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O. K. Bakhtin. Scientific legacy

Iryna Denega, Yaroslav Zabolotnyi

Dedicated to the memory of Professor Oleksandr Bakhtin

Abstract. In the paper we give a brief overview of the Olexandr Bakhtin' scientific results.

Анотація. В роботі дано короткий огляд наукових результатів Олександра Костянтиновича Бахтіна.

Oleksandr Bakhtin (10.11.1948 – 07.10.2021)

Bakhtin Oleksandr Konstantinovych was born on November 10, 1948 in Vladivostok, Primorsky Krai, USSR. Since 1974 he worked in the Institute of Mathematics of NAS of Ukraine. In 1971 he graduated the Far Eastern State University and in 1974 finished postgraduate studies in the Institute of Mathematics of the Academy of Sciences of USSR. Under supervision of P. M. Tamrazov in 1975 O. Bakhtin defended his candidate thesis "Conformal mappings and poles of quadratic differentials" in the specialized

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academic council of the Institute of Mathematics of the Academy of Sciences of USSR. In 2007 he defended his doctoral thesis “Extremal problems and quadratic differentials in geometric function theory of a complex variable” in the specialized academic council of the Institute of Mathematics of NAS of Ukraine and P. M. Tamrazov was his scientific consultant. In 2013 Oleksandr Bakhtin received a scientific title professor in specialty 01.01.01 – mathematical analysis.

O. K. Bakhtin wrote more than 150 publications, including 1 monograph [6]. Scientific research of O. K. Bakhtin was devoted to study of extremal problems of geometric functions theory of a complex variable. In particular, in the direction of studying the coefficients of single-leaf functions of class S, he obtained the following result [1]: *a function of class S, which simultaneously realizes maximum of the modules of two different coefficients, must coincide with the function $e^{-i\theta}K(e^{i\theta}z)$, where $K(z) = \frac{z}{(1-z)^2}$ is the well-known Koebe function, $\theta \in (0, \pi]$.* P. Duren, summarizing this result, formulated a problem on the maximum of the real parts of two linear functionals on the class S of functions, univalent and holomorphic in the unit circle, for which O. K. Bakhtin obtained decision in one partial case.

In his papers the method of “controlling” functionals was proposed, which expand the class of problems about extremal decomposition of the complex plane with free poles on the unit circle [6]. Application of the separating transformation to various extremal problems of geometric function theory was considered. New unimproved inequalities are obtained. It is also given a complete description of the cases of achieving of the upper bound in the estimate of the product of inner radii of mutually non-overlapping domains with respect to the points forming the so-called (n, m)-radial system. A general approach is proposed that allows to transfer most of the results of geometric function theory to the case of an arbitrary multidimensional complex space $\mathbb{C}^n$, $n \geqslant 2$ [3,4]. An effective upper estimates are obtained for the products of inner radii of mutually non-overlapping domains with fixed poles corresponding quadratic differentials [10]. An open problem of finding the maximum of product of inner radii of two domains relative to the points of a unit circle with each power $\gamma \in (0, 2]$ of the inner radius of the domain relative to the origin, provided that all three domains are mutually non-overlapping domains is solved. It is also generalized to the case of arbitrary pair of points of the complex plane [7].

Under the O. K. Bakhtin supervision A. L. Targonskyi (2006) [12–14,41–45], V. E. Vyun (2008) [17–19], R. V. Podvisotskii [5,11], I. Y. Vygovska (2012) [46–48], I. V. Denega (2013) [7–9], Y. V. Zabolotnyi (2014) [20,22,26,27], L. V. Vyhivska (2019) [10,15,16], I. Y. Dvorak (2019) [10,21] defended their candidate theses.

1. Definitions and denotations

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one point compactification, U be the open unit disk in $\mathbb{C}$, and $\mathbb{R}^+ = (0, \infty)$. Let also $\chi(t) = \frac{1}{2}\left(t + t^{-1}\right)$ be the Zhukovskii function.

Suppose that a function $f(z)$ is meromorphic in a disk $\{|z| < 1\}$ and maps univalently the disk $\{|z| < 1\}$ onto the domain $B \subset \overline{\mathbb{C}}$ so that $f(0) = a$, where $a \in B$. Then, the value

$$R(B, a) = |f'(0)|$$

is called the *conformal radius* of the domain B relative to the point $a \in B$. Conformal radius of the domain B with respect to an infinitely distant point is defind as $R(B, \infty) := R(\varphi(B), 0)$, where $\varphi(z) = 1/z$.

A *(classical) Green function* of the domain B with pole at $a \in B$ is a function $g_B(z, a)$ which is continuous in $\overline{\mathbb{C}}$, harmonic in $B \backslash \{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion:

$$g_B(z, a) = -\ln|z - a| + \gamma + o(1), \quad o(1) \to 0, \quad z \to a,$$

(if $a = \infty$, then $g_B(z, \infty) = \ln|z| + \gamma + o(1)$, $o(1) \to 0$, $z \to \infty$).

The inner radius $r(B, a)$ of the domain B with respect to a point a is the quantity e^{γ}. Note, that in the case of simply connected domains, the inner radius of the domain coincides with the conformal radius of the domain.

Let G be a domain in $\overline{\mathbb{C}}$. By a *quadratic differential* in G we mean the expression $Q(z)dz^2$, where $Q(z)$ is a meromorphic function in G (see, for example, [29, 31]).

The class S is the set of regular and single-leaf functions $f(z)$ in U which at the point $z = 0$ are expanded into the following series:

$$f(z) = z + a_2(f)z^2 + a_3(f)z^3 + \dots + a_n(f)z^n + \dots$$

for some $a_i \in \mathbb{C}$.

The class K is the set of all Koebe functions

$$f_\theta(z) = \frac{z}{(1 + e^{i\theta} z)^2},$$

where θ is real number, $\theta \in [0; 2\pi]$.

The following expansion holds for Koebe functions

$$f_\theta(z) = z + 2e^{i\theta} z^2 + 3e^{2i\theta} z^3 + \dots + ne^{(n-1)i\theta} z^n + \dots$$

2. The Bieberbach problem

In 1916 L. Bieberbach [24] proved that for any $f(z) \in S$ the inequality $|a_2(f)| \leqslant 2$ is valid. Also taking into account the following estimates for $z \in U$

$$\frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2},$$
$$\frac{1-|z|}{(1+|z|)^3} \leqslant |f^{'}(z)| \leqslant \frac{1+|z|}{(1-|z|)^3},$$

where equality occurs only for the Koebe functions, he formulated the following problem:

Problem 2.1 (L. Bieberbach, [24]). Prove (or deny) that for any $f(z) \in S$ the inequality $|a_n(f)| \leqslant n$ is true for any natural $n \geqslant 3$.

In the process of solving this problem, the following results were obtained:

- $|a_3(f)| \leqslant 3$ — K. Löwner, 1923;
- $|a_n(f)| \leqslant en,\ n = 2, 3, \ldots$ — D. Littlewood, 1924;
- $|a_3(f) - \alpha a_2^2(f)| \leqslant 2e^{-\frac{2\alpha}{1-\alpha}} + 1,\ 0 < \alpha < 1$ — M. Fekete, G. Szego, 1933;
- $|a_n(f)| \leqslant \frac{3}{2}en,\ n = 2, 3, \ldots$ — G. M. Goluzin, 1948;
- $|a_n(f)| \leqslant \frac{e}{2}n + 1,51$ — I. E. Bazilevich, 1951;
- $|a_4(f)| \leqslant 4$ — P. R. Garabedyan, M. Schiffer, 1955;
- $|a_n(f)| \leqslant 1,243n$ — I. M. Milin, 1967;
- $|a_6(f)| \leqslant 6$ — J. Pedersen, 1968, M. Ozawa, 1969;
- $|a_5(f)| \leqslant 5$ — J. Pedersen, M. Schiffer, 1972;
- $|a_n(f)| < \sqrt{\frac{7}{6}}n < 1,081n$ — C. H. Fitzgerald, 1972;
- $|a_n(f)| \leqslant 1,066n$ — J. L. Horowitz, 1976.

Let $\alpha_n = \max\limits_{f \in S} |a_n(f)|$. Denote by $S(n)$ the set of all functions $f \in S$ such that $\alpha_n = |a_n(f)|$. Also, let $S(n,m) := S(n) \cap S(m)$ for $2 \leqslant n < m$. Then the following theorem holds:

Theorem 2.2. [1] *If $S(n,m) \neq \varnothing$ for $2 \leqslant n < m$, then $S(n,m) = K$.*

This theorem states that only Koebe functions can simultaneously realize the global according to S maximums of the modules of two different coefficients.

For $f(z) \in S$ denote

$$\ln \frac{f(z)}{z} = \sum_{k=1}^{\infty} 2\gamma_k z^k,$$

where the numbers $2\gamma_k$ are logarithmic coefficients of the function $f(z)$.

Problem 2.3 (Milin, [38])**.** Prove (or disprove) that for any $f(z) \in S$, and also for natural k, ν, n the following inequality holds

$$\sum_{\nu=1}^{n}\sum_{k=1}^{\nu} k|\gamma_k|^2 = \sum_{k=1}^{n}(n+1-k)k|\gamma_k|^2 \leqslant \sum_{k=1}^{n}(n+1-k)\frac{1}{k} = \sum_{\nu=1}^{n}\sum_{k=1}^{\nu}\frac{1}{k}.$$

I. M. Milin also proved that the validity of Bieberbach's problem follows from the fulfillment of the hypothesis formulated by him.

The Milin hypothesis was proved by L. de Brange in 1984 [25].

3. Problems of Goluzin and Lebedev

In 1934 M. A. Lavrentiev solved the problem on the maximum of product of conformal radii of two non-overlapping simply connected domains.

Theorem 3.1. [36] *Let a_1 and a_2 be some fixed points of the complex plane $\mathbb{C}$, B_k, $a_k \in B_k$, $k \in \{1,2\}$ be any non-overlapping simply connected domains in $\overline{\mathbb{C}}$. Then the following inequality holds*

$$R(B_1, a_1)R(B_2, a_2) \leqslant |a_1 - a_2|^2. \tag{3.1}$$

Equality in (3.1) *occurs only in the case when the domains B_1 and B_2 are two half-planes, the imaginary axis is their common boundary, and the points a_1, a_2 are symmetric with respect to the common boundary of B_1 and B_2.*

In 1951 G. M. Goluzin [30] obtained the following an exact estimation for $n = 3$:

$$\prod_{k=1}^{3} r(B_k, a_k) \leqslant \frac{64}{81\sqrt{3}}|a_2 - a_1||a_3 - a_1||a_3 - a_2|.$$

If a_1, a_2, and a_3 are three equidistant points on the unit circle $\{|z| = 1\}$, then the equality holds only in the case when the domains B_1, B_2 and B_3 are bounded by rays emanating from the origin at equal angles to each other and containing a_1, a_2, and a_3 on their bisectors.

Later, G. M. Goluzin summarized the problem of M. A. Lavrentiev for the case of an arbitrary finite number $n \geqslant 3$, of mutually non-overlapping simply connected domains B_k, $a_k \in B_k \subset \mathbb{C}$, where $\{a_k\}_{k=1}^{n}$ are arbitrary fixed finite and mutually distinct points of the complex plane.

Problem 3.2 (Goluzin, [30]). Find the maximum of the expression

$$\prod_{k=1}^{n} r(B_k, a_k), \tag{3.2}$$

where $n \in \mathbb{N}$, $n \geqslant 2$, $a_k \in \overline{\mathbb{C}}$, $k = \overline{1,n}$ are some fixed points, and domains B_k, $k = \overline{1,n}$ are such that $a_k \in B_k \subset \overline{\mathbb{C}}$ and $B_i \bigcap B_j = \varnothing$ for $1 \leqslant i, j \leqslant n$, $i \neq j$.

G. V. Kuzmina [35] showed that the problem of maximum of the product (3.2) for $n = 4$ is reduced to the smallest capacity problem in the certain continuum family and obtained the exact inequality

$$\prod_{k=1}^{4} r(B_k, a_k) \leqslant \frac{9}{4^{\frac{8}{3}}} \Big(\prod_{1 \leqslant k < l \leqslant 4} |a_l - a_k| \Big)^{\frac{2}{3}}.$$

Moreover, the equality occurs only in the case when the points -1, 1, and a form a regular triangle, so $a = \pm i\sqrt{3}$.

Problem of maximum of the product (3.2) for $n = 5$ was solved in [29,35] under additional assumption that the quintuple $a_1, \ldots, a_5$ is symmetric with respect to a line or a circle

$$\prod_{k=1}^{5} r(B_k, a_k) \leqslant 4^{\frac{11}{3}} \cdot 3^{-\frac{3}{4}} \cdot 5^{-\frac{25}{6}} \Big(\prod_{1 \leqslant k < l \leqslant 5} |a_l - a_k| \Big)^{\frac{1}{2}}.$$

Equality occurs for points 1, $e^{-\frac{2\pi i}{3}}$, 0, $e^{\frac{2\pi i}{3}}$, ∞ and domains D_k, $k = \overline{1,5}$, which are circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{z^6 + 7z^3 + 1}{z^2(z^3 - 1)^2}\, dz^2.$$

No other ultimate results in the problem of maximum of the product (3.2) for $n \geqslant 5$ are known at present.

In the case of three or more points, many authors considered estimates of a more general Möbius invariant of the form

$$T_n := \frac{\prod\limits_{k=1}^{n} r(D_k, a_k)}{\Big\{ \prod\limits_{1 \leqslant k < p \leqslant n} |a_k - a_p| \Big\}^{\frac{2}{n-1}}},$$

(for an infinitely distant point under the corresponding factor we mean the unit).

Thus, based on the results of works [30, 35, 36], we get the following estimates

$$T_2 \leqslant 1, \qquad T_3 \leqslant \frac{64}{81\sqrt{3}} \approx 0,45618, \qquad T_4 \leqslant \frac{9}{4^{8/3}} \approx 0,22323.$$

O. K. Bakhtin and Y. V. Zabolotnyi [22] obtained the following statement.

Theorem 3.3. [22] *Let $n \in \mathbb{N}$, $n \geqslant 2$, $a_k \in \mathbb{C}$, $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, be some sets of fixed points and domains of the complex plane,respectively, such that $a_k \in B_k$, $k = \overline{1,n}$, $B_i \cap B_j = \varnothing$, $i \neq j$. Then the following inequality holds*

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant (n-1)^{-\frac{n}{4}} \Big(\prod_{1 \leqslant p < k \leqslant n} |a_p - a_k| \Big)^{\frac{2}{n-1}}. \tag{3.3}$$

In the case when one of the points a_k is infinitely distant, then in the inequality (3.3) all factors containing that point will be replaced with unit.

Corollary 3.4. [22] *Suppose that for an arbitrary natural n, $n \geqslant 2$, domains B_k and points a_k, $k = \overline{1,n}$, satisfy all conditions of Theorem 3.3. Then the following inequality holds*

$$T_n \leqslant (n-1)^{-\frac{n}{4}}.$$

In 1952 L. I. Kolbina [32] summarized Lavrentiev's result by taking functions in fixed positive degrees. Namely, she proved that for any finite distinct points a_1, a_2, the maximum of the value of $|f_1'(0)|^\alpha \cdot |f_2'(0)|^\beta$, $(\alpha > 0$, $\beta > 0)$ with respect to all possible systems of functions $f_k(z)$, $f_k(0) = a_k$, $(k \in \{1,2\})$, being regular in the circle U and univalently mapping $\{|z| < 1\}$ onto the domains B_1, B_2 such that $a_1 \in B_1 \subset \mathbb{C}$, $a_2 \in B_2 \subset \mathbb{C}$, is attained for functions which map $\{|z| < 1\}$ onto the circular domains of the following quadratic differential

$$Q(w)dw^2 = -\frac{(a_2 - a_1)[w - a_1 - \alpha(a_2 - a_1)/(\alpha - \beta)]}{(w - a_1)^2 (w - a_2)^2}\, dw^2.$$

More precisely, L. I. Kolbina established the following inequality

$$R^\alpha(B_1, a_1) R^\beta(B_2, a_2) \leqslant \frac{4^{\alpha+\beta}\alpha^\alpha \beta^\beta}{|\alpha - \beta|^{\alpha+\beta}} \left| \frac{\sqrt{\alpha} - \sqrt{\beta}}{\sqrt{\alpha} + \sqrt{\beta}} \right|^{2\sqrt{\alpha\beta}} |a_1 - a_2|^{\alpha+\beta}.$$

Also in [33] the following inequality is proved

$$R^\alpha(B_1, a_1) R^\beta(B_2, a_2) R^\beta(B_3, a_3) \leqslant$$
$$\leqslant A_{\alpha,\beta,\gamma} |a_1 - a_2|^{\alpha+\beta-\gamma} |a_1 - a_3|^{\alpha-\beta+\gamma} |a_2 - a_3|^{-\alpha+\beta+\gamma},$$

where

$$A_{\alpha,\beta,\gamma} = 4^{\alpha+\beta+\gamma} \alpha^\alpha \beta^\beta \gamma^\gamma \times$$

$$\times \frac{\left|2(\alpha\beta+\beta\gamma+\alpha\gamma)-\alpha^2-\beta^2-\gamma^2\right|^{\sqrt{\alpha\beta}+\sqrt{\beta\gamma}+\sqrt{\alpha\gamma}-\frac{1}{2}(\alpha+\beta+\gamma)}}{|(\sqrt{\alpha}+\sqrt{\beta})^2-\gamma|^{2\sqrt{\alpha\beta}}|(\sqrt{\beta}+\sqrt{\gamma})^2-\alpha|^{2\sqrt{\beta\gamma}}|(\sqrt{\alpha}+\sqrt{\gamma})^2-\beta|^{2\sqrt{\alpha\gamma}}}.$$

Some inequalities with respect to conformal radii were also obtained in the work [39]. N. A. Lebedev in the monograph [37] considered the following extremal problem on maximum of the product of conformal radii.

Problem 3.5 (Lebedev, [37]). Let $n \in \mathbb{N}$, $n \geqslant 3$, α_k, $k=\overline{1,n}$, be some positive real numbers and $\{a_k\}$, $k=\overline{1,n}$, be some set of points of the complex plane. Find the maximum of the product

$$\prod_{k=1}^{n} R^{\alpha_k}(B_k, a_k), \tag{3.4}$$

where B_k, $k=\overline{1,n}$, are arbitrary domains such that $a_k \in B_k$, $k=\overline{1,n}$, $B_i \cap B_j = \varnothing$, $i \neq j$.

This Problem 3.5 is a direct generalization of the previous results by M. A. Lavrentiev, G. M. Goluzin, Z. Nehari, J. Jenkins. However, it is not solved for now in general, and its solution is given only in some partial cases. In the paper by O. K. Bakhtin and Y. V. Zabolotnyi [20] following result is proved.

Theorem 3.6. [20] *Let $n \in \mathbb{N}$, $n \geqslant 3$, a_k, $k=\overline{1,n}$, be some fixed points of the complex plane, α_k, $k=\overline{1,n}$, be some positive real numbers, and $\alpha_k \geqslant \frac{\sum_{k=1}^{n}\alpha_k}{2n-2}$ for all $k=\overline{1,n}$. Let also $\lambda := \sum\limits_{k=1}^{n}\alpha_k$. Then for an arbitrary set of domains $B_k \subset \overline{\mathbb{C}}$, $k=\overline{1,n}$, such that $a_k \in B_k$, $k=\overline{1,n}$, and $B_i \cap B_j = \varnothing$, $i \neq j$, the following inequality holds*

$$\prod_{k=1}^{n} r^{\alpha_k}(B_k, a_k) \leqslant (n-1)^{-\frac{1}{4}\sum\limits_{k=1}^{n}\alpha_k} \prod_{i,j=1, i<j}^{n} |a_j - a_i|^{\frac{2}{n-2}\left(\alpha_i+\alpha_j-\frac{1}{n-1}\sum\limits_{k=1}^{n}\alpha_k\right)}.$$

4. «Controlling» functionals method

In 1968 P. M. Tamrazov [40] first attracted the attention to the study of extremal problems associated with quadratic differentials with non-fixed poles possessing a definite "freedom". He solved a significant extremal problem of the geometric function theory of a complex variable with five free simple poles.

In 1975, in accordance with this idea, G. P. Bakhtina [23] in her thesis first considered and solved a number of extremal problems on the class of mutually non-overlapping domains with so-called "free" poles of quadratic differentials.

Theorem 4.1. [23] *For any distinct points a_k, $k = \overline{1,4}$, of the unit circle $\{|z| = 1\}$ and for any mutually simply connected non-overlapping domains D_k, $a_k \in D_k \subset \overline{\mathbb{C}}$, $k = \overline{1,4}$, which are also symmetric relative to the unit circle $\{|z| = 1\}$, the following inequality holds*

$$\prod_{k=1}^{4} \left| f_k' (0) \right| \leqslant 1, \tag{4.1}$$

where $w = f_k (z)$ is a univalent and conformal mapping of the unit disk onto a domains D_k, $a_k = f_k (0)$, $k = \overline{1,4}$. The equality in (4.1) is attained if and only if a_k and B_k are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{w^2}{w^2(w^4-1)^2}\, dw^2.$$

Extremal problems on non-overlapping domains in which the corresponding points a_k have a certain "freedom" are called *extremal problems on non-overlapping domains with free poles.*

Let $n, m \in \mathbb{N}$. The system of points

$$A_{n,m} := \left\{ a_{k,p} \in \mathbb{C} : k = \overline{1,n},\ p = \overline{1,m} \right\}$$

is called (n,m)-*ray*, if for all $k = \overline{1,n}$ and $p = \overline{1,m}$ the following relations are satisfied

$$\begin{aligned} &0 < |a_{k,1}| < \ldots < |a_{k,m}| < \infty; \\ &\arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m} =: \theta_k =: \theta_k(A_{n,m}); \\ &0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi. \end{aligned}$$

For $m = 1$ the $(n,1)$-ray system of points is called an n-*ray.* In this case, we will use the following simpler notations:

$$a_{k,1} =: a_k,\ k = \overline{1,n}, \qquad\qquad A_{n,1} =: A_n.$$

On a set of (n,m)-ray systems consider the following quantities

$$\alpha_k := \alpha_k(A_{n,m}) := \tfrac{1}{\pi} \left[\theta_{k+1}(A_{n,m}) - \theta_k(A_{n,m}) \right],\ k = \overline{1,n},$$
$$\alpha_{n+1} := \alpha_1, \qquad\qquad \alpha_0 := \alpha_n.$$

The quantities $\alpha_k(A_{n,m})$, $k = \overline{1,n}$, are called the *angular parameters* of the system $A_{n,m}$. It is obvious that

$$\sum_{k=1}^{n} \alpha_k(A_{n,m}) = \sum_{k=1}^{n} \alpha_k = 2.$$

For an arbitrary (n,m)-ray system of points $A_{n,m}$ we consider a set of domains $P(A_{n,m}) = \{P_k\}_{k=1}^n$, where

$$P_k := P_k(A_{n,m}) := \big\{w \in \mathbb{C}\backslash\{0\} : \theta_k(A_{n,m}) < \arg w < \theta_{k+1}(A_{n,m})\big\},\ k = \overline{1,n}.$$

For a fixed $R \in \mathbb{R}^+$ and for any (n,m)-ray system of points $A_{n,m}$ we introduce the following "controlling" functional

$$M_R(A_{n,m}) := \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\chi\Big(\Big|\frac{a_{k,p}}{R}\Big|^{\frac{1}{\alpha_k}}\Big)\chi\Big(\Big|\frac{a_{k,p}}{R}\Big|^{\frac{1}{\alpha_{k-1}}}\Big)\right]^{\frac{1}{2}} \cdot |a_{k,p}|.$$

Here, the quantities $\{\mu_k(R)\}_{k=1}^n \subset \mathbb{R}^+$ for the given R are the *coefficients of displacements* of the system $A_{n,m}$. In this case, $\mu_k(R) \leqslant m^{-2m}$, $R \in \mathbb{R}^+$.

Transition from a system of domains $\{B_k\}_{k=1}^n$ to a system of domains $\{\widetilde{B}_k\}_{k=1}^n$ is called the operation of filling non-essential boundary components (see, for example, [6]).

The method of "controlling" functionals was proposed in the book [6]. It enables one to generalize some results on the extremal decomposition of the complex plane. The following result was established in [6].

Theorem 4.2. [6] *Let $R \in \mathbb{R}^+$, $n,m \in \mathbb{N}$ and $n \geqslant 3$. Then, for any (n,m)-ray system of points $A_{n,m} = \{a_{k,p}\}$, $k = \overline{1,n}$, $p = \overline{1,m}$, and for any set of mutually non-overlapping domains $\{B_{k,p}\}$, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, $p = \overline{1,m}$, the following inequality is true*

$$\prod_{k=1}^{n}\prod_{p=1}^{m} r(B_{k,p}, a_{k,p}) \leqslant 2^{nm} \cdot \Big(\prod_{k=1}^{n}\alpha_k\Big)^m \cdot \Big(\prod_{k=1}^{n}\mu_k(R)\Big)^{\frac{1}{2}} \cdot M_R(A_{n,m}).$$

The equality is attained here only in the case when $a_{k,p}$ and $\widetilde{B}_{k,p}$, $k = \overline{1,n}$, $p = \overline{1,m}$, are respectively the poles and circular domains of the following quadratic differential

$$Q(w)dw^2 = -\frac{w^{n-2}(R^n + w^n)^{2m-2}}{\Big[(R^{\frac{n}{2}} - iw^{\frac{n}{2}})^{2m} + (R^{\frac{n}{2}} + iw^{\frac{n}{2}})^{2m}\Big]^2}\,dw^2,$$

and $\operatorname{cap}\Big(\widetilde{B}_{k,p}\backslash B_{k,p}\Big) = 0.$

5. The Dubinin problem

Consider the following problem which was formulated in [28] in the list of unsolved problems.

Problem 5.1 (Dubinin, [28]). Prove that the maximum of the functional

$$I_n(\gamma) = r^{\gamma}(B_0, a_0) \prod_{k=1}^{n} r(B_k, a_k),$$

where $B_0,\ B_1,\ B_2, \ldots, B_n,\ (n \geqslant 2)$ are pairwise non-overlapping domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $k = \overline{1,n}$, $a_k \in B_k$, $k = \overline{0,n}$ and $\gamma \leqslant n$, attains at a configuration of domains B_k and points a_k possessing rotational n-symmetry.

In [28] the above-formulated problem was solved for $\gamma = 1$ and all values of the natural parameter $n \geqslant 2$. Namely, it was shown that the following inequality holds

$$r(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant r\left(D_0, 0\right) \prod_{k=1}^{n} r\left(D_k, d_k\right),$$

where d_k, D_k, $k = \overline{0,n}$, are the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-1)w^n + 1}{w^2(w^n-1)^2}\, dw^2.$$

In [34], L. V. Kovalev obtained its solution for systems of points for which the following inequalities hold

$$0 < \alpha_k \leqslant 2/\sqrt{\gamma}, \quad k = \overline{1,n},\ n \geqslant 5.$$

The following result obtained in [2] complements significantly the statement of the paper [28].

Theorem 5.2. [2] *For an arbitrary $\gamma \in \mathbb{R}^+$ there exists $n_0(\gamma) \in \mathbb{N}$ such that for each $n \geqslant n_0(\gamma)$ the inequality holds*

$$r^{\gamma}(B_0, a_0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant r^{\gamma}(B_0^{(0)}, a_0) \prod_{k=1}^{n} r(B_k^{(0)}, a_k^{(0)})$$

for any system of points $A_n = \{a_k\}_{k=1}^n$, $|a_k| = 1$, $k = \overline{1,n}$, $a_1 = 1$, and for any set of mutually non-overlapping domains $\{B_k\}_{k=0}^n$, $0 \in B_0$, $a_k \in B_k$, $k = \overline{1,n}$. The equality here is attained if and only if $a_k = a_k^{(0)}$, $\widetilde{B}_k = B_k^{(0)}$, $k = \overline{0,n}$, $a_0 = 0$, where $a_k^{(0)}$ and $B_k^{(0)}$ are, respectively, the poles and circular domains of the following quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\, dw^2, \tag{5.1}$$

and $\operatorname{cap}(\widetilde{B}_k \backslash B_k) = 0$, $k = \overline{0,n}$.

Denote

$$I_n^0(\gamma) = \left(\frac{4}{n}\right)^n \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}.$$

We obtained result which shows the limits of the application of the method proposed in [26]. The following theorem characterizes the extremal domains if $0 < \gamma \leqslant n^{\delta}$.

Theorem 5.3. [27] *For any natural $n \geqslant e^9$ and $0 < \gamma \leqslant n^{\frac{2}{3}-\frac{1}{\sqrt{\ln n}}}$ the following inequality holds*

$$r^{\gamma}(B_0, a_0)\prod_{k=1}^{n} r(B_k, a_k) \leqslant I_n^0(\gamma),$$

where B_0, B_1, $B_2, \dots, B_n$, are mutually non-overlapping domains in $\overline{\mathbb{C}}$, $a_0 = 0$, $|a_k| = 1$, $a_1 = 1$, $k = \overline{1,n}$. The equality is attained, for example, if $a_k = a_k^0$, $B_k = D_k$, $k = \overline{0,n}$, where a_k^0, D_k, are, respectively, the poles and circular domains of the quadratic differential (5.1).

The following theorem provides a complete solution of the Dubinin problem for $n = 2$.

Theorem 5.4. [7] *Let $\gamma \in (1; 2]$. Then for any different points a_1 and a_2 of the unit circle and any pairwise non-overlapping domains B_0, B_1, B_2, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_1 \in B_1 \subset \overline{\mathbb{C}}$, $a_2 \in B_2 \subset \overline{\mathbb{C}}$, the following inequality holds*

$$r^{\gamma}(B_0, 0) r(B_1, a_1) r(B_2, a_2) \leqslant I_2^0(\gamma)\left(\frac{1}{2}\left|a_1 - a_2\right|\right)^{2-\gamma}.$$

The sign of equality is attained, when the points a_0, a_1, a_2 and the domains B_0, B_1, B_2 are, respectively, the poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(4-\gamma)w^2+\gamma}{w^2(w^2-1)^2}dw^2.$$

6. Generalized M. A. Lavrentiev's inequality

In [9] there were proposed a method allowing to obtain new estimates for the products of inner radii of mutually non-overlapping domains. In particular, it allowed to generalize and strengthen the result of M. A. Lavrentiev [36] on the maximum of product of conformal radii of two non-overlapping simply connected domains.

Theorem 6.1. [9] *Let $n \in \mathbb{N}$. Then, for any fixed system of mutually distinct points $\{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ and any mutually non-overlapping domains $\{B_k\}_{k=0}^{n}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, the following inequality holds*

$$r^n(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant n^{-\frac{n}{2}} \left(\prod_{k=1}^{n} |a_k| \right)^2. \tag{6.1}$$

Inequality (6.1) will be referred to as the *generalized M. A. Lavrentiev inequality* because, for $n = 1$, we obtain the result of Theorem 3.1. Under additional conditions on the location geometry of a_k, $k = \overline{1,n}$, inequality (6.1) allows a clarification of Theorem 3.1 to be made. Namely, the following result holds.

Corollary 6.2. *Let $n \in \mathbb{N}$. Then, for any fixed system of mutually distinct points $\{a_k\}_{k=1}^{n}$ such that $|a_k| = \rho \in \mathbb{R}^+$, $k = \overline{1,n}$, and a system of any non-overlapping domains $\{B_k\}_{k=0}^{n}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, satisfying*

$$r\left(B_1, a_1\right) = r\left(B_2, a_2\right) = \ldots = r\left(B_n, a_n\right),$$

the following relationship is true

$$r(B_0, 0) r(B_k, a_k) \leqslant \frac{1}{\sqrt{n}} \cdot \rho^2, \quad k = \overline{1,n}.$$

On the other hand, provided all conditions of Corollary 6.2, from the M. A. Lavrentiev' Theorem 3.1, we obtain the inequality

$$r(B_0, 0) r(B_k, a_k) \leqslant \rho^2, \quad k = \overline{1,n}.$$

In the case of infinitely distant point we get

Corollary 6.3. *Let $n \in \mathbb{N}$. Then for any fixed system of mutually distinct points $\{a_k\}_{k=1}^{n}$ such that $|a_k| = R \in \mathbb{R}^+$, $k = \overline{1,n}$, and a set of any non-overlapping domains B_∞, $\{B_k\}_{k=1}^{n}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, satisfying $r\left(B_1, a_1\right) = r\left(B_2, a_2\right) = \ldots = r\left(B_n, a_n\right)$, the following inequality holds*

$$r(B_\infty, \infty) r(B_k, a_k) \leqslant \frac{1}{\sqrt{n}}, \quad k = \overline{1,n}.$$

At the same time, it follows from the corresponding Theorem 3.1 (Lavrentiev) that

$$r(B_\infty, \infty) r(B_k, a_k) \leqslant 1, \quad k = \overline{1,n}.$$

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Iryna Denega
Institute of Mathematics of the National Academy of Science of Ukraine
Tereschenkivs'ka str. 3, Kyiv, Ukraine, 01024
Email: iradenega@gmail.com
ORCID: 0000-0001-8122-4257

Yaroslav Zabolotnyi
Institute of Mathematics of the National Academy of Science of Ukraine,
Tereschenkivs'ka str. 3, Kyiv, Ukraine, 01024
Email: yaroslavzabolotnii@gmail.com
ORCID: 0000-0002-1878-2077