm{cap}_p(f(A), f(C)) \geqslant n\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} [m(f(A))]^{\frac{n-p}{n}}. \tag{2.2}$$

On the other hand, by Proposition 1.1, one gets

$$\mathrm{cap}_p(f(A), f(C)) \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}. \tag{2.3}$$

Combining the inequalities (2.2) and (2.3), we obtain

$$n\Omega_n^{\frac{p}{n}} \left(\frac{p-n}{p-1}\right)^{p-1} [m(f(A))]^{\frac{n-p}{n}} \leqslant \frac{\omega_{n-1}}{\left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{p-1}}.$$

Now, due to $\omega_{n-1} = n\Omega_n$, the last inequality can be rewritten as follows:

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} [m(f(A))]^{\frac{n-p}{n}} \leqslant \left(\int\limits_{r_0}^{R} \frac{dt}{t^{\frac{n-1}{p-1}} q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{1-p}. \tag{2.4}$$

Consider the case when $\alpha \in [0, p-1)$. Then the following estimate follows from the condition (2.1):

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} \left[m(f(A))\right]^{\frac{n-p}{n}} \leqslant$$
$$\leqslant \kappa \left(\frac{p-\alpha-1}{p-1}\right)^{p-1} \left((\ln R)^{\frac{p-\alpha-1}{p-1}} - (\ln r_0)^{\frac{p-\alpha-1}{p-1}}\right)^{1-p}.$$

Therefore,

$$m(f(A)) \geqslant$$
$$\geqslant \Omega_n \kappa^{\frac{n}{n-p}} \left(\frac{p-n}{p-\alpha-1}\right)^{\frac{n(p-1)}{p-n}} \left((\ln R)^{\frac{p-\alpha-1}{p-1}} - (\ln r_0)^{\frac{p-\alpha-1}{p-1}}\right)^{\frac{n(p-1)}{p-n}}.$$

Hence, by Proposition 1.2, we have

$$\operatorname{diam} f(B(x_0, R)) \geqslant$$
$$\geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-\alpha-1}\right)^{\frac{p-1}{p-n}} \left((\ln R)^{\frac{p-\alpha-1}{p-1}} - (\ln r_0)^{\frac{p-\alpha-1}{p-1}}\right)^{\frac{p-1}{p-n}}.$$

Dividing the last inequality by $(\ln R)^{\frac{p-\alpha-1}{p-n}}$ and taking the lower limit for $R \to \infty$, we conclude

$$\liminf_{R\to\infty} \frac{\operatorname{diam} f(B(x_0, R))}{(\ln R)^{\frac{p-\alpha-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-\alpha-1}\right)^{\frac{p-1}{p-n}}.$$

Now we consider the case when $\alpha = p-1$. Then it follows from (2.4) that

$$\Omega_n^{\frac{p}{n}-1} \left(\frac{p-n}{p-1}\right)^{p-1} \left[m(f(A))\right]^{\frac{n-p}{n}} \leqslant \kappa \left(\ln \frac{\ln R}{\ln r_0}\right)^{1-p}.$$

Therefore,

$$m\big(f(B(x_0, R))\big) \geqslant \Omega_n \kappa^{\frac{n}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{n(p-1)}{p-n}} \left(\ln \frac{\ln R}{\ln r_0}\right)^{\frac{n(p-1)}{p-n}}.$$

Hence, by Proposition 1.2, we obtain

$$\operatorname{diam} f(B(x_0, R)) \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}} \left(\ln \frac{\ln R}{\ln r_0}\right)^{\frac{p-1}{p-n}}.$$

Finally, dividing the last inequality by $(\ln\ln R)^{\frac{p-1}{p-n}}$ and taking the lower limit for $R \to \infty$, we conclude

$$\liminf_{R\to\infty} \frac{\operatorname{diam} f(B(x_0, R))}{(\ln\ln R)^{\frac{p-1}{p-n}}} \geqslant 2\kappa^{\frac{1}{n-p}} \left(\frac{p-n}{p-1}\right)^{\frac{p-1}{p-n}}.$$

This completes the proof. □

Remark 2.2. It is easy to construct examples of ring Q-homeomorphisms with respect to p-modulus on which the obtained estimates are attained.

References

[1] L. V. Ahlfors. *Lectures on quasiconformal mappings*, volume 38 of *University Lecture Series*. American Mathematical Society, Providence, RI, second edition, 2006. With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. `doi:10.1090/ulect/038`.

[2] C. J. Bishop, V. Gutlyanskii, O. Martio, and M. Vuorinen. On conformal dilatation in space. *Int. J. Math. Math. Sci.*, (22):1397–1420, 2003. `doi:10.1155/S0161171203110034`.

[3] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York, Inc., New York, 1969.

[4] F. W. Gehring. Rings and quasiconformal mappings in space. *Trans. Amer. Math. Soc.*, 103:353–393, 1962. `doi:10.2307/1993834`.

[5] F. W. Gehring. Lipschitz mappings and the p-capacity of rings in n-space. In *Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, Ann. of Math. Studies, No. 66, pages 175–193. Princeton Univ. Press, Princeton, N.J., 1971.

[6] A. Golberg. Differential properties of (α, Q)-homeomorphisms. *Further Progress in Analysis, Proc. 6th ISAAC Congr.*, pages 218–228, 2009. `doi:10.1142/9789812837332_0015`.

[7] A. Golberg and R. Salimov. Logarithmic Hölder continuity of ring homeomorphisms with controlled p-module. *Complex Var. Elliptic Equ.*, 59(1):91–98, 2014. `doi:10.1080/17476933.2013.823164`.

[8] A. Golberg, R. Salimov, and E. Sevost'yanov. Distortion estimates under mappings with controlled p-module. *Ann. Univ. Buchar. Math. Ser.*, 5:95–114, 2014.

[9] A. Ignat'ev and V. Ryazanov. Finite mean oscillation in the mapping theory. *Ukrainian Math. Bull.*, 2(3):403–424, 2005.

[10] A. Ignat'ev and V. Ryazanov. On the theory of the boundary behavior of spatial mappings. *Ukrainian Math. Bull.*, 3(2):199–211, 301, 2006.

[11] B. Klishchuk. On power-law behavior of some mapping class at infinity. *Journal of Mathematical Sciences*, 268(2):192–198, 2022. `doi:10.1007/s10958-022-06191-2`.

[12] D. Kovtonyuk, V. Ryazanov, R. Salimov, and E. Sevost'yanov. Toward the theory of the Orlicz–Sobolev classes. *Algebra i Analiz*, 25(6):50–102, 2013.

[13] O. Lehto and K. Virtanen. *Quasiconformal mappings in the Plane*. Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[14] O. Martio, S. Rickman, and J. Väisälä. Definitions for quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 448:1–40, 1969.

[15] O. Martio, S. Rickman, and J. Väisälä. Distortion and singularities of quasiregular mappings. *Ann. Acad. Sci. Fenn. Ser. A I No.*, 465:1–13, 1970.

[16] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov. To the theory of Q-homeomorphisms. *Dokl. Akad. Nauk*, 381:20–22, 2001.

[17] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov. Mappings with finite length distortion. *J. Anal. Math.*, 93:215–236, 2004. `doi:10.1007/BF02789308`.

[18] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov. On Q-homeomorphisms. *Ann. Acad. Sci. Fenn. Ser. A1. Math.*, 30:49–69, 2005. `doi:10.1007/978-0-387-85588-2_6`.

[19] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov. *Moduli in Modern Mapping Theory*. Springer Monographs in Mathematics. Springer, New York, 2009.

[20] V. Maz'ya. Lectures on isoperimetric and isocapacitary inequalities in the theory of Sobolev spaces. *Contemp. Math.*, 338:307–340, 2003.

[21] V. Ryazanov and E. Sevost'yanov. Normal families of spatial mappings. *Sib. Elektron. Mat. Izv.*, 3:216–231, 2006.

[22] V. Ryazanov and E. Sevost'yanov. Equicontinuous classes of ring Q-homeomorphisms. *Sib. Math. J.*, 48(6):1093–1105, 2007. `doi:10.1007/s11202-007-0111-4`.

[23] V. Ryazanov, U. Srebro, and E. Yakubov. On ring solutions of Beltrami equations. *J. Anal. Math.*, 96:117–150, 2005. `doi:10.1007/BF02787826`.

[24] R. Salimov. ACL and differentiability of a generalization of quasiconformal maps. *Izvestiya: Mathematics*, 72(5):977–984, 2008. `doi:10.1070/IM2008v072n05ABEH002425`.

[25] R. Salimov. On finitely Lipschitz space mappings. *Sib. Elektron. Mat. Izv.*, 8:284–295, 2011.

[26] R. Salimov. Estimation of the measure of the image of the ball. *Siberian Mathematical Journal*, 53(4):920–930, 2012.

[27] R. Salimov. One property of ring Q-homeomorphisms with respect to a p-module. *Ukrainian Math. J.*, 65(5):728–733, 2013. `doi:10.1007/s11253-013-0818-2`.

[28] R. Salimov and B. Klishchuk. The extremal problem for the area of an image of a disc. *Reports of the National Academy of Sciences of Ukraine*, (10):22–27, 2016. `doi:10.15407/dopovidi2016.10.022`.

[29] R. Salimov and B. Klishchuk. Extremal problem for the area of the image of a disk. *Journal of Mathematical Sciences*, 234(3):373–380, aug 2018. `doi:10.1007/s10958-018-4015-6`.

[30] R. Salimov and B. Klishchuk. An extremal problem for the volume functional. *Matematychni Studii*, 50(1):36–43, 2018. `doi:10.15330/ms.50.1.36-43`.

[31] R. Salimov and B. Klishchuk. Lower bounds for the volume of the image of a ball. *Ukrainian Math. J.*, 71(6):774–785, 2019. `doi:10.1007/s11253-019-01686-9`.

[32] R. Salimov and B. Klishchuk. On the asymptotic behavior at infinity of ring Q-homeomorphisms with respect to p-modulus. *Proceedings of Institute of Applied Mathematics and Mechanics of NAS of Ukraine*, 36(1):26–35, 2022. `doi:10.32782/1683-4720-2022-36-2`.

[33] R. Salimov and B. Klishchuk. On the behavior at infinity of one class of homeomorphisms. pages 173–180, 2022. `doi:10.1007/978-3-030-87502-2_17`.

[34] R. Salimov and E. Smolovaya. On the order of growth of ring Q-homeomorphisms at infinity. *Ukrainian Math. J.*, 62(6):961–969, 2010. `doi:10.1007/s11253-010-0403-x`.

[35] V. Shlyk. On the equality between p-capacity and p-modulus. *Sibirsk. Mat. Zh.*, 34(6):216–221, v, x, 1993. `doi:10.1007/BF00973485`.

[36] J. Väisälä. *Lectures on n-dimensional quasiconformal mappings*, volume 229 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1971.

Received: November 20, 2022, accepted: April 2, 2023.

Ruslan Salimov
Institute of Mathematics of the National Academy of Science of Ukraine
Tereschenkivs'ka str. 3, Kyiv, Ukraine, 01024
Email: ruslan.salimov1@gmail.com
ORCID: 0000-0001-9395-3334

Bogdan Klishchuk
Institute of Mathematics of the National Academy of Science of Ukraine
Tereschenkivs'ka str. 3, Kyiv, Ukraine, 01024
Email: kban1988@gmail.com
ORCID: 0000-0001-9632-2079

Mariia Stefanchuk
Institute of Mathematics of the National Academy of Science of Ukraine
Tereschenkivs'ka str. 3, Kyiv, Ukraine, 01024
Email: stefanmv43@gmail.com